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On Some Approximative Methods for Solving Vekua Complex Differential Equation

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In this paper, some approximative methods for solving the Vekua complex differential equation such as modified areolar series method and Euler method are considered. As some linear and nonlinear complex difference equations play very important role in solving of complex differential equations, the Vekua complex difference equation will be considered.

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1. Introduction

The elliptic system of partial differential equations

$$\begin{aligned} u'_x - v'_y &= a(x,y)u + b(x,y)v + f(x,y), \\ u'_y + v'_x &= c(x,y)u + d(x,y)v + g(x,y), \end{aligned} \tag{1.1}$$

where a, b, c, d, f, g are given continuous real functions of real arguments x and y in a simply connected domain T, plays an important role in many problems in mechanics. If the second equation is multiplied by i and after addition to the first equation, the following complex equation is obtained

$$U'_{\bar{z}} = MU + N\overline{U} + L, \tag{1.2}$$

where $M = \dfrac{a+d+ic-ib}{4}$, $N = \dfrac{a-d+ic+ib}{4}$, $L = \dfrac{f+ig}{2}$, $U = u + iv$. It is known [1] that the function $U_0(z) = \exp\left(-\dfrac{1}{\pi}\iint\limits_T \dfrac{M(\xi,\eta)}{t-z}d\xi d\eta\right)$, $(t = \xi + i\eta)$ is a regular, partial solution of the equation $U'_{\bar{z}} = MU$. By the substitution

$U = w\,U_0$, where w is a new unknown function, equation (1.2) is transformed into the following differential equation:

$$w'_{\bar{z}} = A\overline{w} + B, \tag{1.3}$$

where $A = \dfrac{N\overline{U_0}}{U_0}$, $B = \dfrac{L}{U_0}$. It is shown by Vekua [1] that the general solution of equation (1.3) contains double singular integrals with Cauchy kernel, which cannot be solved in closed form, so it is important to construct some approximative methods for solving equation (1.3). We give the following definitions:

Definition 1.1. Let

$$\overline{z} = g(z) \tag{1.4}$$

be a given complex equation, where $g(z)$ is an arbitrary analytic function. The equation (1.4) can define a closed or open contour, or a set of isolated points. Further, the set of points in the complex plane defined by the equation (1.4) is called $K-$*contour*. In practice, a great number of important contours (line, circle etc) can be presented by (1.4).

Definition 1.2. Let $g(z)$ be a given analytic function and $w = w(z, \bar{z})$ be a continuous complex function which can be developed in a convergent power series in terms of z and $\bar{z}$. Then, the compound function $w = w(z, g(z))$ is an analytic function too, for which we will use the symbol $\alpha_{g(z)}w$. The operator $\alpha_{g(z)}w$ maps the set of continuous complex functions $w = w(z, \bar{z})$ to the set of analytic functions, and the geometrical meaning of this operator is as follows: if $\bar{z} = g(z)$ is an equation of a closed contour, then the functions $w = w(z, \bar{z})$ and $\alpha_{g(z)}w$ have the same boundary values on the mentioned contour.

Let $W(z, \bar{z})$ be a given complex function infinitely differentiable with respect to $\bar{z}$ on the circular ring P: $\sqrt{a} - \delta/2 \le |z| \le \sqrt{a} + \delta/2$. It is supposed that there is a positive number $b = \sup b_k$; $k = n+1, n+2, \ldots$, where b_k are majorants of $|\alpha_{a/z} W_{\bar{z}}^{(k)}|$. It is shown by Canak [2] that the function $W(z, \bar{z})$ can be approximated in P by complex polynomial

$$W(z, \bar{z}) \approx f_0(z) + (\bar{z} - a/z) f_1(z) + \ldots + (\bar{z} - a/z)^n f_n(z). \tag{1.5}$$

The coefficients $f_k(z)$ can be calculated by using the formula

$$f_k(z) = \frac{\alpha_{a/z} W_{\bar{z}}^{(k)}}{k!}, \quad k = 0, 1, \ldots \tag{1.6}$$

In order to estimate the error of the approximation, the inequality

$$|R| \le b \frac{(\delta/2)^{n+1}}{(n+1)!} \exp(\delta/2) \tag{1.7}$$

is used, and $|R| \to 0$ when $n \to \infty$. In the special case when $n = 1$, the approximation is

$$W(z, \bar{z}) \approx \alpha_{a/z} W + (\bar{z} - a/z)\alpha_{a/z} W'_{\bar{z}} \tag{1.8}$$

and

$$|R| \le \frac{b\delta^2}{8} \exp(\delta/2) \tag{1.9}$$

is the error of the approximation.

