Mathematica
Balkanica

New Series Vol. 19, 2005, Fasc. 1-2

# Algorithm for Generating and Map-Coloring of a Special Type of Rectangular Diophantine Carpets

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Different aspects of planar Diophantine figures and carpets were introduced in [3-6]. Their computer implementation is entirely based on arithmetic techniques over integers and data structures. A special type of Diophantine carpet, constructed by a so called Fibonacci triangular decomposiiton (FTD) was considered in [6]. The parametrization of the decomposition and a recursive procedure for its visualization were given too. In this paper we present a generalization of the generating algorithm from [6] and an optimized heuristic algorithm for map-coloring of the carpets. Full computer implementation with Maple.

*AMS Subj. Classification*: 14G99, 68-04

*Key Words*: Diophantine carpet, tiling, map-coloring algorithm

## 1. Introduction

**Definition 1.1.** A *Diophantine figure* $D(P_1, P_2, ..., P_n)$ is defined as a subset of points $P_i \in Z^2 = \{(x_i, y_i) \in Z\}, i = 1, 2..n$ if all distances $d(P_k, P_l)$ $k, l \in 1, 2, ..n$ are nonnegative integer numbers.

**Definition 1.2.** A *Diophantine carpet* $DC(P_1, P_2, ..., P_n)$ where $P_i \in Z^2, i = 1, 2..n$ consists of points $P_i$ some of them connected by line segments with integer length.

**Definition 1.3.** A triple of three nonzero integers $(a, b, c)$ with $a^2+b^2 = c^2$ is called *Pythagorean triple*; a triple is called primitive if these integers are coprime, i.e. $\gcd(a, b, c) = 1$.

Consider any integer - sided right - angled triangle as *Pythagorean triangle* (PT). Pythagorean triangles are represented by Pythagorean triples.

In this paper we consider integer - sided rectangle decomposed into non-overlapping Pythagorean triangles as a DC. Moreover we include also cases with rational coordinates for the points of the DC.

The aim of this work was to develop practical effective algorithms for generating and map-coloring of this type of carpets.

## 2. Fibonacci triangular decomposition and parametrization

In [6] was obtained parametrization of the decomposition of a rectangle with integer sides into four non - overlapping PT as it is shown in Figure 1. Such a decomposition was called *Fibonacci triangular decomposition* (FTD), since it gives a geometric interpretation of the Fibonacci's two - square identity

$$(a_1^2+b_1^2)(a_2^2+b_2^2) = (a_1a_2+b_1b_2)^2+(b_1a_2-a_1b_2)^2 = (a_1a_2-b_1b_2)^2+(b_1a_2+a_1b_2)^2 \tag{2.1}$$

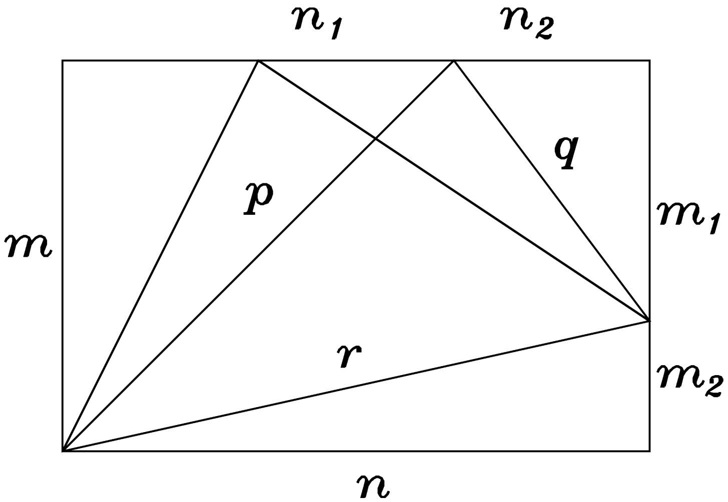


**Figure 1:** Fibonacci triangular decomposition.

The following two theorems were proved :

**Theorem 2.1.[6]** *Any two primitive Pythagorean triples* $(a_1, b_1, c_1)$ *and* $(a_2, b_2, c_2)$ *satisfying*

$$(a_1,b_1)(a_2,b_2) = (a_1a_2-b_1b_2, b_1a_2+a_1b_2), \quad a_1a_2 > b_1b_2 \tag{2.2}$$

$$(b_1,a_1)(a_2,b_2) = (b_1a_2-a_1b_2, a_1a_2+b_1b_2), \quad b_1a_2 > a_1b_2 \tag{2.3}$$

$$(a_1,b_1)(b_2,a_2) = (a_1b_2-b_1a_2, b_1b_2+a_1a_2), \quad b_1a_2 < a_1b_2 \tag{2.4}$$

$$(b_1,a_1)(b_2,a_2) = (b_1b_2-a_1a_2, a_1b_2+b_1a_2), \quad a_1a_2 < b_1b_2 \tag{2.5}$$

*produce exactly one solution of Fibonacci triangular decomposition if* $(a_1, b_1, c_1) \equiv (a_2, b_2, c_2)$ *and exactly two solutions if* $(a_1, b_1, c_1) \neq (a_2, b_2, c_2)$.

