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Asymptotic Solution for an Inverse Parabolic Problem

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This paper deals with a constructive algorithm for asymptotic solution of a one-dimensional inverse parabolic problem. The Wentzel-Kramer-Brillouin (WKB) method is applied for determining an unknown term on the boundary.

Finally, the WKB estimation will be tested by given a numerical example.

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1. Introduction

The effective application of methods based on solving the inverse parabolic problems simulation and in processing the results of tests is determined by the depth of the mathematics required connected with the statement and algorithmic presentation of the problems, by clarifying the specific difficulties in their solution. Misunderstanding of the nature problems that are poorly based can lead to errors in problem solving. Even in cases when a proper method is used its effective application and specific features of the problem may not be fully realized. Such negligence of the formulation of problems of a given class as well as the method of their solving can lead to doubt as to the suitability of the very concept of inverse problems for practical research. Then a procedure to solve an inverse parabolic equation is very important.

In this paper, we consider a parabolic equation in one-dimensional space including non constant coefficient in derivative term, such that values a part of boundary is unknown. Some numerical and theoretical approaches of this type problems, for example in inverse heat conduction problems are summarized in

[1,2,5,6,10,11,12,13]. The unique solution of problem with constant of physically condition to power series form, have been provided by Burggraf in [4]. It has been shown that, if an error is made in known flux of boundary condition, then there will be errors in unknown values of other boundary.

A lower bound of this error can be estimated by $\frac{1}{\sqrt{\Delta t}} \sinh \frac{1}{\sqrt{\Delta t}}$, which Δt is increment of variable t. These results are consistent with earlier observations that small values of time Δt can produce large error in surface flux.

In this paper, we apply a finite difference method of semi-implicit type for $\partial_t u$ and use Wentzel-Kramer-Brillouin approximation (WKB) for finding an asymptotic solution for a system of ordinary differential equations that is produced. Finally, a numerical example is presented.

2. An inverse problem

Let $u(x,t)$ satisfies in linear parabolic equation in the form

$$\partial_t u(x,t) = a(x,t)\partial_{xx}u(x,t), \qquad 0 < x < 1,\ 0 < t < T_f, \tag{1}$$

$$u(x,0) = f(x), \qquad 0 \le x \le 1, \tag{2}$$

$$u(0,t) = g(t), \qquad 0 \le t \le T_f, \tag{3}$$

$$\partial_x u(0,t) = p(t), \qquad 0 \le t \le T_f, \tag{4}$$

where $a(x,t)$ is a known continuous function, so that $a(x,t)$ is positive function and $\partial_t a(x,t)$ don't change sign, T_f is the final time and $g(t)$ and $p(t)$ are piecewise-continuous known functions. Then, we shall determine

$$u(1,t) = h(t), \qquad 0 \le t \le T_f. \tag{5}$$

If we assume that $h(t)$ is a Hölder continuous function, and $g(t)$ and $p(t)$ $\in L^2[0,T_f]$ are known functions, $a(x,t) \in L^\infty([0,1] \times (0,T_f])$ and $a(x,t) > K_1$, where K_1 is a constant number, then there exists a solution $u \in L^2(0,T_f;H_0^1([0,1]))$ for problem (1)-(4) ([5,7,9]).

In the next section, by using the WKB-estimation method is derived an asymptotic approximation of solution for the problem (1)-(4).

3. WKB-estimation for determining asymptotic solution

In the case $f(x) = 0$, we discrete the variable t and identify the solution of (1)-(4).

Let, $M \in \mathrm{N}$, $\Delta t_M = \frac{T_f}{M}$, and $t_i = i\Delta t_M$ for $i \in I_M = \{0,1,2,\cdots,M\}$.

