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On Visualization of Numbers *

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An algorithm for converting real numbers into planar ornament is given. It is based on the continued fraction expansion $x = [a_0; a_1, a_2, \ldots]$ and representation of partial quotients $\{a_i\}$ in the number system with base $b(\geq 2)$, i.e. $a_i = c_0c_1 \ldots c_k$, where $c_j \in \{0, 1, \ldots, b-1\}$, $c_0 \neq 0$. Every digit c_j can be associated with the single movement along a edge of a regular planar grid associated with a regular tessellation of the plane. In this way, the integer number a_i becomes a code for a planar oriented graph g_i while the sequence $\{a_i\}$ will represent a sequence of graphs concatenated to each other, making thereby a supergraph $G(x)$ which is oriented and has all its inner vertices of order two, except the first, and, in finite case, the last one. It is proved that the mapping $G : x \mapsto G(x)$ is bijective.

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1. Continued fractions

Every now and then in the history of mathematics there appear trials to visualize numbers. There exist triangular, quadratic, pentagonal etc. (natural) numbers. Also, there are pyramidal, octahedral, dodecahedral etc. numbers that also belongs to $\mathbb{N}$. But, if the real numbers are in account, the visualization is not so easy. This note is a step towards having a real number visualized through a graphical representation of its continued fraction expansion. Starting point for such a visualization is *continued fraction expansion* of a number. The continued fraction expansion is preferable due to its intrinsic property. In other words, the continued fraction expansion of a number contains deeper characteristics of a number than its decimal representation ([2],[4]).

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It is well known that every $x \in \mathbb{R}$ has finite or infinite continued fraction representation

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = [a_0;\, a_1,\, a_2,\, a_3, \dots], \tag{1.1}$$

depending on the sequence $a_0, a_1, a_2, \dots$ of *partial quotients* being finite or infinite. Restriction to the case $x > 0$ implies $a_0 \in \mathbb{N}_0$ and $a_i \in \mathbb{N}$, $i \geq 1$.

Expansion (1.1) is finite if and only if x is rational. In fact, the following well known theorem refers to this case.

Theorem A *Any positive rational number can be uniquely expressed by a finite continued fraction* (1.1) *with* $a_0 \in \mathbb{N}_0$, $a_i \in \mathbb{N}$, *and for* $n \geq 1$, $a_n \geq 2$.

The finite expansion is shortened to $x = [a_0;\, a_1,\, a_2, \dots,\, a_n]$.

Note that the purpose of the restriction on a_n is to exclude ambiguity rising from the obvious identity $[a_0;\, a_1,\, a_2, \dots,\, a_n] = [a_0;\, a_1,\, a_2, \dots,\, a_n - 1,\, 1]$, that takes place for $a_n \geq 2$, $n \geq 1$.

Infinite continued fraction expansions can be periodic or non-periodic.

Theorem B *Expansion* (1.1) *is periodic if and only if x is a quadratic irrational, i.e.,* $x = \alpha + \sqrt{\beta}$, $\alpha \in \mathbb{R}$, $\beta > 0$.

A periodic continued fraction with endlessly repeated subsequence of partial quotients $a_p, \dots a_q$ will be shortened as $[a_0;\, a_1,\, a_2, \dots,\, a_{p-1},\, (a_p, \dots,\, a_q)^\infty]$.

It follows by Theorems A and B that $x \in \mathbb{R}$ has infinite, non-periodic continued fraction expansion if and only if x is not rational or quadratic irrational.

2. Continued fraction's graphs

Consider the representation of a positive integer p in base $b \geq 2$

$$p = \sum_{j=0}^{k} c_j b^{k-j}, \quad c_0 \neq 0, \quad c_j \in \{0, 1, \dots, b-1\}, \quad k \geq 0, \tag{2.1}$$