**Theorem 2.2.[6]** *The general parametrization of the Fibonacci triangular decomposition is given by the formulas (2.6) - (2.9).*

$$\begin{array}{ll} m = (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) & n = 2\alpha\beta(\gamma^2 - \delta^2) + (\alpha^2 - \beta^2)2\gamma\delta \\ n_1 = 2\alpha\beta(\gamma^2 - \delta^2) & n_2 = (\alpha^2 - \beta^2)2\gamma\delta \\ m_1 = 4\alpha\beta\gamma\delta & m_2 = (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) - 4\alpha\beta\gamma\delta \\ p = (\alpha^2 + \beta^2)(\gamma^2 - \delta^2) & q = (\alpha^2 + \beta^2)2\gamma\delta \\ r = (\alpha^2 + \beta^2)(\gamma^2 + \delta^2) & \end{array} \tag{2.6a}$$

*where parameters* $\alpha, \beta, \gamma, \delta$ *satisfy*

$$(\alpha^2 - \beta^2)(\gamma^2 - \delta^2) > 4\alpha\beta\gamma\delta, \quad \alpha, \beta, \gamma, \delta \in Z \tag{2.6b}$$

$$\begin{array}{ll} m = (\alpha^2 - \beta^2)2\gamma\delta & n = 4\alpha\beta\gamma\delta + (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) \\ n_1 = 4\alpha\beta\gamma\delta & n_2 = (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) \\ m_1 = 2\alpha\beta(\gamma^2 - \delta^2) & m_2 = (\alpha^2 - \beta^2)2\gamma\delta - 2\alpha\beta(\gamma^2 - \delta^2) \\ p = (\alpha^2 + \beta^2)2\gamma\delta & q = (\alpha^2 + \beta^2)(\gamma^2 - \delta^2) \\ r = (\alpha^2 + \beta^2)(\gamma^2 + \delta^2) & \end{array} \tag{2.7a}$$

*where parameters* $\alpha, \beta, \gamma, \delta$ *satisfy*

$$(\alpha^2 - \beta^2)2\gamma\delta > 2\alpha\beta(\gamma^2 - \delta^2) \tag{2.7b}$$

$$\begin{array}{ll} m = 2\alpha\beta(\gamma^2 - \delta^2) & n = (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) + 4\alpha\beta\gamma\delta \\ n_1 = (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) & n_2 = 4\alpha\beta\gamma\delta \\ m_1 = (\alpha^2 - \beta^2)2\gamma\delta & m_2 = 2\alpha\beta(\gamma^2 - \delta^2) - (\alpha^2 - \beta^2)2\gamma\delta \\ p = (\alpha^2 + \beta^2)(\gamma^2 - \delta^2) & q = (\alpha^2 + \beta^2)2\gamma\delta \\ r = (\alpha^2 + \beta^2)(\gamma^2 + \delta^2) & \end{array} \tag{2.8a}$$

*where parameters* $\alpha, \beta, \gamma, \delta$ *satisfy*

$$(\alpha^2 - \beta^2)2\gamma\delta < 2\alpha\beta(\gamma^2 - \delta^2), \quad \alpha, \beta, \gamma, \delta \in Z \tag{2.8b}$$

$$\begin{array}{ll} m = 4\alpha\beta\gamma\delta & n = (\alpha^2 - \beta^2)2\gamma\delta + 2\alpha\beta(\gamma^2 - \delta^2) \\ n_1 = (\alpha^2 - \beta^2)2\gamma\delta & n_2 = 2\alpha\beta(\gamma^2 - \delta^2) \\ m_1 = (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) & m_2 = 4\alpha\beta\gamma\delta - (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) \\ p = (\alpha^2 + \beta^2)2\gamma\delta & q = (\alpha^2 + \beta^2)(\gamma^2 - \delta^2) \\ r = (\alpha^2 + \beta^2)(\gamma^2 + \delta^2), & \end{array} \tag{2.9a}$$