Now, if u is the solution of problem (1)-(4), then for any $i \in I_M$ the vector

$$\mathbf{u} = [u_1(x),\ldots,u_M(x)]^T,$$

is defined by

$$u_i(x) = u(x, t_i), \qquad i \in I_M.$$

Now, if $\mathbf{u}$ is approximated by $\hat{\mathbf{u}}$, then by putting

$$\hat{u}_{i+1}(x) = \hat{u}_i(x) + (\theta \partial_t \hat{u}(x, t_i) + \theta' \partial_t \hat{u}(x, t_{i+1}))\Delta t_M, \tag{6}$$

for $0 \leq \theta < 1$ and $\theta' = 1 - \theta \geq 0$, and substituting (6) into (1)-(4), we obtain a system linear of second order differential equations with given initial conditions in the form

$$\Delta t_M(\theta' a(x, t_{i+1})\hat{u}''_{i+1}(x) + \theta\ a(x, t_i)\hat{u}''_i(x)) = \hat{u}_{i+1}(x) - \hat{u}_i(x), \quad i \in I_{M-1} \tag{7}$$
$$\hat{u}_i(0) = g(t_i) = g_i, \quad i \in I_M - \{0\}, \tag{8}$$
$$\hat{u}'_i(0) = p(t_i) = p_i, \quad i \in I_M - \{0\}, \tag{9}$$

where $h_i = h(t_i)$ and $\hat{u}_i(x)$, for any $i \in I_M - \{0\}$ are unknown values and functions, respectively. Clearly $\hat{u}_0(x) = u_0(x) = f(x) = 0$. By this assumptions, when Δt_M vanishes, then this is the case of non regular dependence of the equation on the parameter.

We know that, for any $i \in I_{M-1}$ equation (7) is an equation with small parameter in the highest derivative. Whereas in the case of regular dependence the limit equation ($\Delta t_M = 0$) allows us to find first approximation to the solution, in the present case the limit equation $\hat{u}_{i+1}(x) - \hat{u}_i(x) = 0$, for any $i \in I_M$ contains no information about the solution of (7).

Now, we investigate the asymptotic form non-trivial solution for the problem (7)-(9). We can write the system of equations (7)-(9)

$$\frac{d^2\hat{\mathbf{u}}}{dx^2} = \lambda^2 A\hat{\mathbf{u}}, \tag{10}$$

where $A = B^{-1}.C$, such that

$$B = \begin{pmatrix} \theta' a_1(x) & 0 & 0 & \dots & 0 \\ \theta a_1(x) & \theta' a_2(x) & 0 & \dots & 0 \\ 0 & \theta a_2(x) & \theta' a_3(x) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \theta' a_M(x) \end{pmatrix}_{M\times M} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$a_i(.) = a(., t_i)$ for any $i \in I_M$ and $\lambda = (\Delta t_M)^{(-\frac{1}{2})}$. Thus $\hat{\mathbf{u}}(0) = [g_1, \dots, g_M]^T$ and $\hat{\mathbf{u}}'(0) = [p_1, \dots, p_M]^T$.

Obviously, since $a(., t)$ and $\partial_t a(., t)$ are positive functions in their domain, consequently, the characteristic equation (10), has not turning points in $[0, 1]$ ([6]).

Let $p_1(x), \cdots, p_M(x)$ be the eigenvalues of A, then $\pm\sqrt{p_1(x)}, \cdots, \pm\sqrt{p_M(x)}$ are roots of the characteristic equation in the form $det(A(x) - p^2 I) = 0$ ([6]). Now, we are in position to determine the formal asymptotic solution(FAS). This solution may be in the form

$$\hat{\mathbf{u}}(x) = \exp(\lambda S(x))(f_0(x) + \lambda^{-1} f_1(x) + \lambda^{-2} f_2(x) + \cdots), \tag{11}$$

where $S(x)$ is an unknown function and $f_0(x), f_1(x), \cdots$ are unknown vector-functions ([3,6]).

By substituting (11) into the system, cancel the exponential term and equating to zero the coefficients of power of λ^{-1}, then we obtain a recurrent system of equations

$$(A - S'^2(x)I) f_0(x) = 0, \tag{12}$$

$$(A - S'^2(x)I) f_1(x) = f_0(x)S''(x) + 2f_0'(x)S'(x), \tag{13}$$

$$(A - S'^2(x)I) f_k(x) = f_{k-1}(x)S''(x) + 2f_{k-1}'(x)S'(x) + f_{k-2}''(x), \quad k \geq 2. \tag{14}$$