that is shortened to $p = c_0 c_1 \dots c_k$. Then, one way to visualize p is to attach a graphical content to every c_j. An idea from [1] of using a regular plane tessellation is further developed in [3]. One can consider *infinite regular graphs* with degrees 3, 4, 6 and 8 that will be denoted by $\mathcal{M}_i$, $i = 3, 4, 6, 8$. The norm of a grid $\mathcal{M}_i$, $\|\mathcal{M}_i\|$ is the length of polygon's edge. Vertices of polygons are *nodes* of a grid. Also, we are going to include a hybrid grid $\mathcal{M}_8$ by overlapping two $\mathcal{M}_4$ grids, with norms 1 and $\sqrt{2}/2$ and sharing nodes with the norm-1 grid. In the terms of graph theory (we will use graph interpretation of a grid as long as we want to stress its topological rather than metric properties), $\mathcal{M}_3$, $\mathcal{M}_4$, $\mathcal{M}_6$ and $\mathcal{M}_8$ are *infinite regular graphs* with degrees 3, 4, 6 and 8. The obvious concequence of this fact is

Lemma 2.1 *A moving particle situated in any node of the grid* $\mathcal{M}_b$, $b \in \{3, 4, 6, 8\}$, *can escape the node in* b *directions.*

Therefore, escaping directions of a $\mathcal{M}_b$ grid can be coded by the digits of the number, expressed in base b, i.e., by $\{0, 1, \ldots, b-1\}$.

Note that the grid $\mathcal{M}_3$ has two types of nodes being mirror symmetric to each other: the node of configuration $-\!\!<$, that will be called A-type node, and B-type, $>\!\!-$ (shortly, A-node and B-node).

Supply a Descartes coordinate system to $\mathcal{M}_b$ such that

a) regular polygons consisting $\mathcal{M}_b$ have at least one side parallel to x-axis;

b) the origin coincide with any node of $\mathcal{M}_4$, $\mathcal{M}_6$ and $\mathcal{M}_8$ and A-node of $\mathcal{M}_3$;

Then, the following definition defines the coding procedure.

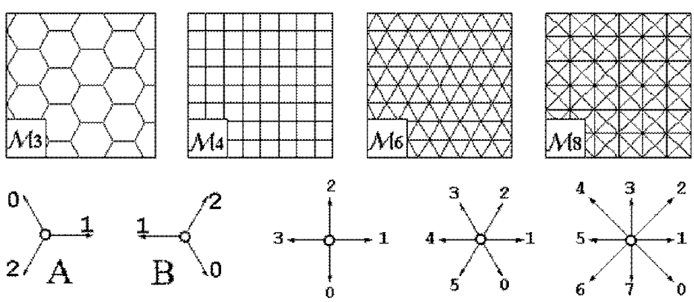

Figure 1. Grids $\{\mathcal{M}_b\}$ (above) and escaping directions codes (below).

Definition 2.1 The escaping directions from any node of the grid $\mathcal{M}_b$, $b \in \{4, 6, 8\}$ and any A-node of the grid $\mathcal{M}_3$, are coded by the sequence $1, 2, \ldots, b-1, 0$, going anti-clockwise, starting from x-axis direction. Coding of the escaping directions from B-node is obtained by anti-clockwise rotation of A-node for π radians (see Figure 1).

Let $p \in \mathbb{N}_0$ have representation (2.1), in the basis b, i.e., $p = c_0c_1\ldots c_k$, $c_j \in \{0, 1, \ldots, b-1\}$. Let $\hat{c}_j$ be the escaping edge of the grid $\mathcal{M}_b$ coded by c_j.

Definition 2.2 The graph, rooted at the node $\mathbf{v} \in \mathcal{M}_b$, $g(\mathbf{v}, p) = g(\mathbf{v}, p; b) = \cup_{j=0}^{k}\hat{c}_j$ is the partial graph of the number p with respect to the grid

$\mathcal{M}_b$. If $\mathbf{v} = \{0, 0\}$, it will be omitted from g.

The following lemma is obvious:

Lemma 2.2 *A moving particle situated in a node of $\mathcal{M}_b$ graph, $b \in \{3, 4, 6, 8\}$, may escape to an adjacent node in exactly b directions.*

The grid $\mathcal{M}_3$ is a bit peculiar.