*where parameters* $\alpha, \beta, \gamma, \delta$ *satisfy*

$$(\alpha^2 - \beta^2)(\gamma^2 - \delta^2) < 4\alpha\beta\gamma\delta, \quad \alpha, \beta, \gamma, \delta \in Z \tag{2.9b}$$

## 3. Generating Diophantine carpets by tiling procedure

**Definition 3.1.** A *tiling* is a complete covering of the surface by non - overlapping polygons, called tiles.

In this paper we consider tiling of a given FTD with Pythagorean triangles. To do this we proceed with decomposition of each of the four PT of a given FTD into PT using altitudes in order to construct a Diophantine carpet (see Figure 1). It's important to be mentioned that not every tiling of FTD leads to Diophantine carpet in the sense of Definition 1.2. since there are cases where not all of the points have integer coordinates - some of them are rational. Anyway the other condition is always fulfilled - all line segments have integer length.

At this point the solution of the above problem is reduced to decomposition of a given PT by it's altitude to hypotenuse into two PT. The answer is given by Buchholz with the following lemma:

**Lemma 3.1.[2]** *If* $a, b, c, \gamma, \in \mathbb{N}$ *and* $c_1$ *and* $c_2$ *are the distances from the foot of the altitude* $\gamma$ *to the endpoints of the side* $c$ *then* $c_1, c_2 \in \mathbb{N}$*.*

Now since $\gamma = ab/c$ scaling up the rectangle by $c$ ensures that $\gamma \in \mathbb{N}$. This could be repeated as long as we obtain so integer length altitudes (respectively so PT) as we need.

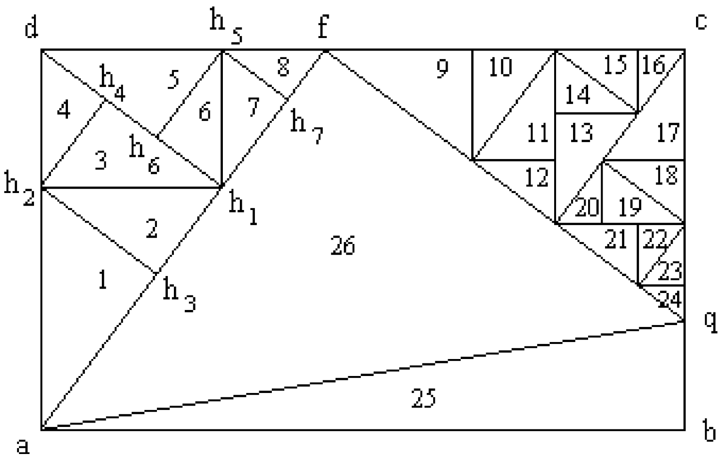


**Figure 2:** Example of FTD tiled with Pythagorean trianglrs Fibonacci triangular decomposition.

The tree structure, representing the generation of the above Diophantine carpet is shown in Figure 3 and the corresponding graph representation is shown in Figure 4.