2. Approximative solving of the Vekua complex differential equation using modified areolar polynomials

The integral operator $⨍$, which is an inverse to the operator $\partial/\partial\bar{z}$, is introduced by Theodorescu and Fempl ([3], [4]), and they have made some investigations of its properties. This operator is often addressed as *areolar integral.*

Further, we consider the continuous complex function $w(z, \bar{z})$, expandable in convergent series z and $\bar{z}$ in some domain T. The set of such functions is denoted by W_T. These functions can be integrated term by term and its areolar integral is a common indefinite integral on variable $\bar{z}$. It is possible to define areolar definite integral by

$$⨍_a^b w(z, \bar{z}) \overset{def}{=} \alpha_b W - \alpha_a W, \tag{2.1}$$

where

$$⨍ w(z, \bar{z}) = W(z, \bar{z}), \quad (W'_{\bar{z}} = w(z, \bar{z})). \tag{2.2}$$

If the function $w(z, \bar{z})$ belongs to the class W_T, then the right-hand side of (2.1) is an analytic function. The following theorem is proved by Canak [5]:

Theorem 2.1. *Let $w_0(z)$ be an analytic function in a finite simply connected domain T. The Vekua complex differential equation (1.3) with the initial condition*

$$\alpha_0 w(z, \bar{z}) = w_0(z) \tag{2.3}$$

is equivalent to the areolar integral equation

$$w = w_0 + ⨍_0^{\bar{z}} (A\bar{w} + B), \quad \bar{z} \in T, \quad w \in W_T. \tag{2.4}$$

Since the functions $A(z)$, $B(z)$ and $w_0(z)$ are analytic in T, then there exist such positive real numbers $\alpha, \gamma, \delta, r$, satisfying

$$|w_0(z)| \leq \alpha, \ \ |A(z)| \leq \gamma, \ \ |B(z)| \leq \delta, \ \ |z| \leq r, \ \ (z \in T).$$

The following sequences of functions will be useful:

$$\varphi_0(z) = 1, \ \varphi_1(z) = \int_0^z w(z)dz, \ \ldots, \ \varphi_n(z) = \int_0^z A(z)\varphi_{n-2}(z)dz, \tag{2.5}$$

$$\beta_0(z) = z, \ \beta_1(z) = \int_0^z B(z)dz, \ \ldots, \ \beta_n(z) = \int_0^z A(z)\beta_{n-2}(z)dz. \tag{2.6}$$

Then we form the so-called modified areolar polynomials $\sum_{k=1}^{n} \varphi_{k-1}\overline{\varphi}_k$ and $\sum_{k=1}^{n} \beta_{k-1}\overline{\beta}_k$. For their coefficients the following estimations are valid:

$$|\varphi_1(z)| = \left|\int_0^z w_0(z)dz\right| \leq \alpha|z| \leq \alpha r \ , \ldots, \ |\varphi_{2j-1}(z)| \leq \gamma^{j-1}\alpha r^j; \ |\varphi_{2j}(z)| \leq \gamma^j r^j,$$

$$|\beta_1(z)| = \left|\int_0^z B(z)dz\right| \leq \delta r \ , \ldots, \ |\beta_{2j-1}(z)| \leq \delta\gamma^{j-1}r; \ |\beta_{2j}(z)| \leq \gamma^j r^{j+1}.$$

By using the previous inequalities, one obtains

$$\left|\sum_{k=1}^{n} \varphi_{k-1}\overline{\varphi}_k\right| \leq \alpha r + \alpha\gamma r^2 + \ldots + \alpha\gamma^{n-1}r^n = \alpha r\frac{1-(\gamma r)^n}{1-\gamma r}, \tag{2.7}$$

$$\left|\sum_{k=1}^{n} \beta_{k-1}\overline{\beta}_k\right| \leq \delta r^2 + \delta\gamma r^3 + \ldots + \delta\gamma^{n-1}r^{n+1} = \delta r^2\frac{1-(\gamma r)^n}{1-\gamma r}. \tag{2.8}$$

