Mathematica Balkanica

New Series Vol. 19, 2005, Fasc. 1-2

γ-Sets, γ_k-Sets and Hyperspaces *

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We consider relationships between γ-set property and hyperspaces. In particular, we show that each open subset of a space X is a γ-set if and only if the hyperspace 2^X of closed subsets of X with the Z^+-topology is (strongly) Fréchet-Urysohn. We also introduce and study two classes of spaces related to γ-sets.

AMS Subj. Classification: 54B20, 54D20, 54D55

Key Words: Selection principles, hyperspace, ω-cover, k-cover, γ-cover, γ_k-cover, γ-set, γ_k-set, γ_k'-set, Z^+-topology, upper Fell topology

1. Introduction

The notation and terminology that we use in this paper are standard as in [6]. Let us fix some other notation and terminology. X will denote an infinite Hausdorff topological space. By 2^X we denote the family of all closed subsets of X. For a subset A of X and a family $\mathcal{A}$ of subsets of X we put

$$A^c = X \setminus A \text{ and } \mathcal{A}^c = \{A^c : A \in \mathcal{A}\},$$
$$A^+ = \{F \in 2^X : F \subset A\}.$$

We consider the following two topologies on 2^X (compare with [17], [13]):

1. the Z^+-*topology* whose basic sets are of the form $(A^c)^+$, where A is a finite subset of X, and

2. the F^+-*topology* (known as the *upper Fell topology* [7] or *co-compact topology*) having basic sets of the form $(A^c)^+$ with A a compact set in X.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose elements are families of subsets of an infinite set X. Then (see [18], [12]):
$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection principle:

*Supported by MNTR of Serbia, grant N[0] 1233

> *For each sequence* $(A_n : n \in \mathbb{N})$ *of elements of* $\mathcal{A}$ *there is a sequence* $(b_n : n \in \mathbb{N})$ *such that for each* $n \in \mathbb{N}$, $b_n \in A_n$, *and* $\{b_n : n \in \mathbb{N}\}$ *is an element of* $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis:

> *For each sequence* $(A_n : n \in \mathbb{N})$ *of elements of* $\mathcal{A}$ *there is a sequence* $(B_n : n \in \mathbb{N})$ *of finite sets such that for each* $n \in \mathbb{N}$, $B_n \subset A_n$, *and* $\bigcup_{n\in\mathbb{N}} B_n$ *is an element of* $\mathcal{B}$.

The symbol $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ [18] denotes an infinitely long game for two players, ONE and TWO, which play a round for each positive integer. In the n-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \cdots; A_n, b_n; \cdots)$ if $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

An open cover $\mathcal{U}$ of a space X is an *ω-cover* [8] (resp. *k-cover* [16]) if X does not belong to $\mathcal{U}$ and every finite (resp. compact) subset of X is contained in a member of $\mathcal{U}$. Therefore, we assume that spaces we consider are non-compact. We shall also suppose that all covers are countable. An open cover $\mathcal{U}$ of X is called a *γ-cover* [8] if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Notice that it is equivalent to the assertion: Each finite subset of X belongs to all but finitely many members of $\mathcal{U}$. In a natural way we introduce a stronger notion.

Definition 1.1. An open cover of a space X is called a *γ_k-cover* of X if each compact subset of X is contained in all but finitely many elements of $\mathcal{U}$ and X is not a member of the cover.

For a topological space X we denote:

1. Ω – the family of ω-covers of X;
2. $\mathcal{K}$ – the family of k-covers of X;
3. Γ – the family of γ-covers of X;
4. Γ_k – the family of γ_k-covers of X.

Let us observe that we have

$$\Gamma_k \subset \Gamma \subset \Omega, \quad \Gamma_k \subset \mathcal{K} \subset \Omega.$$

In [8], Gerlits and Nagy introduced the following notion: a space X is a *γ-space* (or a *γ-set*) if each ω-cover $\mathcal{U}$ of X contains a countable family $\{U_n : n \in \mathbb{N}\}$ which is a γ-cover of X. They have also proved that the γ-set property of a space X is equivalent to the statement that X satisfies the selection property $\mathsf{S}_1(\Omega,\Gamma)$. It was shown in [12] that the γ-set property is equivalent also to the selection hypothesis $\mathsf{S}_{fin}(\Omega,\Gamma)$.

A space X is said to be *Fréchet-Urysohn* if for each $A \subset X$ and each $x \in \overline{A}$ there is a sequence $(x_n : n \in \mathbb{N})$ in A converging to x. X is *strongly Fréchet-Urysohn* if for each sequence $(A_n : n \in \mathbb{N})$ of subsets of X and each point $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ there is a sequence $(x_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $x_n \in A_n$ and $(x_n : n \in \mathbb{N})$ converges to x.

For a Tychonoff space X $C_p(X)$ denotes the space of all continuous real-valued functions on X endowed with the pointwise topology. Gerlits and Nagy [8] have shown the following result (see also [1]).

Theorem 1.1. *For a Tychonoff space X the following statements are equivalent:*

(*a*) *$C_p(X)$ is a Fréchet-Urysohn space;*

(*b*) *$C_p(X)$ is a strongly Fréchet-Urysohn space;*

(*c*) *X is a γ-set.*

The investigation of the same kind for function spaces $C_k(X)$ of continuous real-valued functions on X with the compact-open topology has been done in [16] and [15].