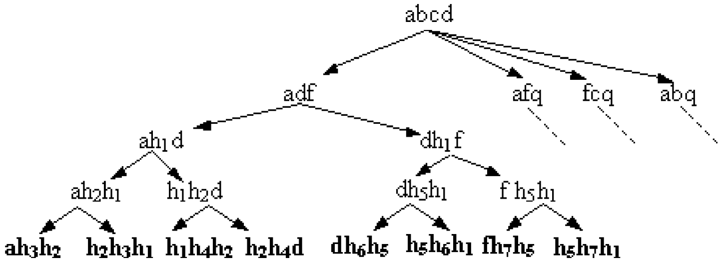


**Figure 3:** The "tiling" tree of the DC from Figure 2.

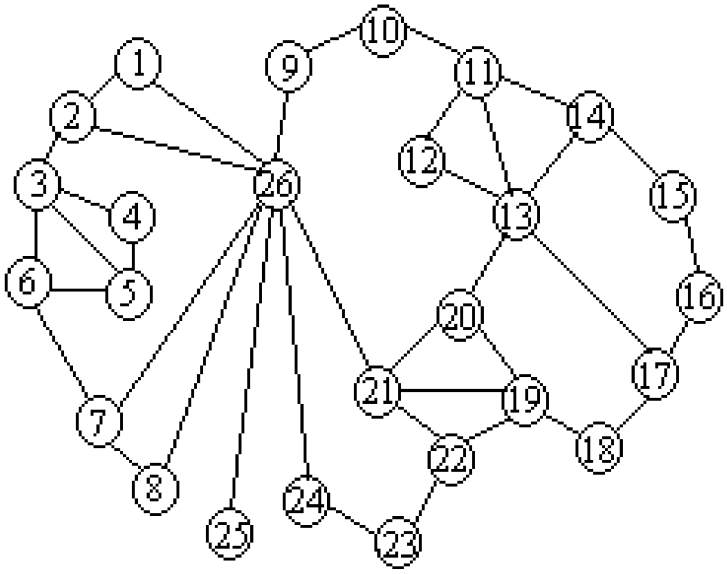


**Figure 4:** Graph representation of thr DC from Figure 2.

#### 4. Coloring algorithm

For coloring of the above generated DC we propose a practical heuristic algorithm for minimal graph coloring so that no two adjacent vertices are given the same color.

1. Assign every vertex a set of three "free" colors, for instance {1,2,3}.
2. Choose the vertex with maximal degree from uncolored vertices.
3. Assign the current vertex a color already forbidden for the most of its neighbours if possible, else assign a color with minimal number from its set of colors.

4. Forbid the used color for all the neighbours of the current vertex, decrease the number of their neighbours with 1.

5. Mark the current vertex as colored.

6. Examine the list of neighbours of the current vertex for vertices with set of colors consisting of only one "free" color.

7. If any then queue these vertices and make them consequently *current vertex* and execute steps 4, 5, 6, 7, else execute step 2 and following while there are still uncolored vertices.

## 5. Discussion

The proposed heuristic algorithm can be classified as greedy algorithm, implemented by recursion [1]. The color picked by the above procedure is a "greedy choice" in sense that, intuitively, it leaves as much opportunity as possible for the remaining vertices to be colored.

A greedy algorithm obtains an optimal solution by making a sequence of choices. For each decision point in the algorithm, the choice that seems best at the moment is taken. This heuristic strategy does not always produce an optimal solution, but that risk worth since with the proposed algorithm coloring will occur in a single attempt, i.e. this strategy leads to faster algorithms.

In our work we consider 3 - graph coloring as an optimum solution and 4 - graph coloring as an non-optimum solution. In Figures 5-8 are shown some examples of 3 - colored Diophantine carpets. Computer implementation was performed with Maple.

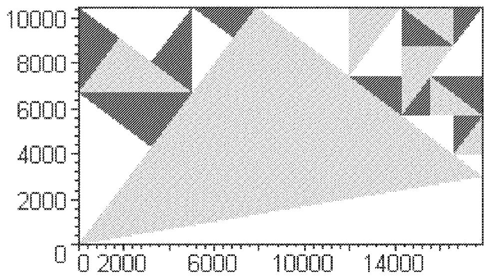


**Figure 5:** 3-coloring of the example from Figure 2.

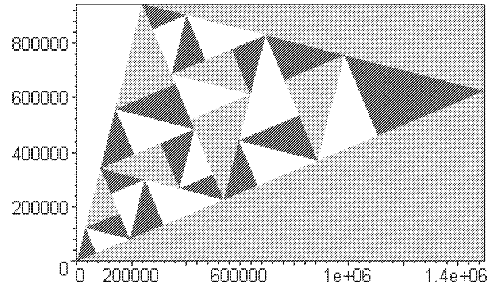


**Figure 6:** Example of 3-colored DC.

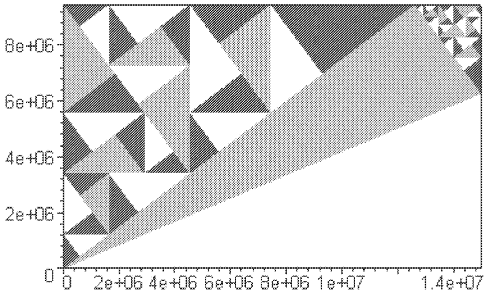


**Figure 7:** Example of 3-colored DC.

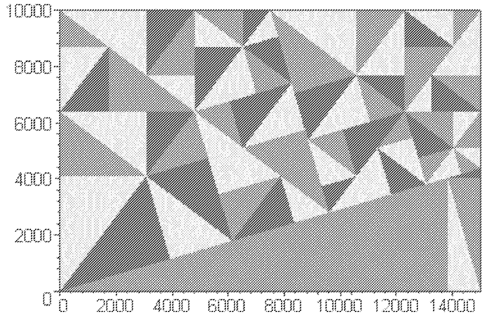


**Figure 8:** Example of 3-colored DC.

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*Received 30.09.2003*