It follows from (2.7) and (2.8) that the corresponding areolar series

$$\sum_{k=1}^{\infty} \varphi_{k-1}\overline{\varphi}_k = \lim_{n\to\infty}\sum_{k=1}^{n} \varphi_{k-1}\overline{\varphi}_k; \qquad \sum_{k=1}^{\infty} \beta_{k-1}\overline{\beta}_k = \lim_{n\to\infty}\sum_{k=1}^{n} \beta_{k-1}\overline{\beta}_k$$

are absolutely and uniformly convergent for $\gamma r \leq q < 1$ because

$$\left|\sum_{k=1}^{\infty} \varphi_{k-1}\overline{\varphi}_k\right| \leq \frac{\alpha r}{1-\gamma r}; \qquad \left|\sum_{k=1}^{\infty} \beta_{k-1}\overline{\beta}_k\right| \leq \frac{\delta r^2}{1-\gamma r}. \tag{2.9}$$

These series define two continuous complex functions

$$(2.10)\qquad \varphi(z,\bar z)=\sum_{k=1}^{\infty}\varphi_{k-1}\overline{\varphi}_k,\quad \beta(z,\bar z)=\sum_{k=1}^{\infty}\beta_{k-1}\overline{\beta}_k.$$

On the base of the formula (2.4) the recurrent sequence of functions can be constructed: $w_n = w_0 + \int_0^{\bar z}(A\overline{w}_{n-1}+B)d\bar z$, $n=1,2\ldots$. Thus, it is easy to obtain the sequence of approximative solutions of the equation (1.3) as:

$$(2.11)\qquad \begin{aligned} w_1 &= w_0+\int_0^{\bar z}(A\overline{w}_0+B)d\bar z = w_0 + A\varphi_0\overline{\varphi}_1 + B\bar z\\ w_2 &= w_0+\int_0^{\bar z}(A\overline{w}_1+B)d\bar z = w_0 + A\varphi_0\overline{\varphi}_1 + A\varphi_1\overline{\varphi}_2 + B\bar z + A\beta_0\overline{\beta}_1\\ &\vdots\\ w_n &= w_0 + A\sum_{k=1}^{n}\varphi_{k-1}\overline{\varphi}_k + B\bar z + A\sum_{k=2}^{n}\beta_{k-2}\overline{\beta}_{k-1}.\end{aligned}$$

As the modified areolar series (2.10) are uniformly convergent, the sequence of functions (2.11) is uniformly convergent as well, i.e.

$$(2.12)\ \lim_{n\to\infty} w_n(z,\bar z) = w(z,\bar z) = w_0 + B\bar z + A\sum_{k=1}^{\infty}\varphi_{k-1}\overline{\varphi}_k + A\sum_{k=2}^{\infty}\beta_{k-2}\overline{\beta}_{k-1}.$$

The function $w(z,\bar z)$ is an exact solution of the Vekua equation (1.3). Indeed, if the value of (2.12) and its derivative $w'_{\bar z} = B + A\sum_{k=1}^{\infty}\varphi_{k-1}\overline{\varphi}'_k + A\sum_{k=2}^{\infty}\beta_{k-2}\overline{\beta}'_{k-1}$ are substituted in the equation (1.3), we can see that the coefficients of the modified areolar series on the left side are equal to the corresponding coefficients on the right side, i.e. the function $w(z,\bar z)$ satisfied the equation (1.3). Further, we can see that the function (2.12) satisfies the initial condition (2.3).