It follows from (12) that $S'^2(x)$ is an eigenvalues, and $f_0(x)$ is an eigenvectors of A. Let $\{e_1(x), \cdots, e_M(x)\}$ be a base of eigenvectors. Putting

$$S_j(x) = \pm \int_0^x \sqrt{p_j(s)} ds, \qquad 1 \leq j \leq M,$$

and

$$f_{0,j}(x) = \alpha_{0,j}(x) e_j(x), \qquad 1 \leq j \leq M.$$

For any $i = 0, 1, 2, \cdots, M$ the function $\alpha_{0,j}(x)$ may be found from the following equation

$$(A(x) - p_j(x)I) f_1(x) = \pm \frac{4f_{0,j}'(x)p_j(x) + f_{0,j}(x)p_j'(x)}{2\sqrt{p_j(x)}}. \tag{15}$$

Clearly, the matrix of this system is a degenerate matrix. It is will known that, a necessary and sufficient condition for the system (15) to be soluble is the following orthogonality condition

$$e_j^*(x).(4f_{0,j}'(x)p_j(x) + f_{0,j}(x)p_j'(x)) = 0, \tag{16}$$

to be satisfied. Here, $\{e_j^*(x)\}_{j=1}^M$ is a base of left eigen-row-vector defined by

$$e_j^*(x).A(x) = p_j(x)e_j^*(x).$$

Then, we obtain,

$$(17)\quad \begin{aligned} 4p_j(x)\alpha'_{0,j}(x)e_j^*(x).e_j(x) + 4p_j(x)\alpha_{0,j}(x)e_j^*(x).e'_j(x) \\ +p'_j(x)\alpha_{0,j}(x)e_j^*(x).e_j(x) = 0. \end{aligned}$$

Consequently, (17) gives

$$\alpha_{0,j}(x) = \exp\{\int_0^x P_{0,j}(s)ds\},$$

where

$$P_{0,j}(x) = -\frac{e_j^*(x).(4p_j(x)e'_j(x) + p'_j(x)e_j(x))}{4p_j(x)e_j^*(x).e_j(x)}.$$

Now, we write

$$f_{1,j}(x) = \sum_{k=1}^{M} \alpha_{k,j}(x)e_k(x),$$

then, from (13) and (14) we obtain

$$\alpha_{k,j}(x) = \frac{2p_j(x)\alpha_{0,j}(x)e_j^*(x).e'_j(x)}{p_k(x) - p_j(x)}, \quad k \neq j$$

and

$$\alpha_{j,j}(x) = \alpha_{0,j}(x)\int_0^x \frac{R_j(s)}{\alpha_{0,j}(s)}ds,$$

where

$$R_j(x) = -\frac{e_j^*(x).(2\sqrt{p_j(x)}f''_{0,j}(x) + 4p_j(x)\sum_{\substack{k=1 \\ k\neq j}}^{M} \alpha_{k,j}(x)e'_k(x))}{4p_j(x)e_j^*(x).e_j(x)}.$$

Similarly, The vector functions $f_{2,j}, f_{3,j}, \cdots$, will be found for $1 \leq j \leq M$. Thus (16) has $2M$ FAS solutions of the form (11).

Let us write out the leading terms. For this purpose,let us introduce the notation

$$\tilde{u}_j^{(\pm)}(x;\lambda) = \exp\{\pm\lambda\int_0^x \sqrt{p_j(s)}ds + \ln(\alpha_{0,j}(x))\}, \;\; 1 \leq j \leq M,$$

then

$$\hat{u}_j^{(\pm)}(x) = \tilde{u}_j^{(\pm)}(x;\lambda)(e_j(x) + \mathbf{o}(\lambda^{-1})).$$

Now, if for any $1 \le k \le M, Re(p_j(x) - p_k(x))$, do not change sign for $x \in [0,1]$, then the system of equations in the form (10) has two solutions $\hat{u}_j^{(\pm)}(x)$ satisfying (10), as $\lambda \longrightarrow +\infty$ uniformly in $x \in [0,1]$ ([8]).

In order to find a solution from (10) without the term exponentially of λ, a nicely done ideas is $p_j(x) = -y^2(x)$, where, it means of that $\theta > 1$ or $\theta' < 0$ ([13]). Then

$$\hat{u}(x) = \sum_{j=1}^{M} c_j^{(+)}\hat{u}_j^{(+)}(x) + \sum_{j=1}^{M} c_j^{(-)}\hat{u}_j^{(-)}(x), \tag{18}$$

such that $c_j^{(\pm)}$are derived of the initial conditions (8) and (9), are a solution of the system (10).