Lemma 2.3 *The node (a, b) of the unit grid $\mathcal{M}_3$ is an A-type node if $(2a-1)/3 \in \mathbb{Z}$. Otherwise, it is of B-type.*

Proof. Note that, according to Definition 2.1, the coordinate origin is set to A-type node and the positive direction of x-axis will be usually headed. Taking into account that the sides of hexagonal cells are $\|\mathcal{M}_3\| = 1$, the rest of the proof follows by a simple calculation ∎

Definition 2.3 Let $x \in \mathbb{N}_0$ be represented by (2.1) i.e., $x = c_0c_1 \ldots c_k$. The path of a particle, starting at the origin and moving along the sides of $\mathcal{M}_b$ grid, directed by the digit c_j at the j-th step, will be a graph associated with x on the grid $\mathcal{M}_b$ and will be denoted by $g(x) = g(x; b)$.

Note that $g(x)$ is oriented (following the direction of a moving particle) and have $k+1$ vertices (thus, et least two). The inner vertices are at least of degree 2 while the end vertices are at least of of degree 1. In this manner $g(x)$ can be seen as a chain of vertices, in notation $\langle v_1, v_2, , v_{k+1}\rangle$ or simply $\langle 1, 2, , k+1\rangle$.

Example 2.1 The number $x = 873$ has the following representations in bases 3, 4, 6 and 8:

$$(x)_3 = 1012100, \quad (x)_4 = 31221, \quad (x)_6 = 4013, \quad (x)_8 = 1551;$$

These induce graphs: $g(x,3) = \langle (0,0), (1,0), (3/2, -\sqrt{3}/2), (5/2, -\sqrt{3}/2), (3,0), (4,0), (9/2, -\sqrt{3}/2), (4,0)\rangle$; $g(x,4) = \langle (0,0), (-1,0), (0,0), (0,1), (0,2), (1,2)\rangle$; $g(x,6) = \langle (0,0), (-1,0), (-1/2, -\sqrt{3}/2), (1/2, -\sqrt{3}/2), (0,0)\rangle$; $g(x,8) = \langle (0,0), (1,0), (0,0), (-1,0), (0,0)\rangle$.

3. Rational numbers

The consequence of Theorem A is that every $x \in \mathbb{Q}$ $(x \geq 0)$ can be expressed as a sum of *integral* and *fractional part* i.e., $x = a_0 + [0;\, a_1,\, a_2, \ldots,\, a_n]$ $(a_0 \in \mathbb{N}_0,\, a_i \in \mathbb{N},\, a_n \geq 2,\, n \geq 1)$.

Definition 3.1 The graph $g_i = g(a_i, b)$ corresponding to the partial quotient a_i, from the fractional part of $x \in \mathbb{Q}$ will be called *partial graph* of x

generated by a_i. The graph $g_0 = g(a_0, b)$, that corresponds to a_0, will be called *root graph* of x.

Definition 3.2 The graph $G_b(x) = [g_0; g_1, \ldots, g_n]$, composed of the root graph g_0 and partial graphs g_i of $x \in \mathbb{Q}$, such that the last vertex of g_{i-1} is the first vertex of g_i, $i = 1, 2, \ldots, n$ will be called *graph associated with* x on the grid $\mathcal{M}_b$.

Vertices of $G_b(x)$ will be labeled by two indices such that the j-th vertex of the subgraph g_i is the (i, j)-th vertex of $G_b(x)$.

Note that $G_b(x)$ is oriented, following the direction of its subgraphs. The proof of the following lemma is elementary.

Lemma 3.1 *If* $G_b(x) = [g_0; g_1, \ldots, g_n]$ *and* g_i *have* r_i *vertices than* $G_b(x)$ *has* $\sum_i r_i - n$ *vertices. Moreover, the* i_k*-th vertex of* g_k *is* $(\sum_{j=0,k-1} r_i + i_k - 2)$*-th vertex of* $G_b(x)$.

Example 3.1 Consider the rational number

$$p = \frac{6193062229}{2043355821} \approx 3.03082907311,$$ and its expansion $p = [3; 32, 2, 3, 2, 6, 4, 3, 32, 2, 3, 6, 4, 3]$. Note that p has 15 convergents: $p_1 = [3;]$, $p_2 = [3; 32]$, $p_3 = [3; 32, 2]$,..., $p_{15} = p$. Figure 2 shows the convergents p_1, p_4, p_8 and p_{15} in $\mathcal{M}_8$ (vertices are not marked).