Finally, it is possible to estimate the absolute value of the error of the $n-$th approximative solution w_n of the equation (1.3) by

$$\begin{aligned} |R| = |w-w_n| &= \left|A\sum_{k=n+1}^{\infty}\varphi_{k-1}\overline{\varphi}_k + A\sum_{k=n+1}^{\infty}\beta_{k-2}\overline{\beta}_{k-1}\right|\\ &\le A\alpha r\left[(\gamma r)^n + (\gamma r)^{n+1}+\ldots\right] + A\delta r^2\left[(\gamma r)^n+(\gamma r)^{n+1}+\ldots\right]\\ &\le (\gamma\alpha r+\gamma\delta r^2)(\gamma r)^n\frac{1}{1-\gamma r} = (\alpha+\delta r)\frac{(\gamma r)^{n+1}}{1-\gamma r}.\end{aligned}$$

If the assumption $\gamma r\le q<1$ is valid, then the error of the estimation can be always made arbitrary small.

3. A method of Euler type for approximative solving of the Vekua complex differential equation

Let

$$w'_{\bar z} = A(z,\,\bar z)\bar w + B(z,\bar z) \tag{3.1}$$

be a given complex differential equation of Vekua type with the initial condition

$$\alpha_{a/z} w = w_0(z). \tag{3.2}$$

It is supposed that $A(z,\,\bar z)$ and $B(z,\,\bar z)$ are given continuous functions of the circular ring P: $\sqrt{a}-\delta/2 \le |z| \le \sqrt{a}+\delta/2$ and $w_0(z)$ is a given analytic function of the same ring.

Let$[a,A]$, $(a, A \in R)$ be a given finite interval. We will introduce the mesh $x_i = x_0 + ih,\ h = (A-a)/n,\ x_0 = a,\, x_n = A$ and construct the system of circles $K_i : \bar z = x_i/z,\ (i = 0,1,2,\dots,n)$. An analytic function of a complex variable can be presented in an explicit form by $W = W(z,\bar z)$, or in tabular form, if its boundary values on a given circles $W(z,\bar z)/_{K_i}$ are known. By using operator α it is possible to construct a sequence of analytic functions $w_0(z), w_1(z),\dots,w_n(z)$, if their boundary values on K_i are equal to the corresponding boundary values of $W(z,\bar z)$. The presentation of the analytic function $W(z,\bar z)$ in a tabular form

(3.3)

K_i	$K_0 : \bar z = x_0/z$	$K_1 : \bar z = x_1/z$	$\dots$	$K_n : \bar z = x_n/z$
$w_i(z)$	$w_0(z)$	$w_1(z)$	$\dots$	$w_n(z)$

is called *α-representation* of the function on the system of circles K_i.

If $\alpha_{x_i/z}\overline{W}(z,\bar z) = w_i(z)$, $(i = 0,1,\dots,n)$, then the representation (3.3) is unique for each value $\bar z = x_i/z$ corresponds one and only one analytic function $w_i(z)$.

We are looking for an approximative solution of the problem (3.1)–(3.2) in the form (3.3). We are trying to find a sequence of analytic functions $w_i(z)$, $(i = 0,1,\dots,n)$ which boundary values on the circles $\bar z = x_i/z$, $(x_{i+1} - x_i = h = (A-a)/n)$ are approximatively equal to the limit values of the exact solutions.

Using (3.1), the formula (1.8) is transformed into

$$W(z,\bar z) \approx \alpha_{a/z} W + (\bar z - a/z)\alpha_{a/z}(A\overline{w} + B). \tag{3.4}$$

By using the operator $\alpha_{x_1/z}$ on (3.4), the first approximative value is $w_1 = w_0 + \dfrac{h}{z}\alpha_{a/z}(A\overline{w}_0 + B)$. On the circle $\bar z = x_1/z$, the formula (3.4) is transformed into

$$W(z,\bar z) \approx \alpha_{x_1/z} W + (\bar z - x_1/z)\alpha_{x_1/z}(A\overline{w} + B). \tag{3.5}$$

The second approximative value w_2 is obtained by using the operator $\alpha_{x_2/z}$ on (3.5), i.e. $\alpha_{x_2/z}W = w_2 = w_1 + \frac{h}{z}\alpha_{x_1/z}(A\overline{w}_1 + B)$. By resuming this process in the same way, the recurrent formula

$$w_{i+1} = w_i + \frac{h}{z}\alpha_{x_i/z}(A\overline{w}_i + B) \tag{3.6}$$

is obtained, and it is an approximative solution of the problem (3.1)–(3.2) in the tabular form (3.3).