Finally, if $f(x) \neq 0$, then using the superposition principle, putting $u(x,t) = v(x,t) + w(x,t)$, then we find that

$$\begin{aligned}
\partial_t v(x,t) &= a(x,t)\partial_{xx} v(x,t), \; 0 < x < 1, \; 0 < t < T_f \\
v(x,0) &= 0, \qquad 0 \le x \le 1, \\
v(0,t) &= g(t), \qquad 0 \le t \le T_f, \\
\partial_x v(0,t) &= p(t), \qquad 0 \le t \le T_f,
\end{aligned}$$

and

$$\begin{aligned}
\partial_t w(x,t) &= a(x,t)\partial_{xx} w(x,t), \; 0 < x < 1, \; 0 < t < T_f \\
w(x,0) &= f(x), \qquad 0 \le x \le 1, \\
w(0,t) &= 0, \qquad 0 \le t \le T_f, \\
w(1,t) &= v(1,t), \qquad 0 \le t \le T_f,
\end{aligned}$$

and we can find an asymptotic solution for problem(1)-(4).

The above result may be summarized in the following statement.

Theorem. *Let $a(x,t)$ and $\partial_t a(x,t)$ are strictly positive functions and differentiable for any $x \in [0,1]$ for a given fixed t, then the ordinary second order system of differential equations (10) possesses two asymptotic solutions*

$$\hat{u}_j^{(\pm)}(x) = \tilde{u}_j^{(\pm)}(x;\lambda)(e_j(x) + \boldsymbol{o}(\lambda^{-1})),$$

where for any $j = 0, 1, \cdots, M$,

$$\tilde{u}_j^{(\pm)}(x; \lambda) = \exp\{\pm\lambda \int_0^x \sqrt{p_j(s)}ds + \ln(\alpha(x))\},$$

and

$$\alpha_{0,j}(x) = \exp\{-\int_0^x \frac{e_j^*(s).(4p_j(s)e_j'(s) + p_j'(s)e_j(s))}{4p_j(s)e_j^*(s).e_j(s)}ds\},$$

$p_j(x), e_j(x)$, *and* $e_j^*(x)$ *are eigenvalues, eigenvectors, and left-eigen-row-vectors matrix* A, *respectively.*

Proof. See the analysis preceding the above theorem statement. ■

4. Numerical sample

In this section, to illustrate the results of this paper we have chosen an example with linear flux. The following example demonstrate the application of the proposed approach. The exact values for u satisfies in problem (1)-(4). Detailed descriptions for some $\Delta t_M = 0.2, .01$, and $T_f = 1$ at $x = 1$ are shown as follows.

Example. Assume that

$$\begin{aligned} a(x,t) &= \frac{1}{t}, \\ f(x) &= 0, \\ g(t) &= t, \\ p(t) &= -t. \end{aligned}$$

Obviously, $u(x,t) = t\exp(-x)$ and $h(t) = -\frac{t}{e}$ are exact solutions for the above problem. Now we use WKB method to this problem. For $\Delta t_M = .2, .1$ and $\theta = 0.1$, at $x = 1$, the result are given in the following table.

t	$h(t) = u_x(1,t)$	$\Delta t_M = 0.2$		$\Delta t_M = 0.1$	
		$\hat{u}_x(1,t)$	$\frac{\lvert u_x(1,t)-\hat{u}_x(1,t)\rvert}{\lvert u_x(1,t)\rvert}$	$\hat{u}_x(1,t)$	$\frac{\lvert u_x(1,t)-\hat{u}_x(1,t)\rvert}{\lvert u_x(1,t)\rvert}$
.2	-0.0735759	-0.0560657	%23.7	-0.0802724	%9.1
.4	-0.147152	-0.160545	%9.10	-0.148848	%1.15
.6	-0.220728	-0.213328	%3.3	-0.220985	%0.22
.8	-0.294304	-0.297695	%1.15	-0.294333	%0.29
1	-0.367879	-0.366499	%0.3	-0.367882	%0.37

Table 1. Exact and approximate solution $u_x(1,t)$ with $\theta = 0.1$

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