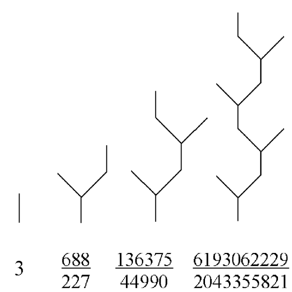

Figure 2. Growing tree of convergents p_1, p_4, p_8, $p_{15} = p$.

The main question is if the mapping $x \mapsto G_b(x)$ for rational x is bijective, and the answer is given in the following theorem:

Theorem 3.1 *The mapping* $G_b : x \mapsto G_b(x)$ *is bijective. In other words, any nonnegative rational number has one and only one graph representation in the sense of Definition* 3.2.

Proof. Let $x \in \mathbb{Q}$, $x \geq 0$. The mapping $G_b : x \mapsto G_b(x)$ is given by the next steps

i. Expand x in continued fraction, using base b, $x = [a_0;\, a_1,\, a_2, \ldots,\, a_n]_b$;

ii. Expand partial quotients in base b, $a_i = c_0 c_1 \ldots c_k$, $c_i \in \{0, \ldots, b-1\}$;

iii. Associate to a_0 the root graph $g(a_0, b)$ having the first vertex in the origin of $\mathcal{M}_b$;

iv. For $i = 1, \ldots, n$ associate to a_i the partial graph $g(a_i, b)$, having the first vertex in the last vertex of $g(a_{i-1}, b)$;

v. Construct the supergraph $G_b(x) = [g(a_0, b); g(a_1, b), \ldots, g(a_n, b)] = [g_0, g_1, \ldots, g_n]$ (Definition 3.2) and label its vertices.

All steps, $i. - v.$ can be performed in one and only one way. This means that every step can be inverted, so G_b is bijective. ∎

4. Ornaments and picture compression

From the theory of continued fractions it is known that if $a_0 \in \mathbb{N}_0$, $a_i \in \mathbb{N}$, the sequence of convergents

$$[a_0;],\ [a_0;\ a_1],\ [a_0;\ a_1, a_2],\ [a_0;\ a_1, a_2, a_3], \ldots, [a_0;\ a_1, a_2, , a_n], \ldots,$$

where partial quotients obey the requirements of Theorem A, always converges to its limit. This must be a nonnegative irrational number since by Theorem A, the rational number can be expanded only in a finite continued fraction.

Note that Theorem B deals with the case of periodic continued fraction expansion which yields a quadratic irrational. An interesting consequence is that the corresponding graph would follow a periodic pattern.

Example 4.1 (*Heawood numbers*) It is known that the number of colors $c(p)$ needed to color any map on a sphere with $p(\geq 0)$ handles is given by the integer part of so called Heawood numbers

$$h_p = \frac{1}{2}\left(7 + \sqrt{1 + 48p}\right), \quad p = 0, 1, 2, \ldots.$$

There are continued fraction expansion (in base 10) for some h_p's:

$$h_2 = [8; (2, 2, 1, 4, 4, 1, 2, 2, 9)^\infty], \quad h_8 = [13; (3, 4, 1, 1, 2, 1, 2, 1, 1, 4, 3, 19)^\infty],$$

$$h_9 = [13; (1, 9, 2, 4, 1, 2, 1, 1, 1, 6, 3, 3, 6, 1, 1, 1, 2, 1, 4, 2, 9, 1, 19)^\infty].$$

The corresponding non labeled graphs are presented in Figure 3. Note that some of these linear ornaments are fragments of infinite graphs, like $G_4(h_2)$, $G_4(h_8)$ and $G_3(h_8)$, while $G_3(h_9)$ is a cyclic graph, and presents a finite ornament thou.