4. On a Vekua complex difference equation

Some linear and nonlinear complex difference equations are considered by Čanak [6] and he has shown that they play very important role in solving of complex differential equations. In this chapter, the Vekua complex difference equation will be considered.

Definition 4.1. Let $L_0 : \bar{z} = g_0(z),\ L_1 : \bar{z} = g_1(z), \dots, L_n : \bar{z} = g_n(z)$ be given closed contours which limit the domains $G_0, G_1, \dots, G_n$ and let $G_0 \subset G_1 \subset \dots \subset G_n$. In [6] Čanak introduced areolar ψ-differences as:

$$\frac{\alpha_{g_1}w - \alpha_{g_0}w}{g_1 - g_0} = \psi(g_0, g_1),\ \dots,\ \frac{\alpha_{g_n}w - \alpha_{g_{n-1}}w}{g_n - g_{n-1}} = \psi(g_{n-1}, g_n)$$

$$\begin{aligned} &\frac{\psi(g_1, g_2) - \psi(g_0, g_1)}{g_2 - g_0} = \psi(g_0, g_1, g_2), \\ &\qquad\vdots \\ &\frac{\psi(g_{n-1}, g_n) - \psi(g_{n-2}, g_{n-1})}{g_n - g_{n-2}} = \psi(g_{n-2}, g_{n-1}, g_n) \\ &\qquad\vdots \\ &\frac{\psi(g_1, g_2, \dots, g_n) - \psi(g_0, g_1, \dots, g_{n-1})}{g_n - g_0} = \psi(g_0, g_1, \dots, g_{n-1}, g_n). \end{aligned} \tag{4.1}$$

On the base of the mentioned differences, the sequence of functions

$$\begin{aligned} &\frac{\alpha_{g_0}w - w(z, \bar{z})}{g_0 - \bar{z}} = \psi(\bar{z}, g_0) \\ &\frac{\psi(g_0, g_1) - \psi(\bar{z}, g_0)}{g_1 - \bar{z}} = \psi(\bar{z}, g_0, g_1) \\ &\qquad\vdots \\ &\frac{\psi(g_0, g_1, \dots, g_n) - \psi(\bar{z}, g_0, g_1, \dots, g_{n-1})}{g_n - \bar{z}} = \psi(\bar{z}, g_0, g_1, \dots, g_n) \end{aligned} \tag{4.2}$$

is constructed.

These differences play an important role in the theory of interpolation of the nonanalytic functions. The equations that contain unknown complex function and its differences of the type (4.2) for the system of contours $\bar{z} = g_i(z), (i = 0, 1, \ldots, n)$ are called *complex areolar ψ-difference equations.*

Let

$$\psi_w(\bar{z}, g) = b(z, \bar{z})\overline{w} + c(z, \bar{z}) \tag{4.3}$$

be a complex difference equation of the Vekua type, where $\bar{z} = g(z)$ is an equation of a simple smooth closed contour, and $b(z, \bar{z}), c(z, \bar{z})$ are given continuous complex functions. It follows from (4.3) that

$$\psi_w(\bar{z}, g) = \frac{\alpha_g w - w}{g - \bar{z}} = b\overline{w} + c \Rightarrow w + (g - \bar{z})b\overline{w} = \alpha_g w - (g - \bar{z})c. \tag{4.4}$$

The conjugated equation

$$\overline{w} + (\overline{g} - z)\bar{b}w = \overline{\alpha_g w} - (\overline{g} - z)\overline{c} \tag{4.5}$$

will also be considered. The solution of the system (4.4)–(4.5) is

$$w(z, \bar{z}) = \frac{\alpha_g w - (g - \bar{z})\Big[c + b\overline{\alpha_g w} - (\overline{g} - z)b\overline{c}\Big]}{1 - b\bar{b}(g - \bar{z})(\overline{g} - z)}. \tag{4.6}$$

Since the general solution of difference equation (4.3) contains an arbitrary analytic function $\varphi(z)$, we make an assumption, on the base of (4.6), that it should be in the following form

$$w(z, \bar{z}) = \frac{\varphi(z) - (g - \bar{z})\Big[c + b\overline{\varphi(z)} - (\overline{g} - z)b\overline{c}\Big]}{1 - b\bar{b}(g - \bar{z})(\overline{g} - z)}. \tag{4.7}$$

It means that the unknown analytic function $\alpha_g w$ is included into the arbitrary analytic function $\varphi(z)$. As the function (4.7) satisfies the equation (4.3), it is its general solution. We can see that the process of solving of the Vekua difference equation (4.3) is easier than the corresponding differential equation (1.3). The substitution of the first derivation $w'_{\bar{z}}$ by the approximative value of ψ-difference $\psi_w(\bar{z}, g) = \dfrac{\alpha_g w - w}{g - \bar{z}}$ together with the formula (3.4) open the possibility of approximative solving the Vekua differential equation via difference equations. In some particular cases, the difference equation (4.3) enables the exact solving of the differential equation (1.3). At first, we can state a question: for which class of functions $w(z,\bar{z})$ its first derivative $w'_{\bar{z}}$ is equal to its ψ-difference $\psi_w(\bar{z},g)$? Answering the question, we introduce the new differential equation

$$w'_{\bar{z}} = \frac{\alpha_g w - w}{g - \bar{z}}. \tag{4.8}$$

As the contour $\bar{z} = g(z)$, i.e. the analytic function $g(z)$ is an arbitrary function, we can put $\alpha_g w = \xi(z)$, where $\xi(z)$ is an arbitrary analytic function. So, the differential equation (4.8) transforms into $w'_{\bar{z}} - \dfrac{w}{\bar{z}-g} + \dfrac{\varphi(z)}{\bar{z}-g} = 0$ and its general solution [7] is bi-analytic function $w = (\bar{z} - g)\varphi(z) + \xi(z)$ or $w = \bar{z}\varphi(z) + \xi_1(z)$, $(\xi_1 = -g\varphi + \xi)$, where $\varphi(z)$, $\xi(z)$ are arbitrary analytic functions.

We can conclude that if the difference equation (4.3) has a bi-analytic function as a solution, then it is a solution of the Vekua differential equation as well, and the process of solving of the differential equation can be substituted by the solving the corresponding difference equation. Now we can determine the condition upon it is possible.

Let the difference equation

$$\psi_w(\bar{z},g) = A\overline{w} + B \tag{4.9}$$

have a solution of the form $w = \bar{z}\varphi(z) + \xi(z)$, where $\varphi(z)$, $\xi(z)$ are arbitrary analytic functions. By the substitutions

$$\psi_w(\bar{z},g) = \frac{\alpha_g w - w}{g - \bar{z}} = \frac{\alpha_g\Big[\bar{z}\varphi(z) + \xi(z)\Big] - \Big[\bar{z}\varphi(z) + \xi(z)\Big]}{g - \bar{z}}$$

$$= \frac{g\varphi(z) + \xi(z) - \bar{z}\varphi(z) - \xi(z)}{g - \bar{z}} = \frac{\varphi(z)(g - \bar{z})}{g - \bar{z}} = \varphi(z)$$

and $\overline{w} = z\overline{\varphi(z)} + \overline{\xi(z)}$, from (4.9), we obtain $\varphi(z) = A\Big[z\overline{\varphi(z)} + \overline{\xi(z)}\Big] + B$ or

$$B = \varphi(z) - A\Big[z\overline{\varphi(z)} + \overline{\xi(z)}\Big]. \tag{4.10}$$

So, we can formulate the following

Theorem 4.1. *The Vekua complex differential equation (1.3) and its corresponding difference equation (4.9) have a solution in the form of a bi-analytic function for any coefficient $A = A(z,\bar{z})$, if the coefficient $B = B(z,\bar{z})$ is in the form of (4.10).*

Example. Find a particular solution of the Vekua differential equation

$$w'_{\bar{z}} = \frac{1}{\bar{z}}\overline{w} - \bar{z}. \tag{4.11}$$

Solution. In this example, the coefficients $A(z,\bar{z}) = 1/\bar{z}$ and $B(z,\bar{z}) = -\bar{z}$. The condition (4.10) is satisfied if $\varphi(z) = z$ and $\xi(z) = z^2$. It means that the equation (4.11) has a particular solution in the form of a bi-analytic function $w(z,\bar{z}) = z\bar{z} + z^2$.

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