Example 4.1 shows that periodic expansions, i.e. quadratic irrationals makes powerful tool in creating patterns with gliding or cyclic symmetry. In fact, as a rule, it is more economical to store periodic ornaments using a quadratic irrational number than using a rational number. It is easy to see by comparing the quadratic irrational $x = (19368 + \sqrt{515698682})/13883$ by its 15-th convergent, the rational number $p = 6193062229/2043355821$ from Example 3.1. The rational number p is expressed by 20 digits, while x has 19. Therefore, x contains less information than all rational convergents with index ≥ 15 but in spite of this it can generate the tree pattern shown in Figure 2 of arbitrary height.

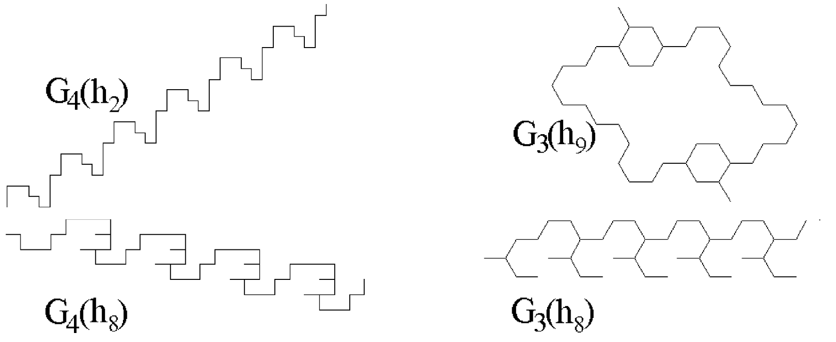

Figure 3. Ornaments generated by Heawood numbers.

The rational number periodic pattern (segment of the graph $G_4(h_2)$) shown in the left-top corner of Figure 3 needs 49 digits (in decimal notation), while the number $h_8 = (7+\sqrt{385})/2$, used to produce this pattern has only five digits. The information of the length of the chain has to be added, which needs two more digits (in the particular case it is 57). It gives 7 as the compression ratio.

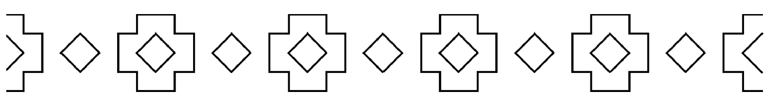

Figure 4. A simple frieze.

Example 4.2 (Friezes) In some cases, an ornament is made out of a few simple figures (motifs) that repeat periodically by gliding transformation (translation). Such ornament is usually called frieze. The motif figure can be represented by a rational or a quadratic irrational number. Except this number, the location of the motif needs an extra pair if coordinates for determining the beginning of the associated graph. In the case of irrational number, one also needs to supply the information of the length of expansion used. In the case of the frieze displayed in Figure 4, two motifs are present: a) The cross-like figure is represented by the sixteen base 4 digits (in the $\mathcal{M}_4$ grid): 2, 1, 2, 1, 1, 0, 1, 0 , 0, 3, 0, 3, 3, 2, 3, 2. This sequence can be organized in partial quotients in many different ways. The simplest one is: 2, 1, 2, 1, 10, 100, 30, 3, 3, 2, 3, 2 and the corresponding base-10 continued fraction gives:

$$x = [(2,1,2,1,4,16,12,3,3,2,3,2)^{\infty}] = \frac{4(66079+6\sqrt{230797286})}{229785}.$$

b) The second motif of the frieze in Figure 4 is a small square with diagonals parallel to the coordinate axes. It is created in $\mathcal{M}_8$ grid by the sequence

2, 0, 6, 4, i.e., by the integer number y = 2064, which is more economical since the periodic continuous fraction gives $[(20,6,4)^\infty] = (199+\sqrt{42026})/25$.

5. Conclusion

In this experimental paper, the author's aim is to offer a method for visualization of numbers, using a kind of guided Brownian motion. A particle travels along the plane regular grid, choosing the new direction at each node. The set of feasible directions is finite and is encoded by the sequence of partial quotients of a continuous fraction expansion of a real number. Oriented graphs offer the most exact description of obtained patterns. This paper discusses just limited number of possibilities like image compression, creation of periodic patterns and friezes. A lot of themes for further investigations are ahead. For example, question of concatenation of graphs, simple geometric transformations of patterns, investigation of irrational and transcendental numbers in their deeper nature.

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