In [18], it was proved:

Theorem 1.2. *For a space X the following two statements are equivalent:*

(*a*) *X is a γ-set.*

(*b*) *ONE has no winning strategy in the game $\mathsf{G}_1(\Omega,\Gamma)$ on X.*

We shall prove here that similar results concerning γ-sets and their variations can be obtained in terms of special hyperspace topologies on 2^X. Let us mention that the Fréchet-Urysonh property in hyperspaces was studied in [3] and [9].

In Section 2 we consider the space 2^X with the Z^+-topology. In Section 3 we define two classes of sets related to γ-sets and study some their properties. In Section 4 we study another class of spaces related to γ_k-covers.

2. γ-sets and the space $(2^X, \mathsf{Z}^+)$

The first our result in this section is similar to Theorem 1.1.

Theorem 2.1. *For a space X the following statements are equivalent:*

(1) $(2^X, \mathsf{Z}^+)$ *is a Fréchet-Urysohn space;*

(2) $(2^X, \mathsf{Z}^+)$ *is a strongly Fréchet-Urysohn space;*

(3) *Each open set* $Y \subset X$ *is a* γ*-set.*

Proof. (1) $\Rightarrow$ (3): Let Y be an open subset of X and let $\mathcal{U}$ be an ω-cover of Y. Put $\mathcal{A} := \mathcal{U}^c$. Then $\mathcal{A} \subset 2^X$ and $Y^c \in Cl_{\mathsf{Z}^+}(\mathcal{A})$. Indeed, let $(F^c)^+$ be a Z^+-neighborhood of Y^c. Then F is a finite subset of Y so that there is a $U \in \mathcal{U}$ such that $F \subset U$. Thus $Y^c \subset U^c \subset F^c$, i.e. $U^c \in (F^c)^+ \cap \mathcal{A}$. Therefore, $Y^c \in Cl_{\mathsf{Z}^+}(\mathcal{A})$. Since, by (1), $(2^X, \mathsf{Z}^+)$ is Fréchet-Urysohn, there is a sequence $(A_n : n \in \mathbb{N})$ in $\mathcal{A}$ converging to Y^c. Let for each $n \in \mathbb{N}$, $U_n = A_n^c$ and show that $\{U_n : n \in \mathbb{N}\} \subset \mathcal{U}$ is a γ-cover of Y.

Let $y \in Y$. Then $(\{y\}^c)^+$ is a Z^+-neighborhood of Y^c and thus there is $n_0 \in \mathbb{N}$ such that for each $n > n_0$ one has $A_n \in (\{y\}^c)^+$. This implies $y \in A_n^c = U_n$ for all $n > n_0$, which means that $\{U_n : n \in \mathbb{N}\}$ is a selector for $\mathcal{U}$. Hence (3) holds.

(3) $\Rightarrow$ (1): Let $\mathcal{A}$ be a subset of 2^X and let $S \in Cl_{\mathsf{Z}^+}(\mathcal{A})$. Then $\mathcal{U} := \mathcal{A}^c$ is an ω-cover of the open set $S^c \subset X$, as can be easily shown. Applying (3) to S^c and $\mathcal{U}$ we find a countable family $\{U_n : n \in \mathbb{N}\} \subset \mathcal{U}$ which is a γ-cover of S^c. We claim that the sets $A_n = U_n^c$, $n \in \mathbb{N}$, form a sequence in $\mathcal{A}$ which Z^+-converges to S. Suppose that $(F^c)^+$ is a Z^+-neighborhood of S. Then F is a finite subset of S^c, hence there exists $n_0 \in \mathbb{N}$ such that $F \subset U_n$ whenever $n > n_0$. It follows that for each $n > n_0$ we have $A_n \in (F^c)^+$, i.e. that $(A_n : n \in \mathbb{N})$ really converges to S and $(2^X, \mathsf{Z}^+)$ is a Fréchet-Urysohn space.

(2) $\Leftrightarrow$ (3): It is shown similarly by using the fact that the γ-set property is equivalent to $\mathsf{S}_1(\Omega, \Gamma)$. ∎

It is natural to ask what happens if X is a γ-set, i.e. to find out a property of $(2^X, \mathsf{Z}^+)$ that is equivalent to X is a γ-set.

Theorem 2.2. *If a space* X *is a* γ*-set, then each dense subset of* $(2^X, \mathsf{Z}^+)$ *is sequentially dense.*

Proof. Let $\mathcal{D}$ be a dense subset of $(2^X, \mathsf{Z}^+)$ and let $\mathcal{U} := \mathcal{D}^c$. Then $\mathcal{U}$ is an ω-cover of X; it is proved similarly to the corresponding part of the proof (3) implies (1) in the previous theorem. Since X is a γ-set there is a set $\{U_n : n \in \mathbb{N}\} \subset \mathcal{U}$ which is a γ-cover of X. For each $n \in \mathbb{N}$, let $D_n = U_n^c$. We show that the set $\mathcal{S} := \{D_n : n \in \mathbb{N}\} \subset \mathcal{D}$ is sequentially dense in $(2^X, \mathsf{Z}^+)$.

Let $A \in (2^X, \mathsf{Z}^+)$. Suppose, to the contrary, that no sequence in $\mathcal{S}$ converges to A. Let $(D_{n_m} : m \in \mathbb{N})$ be a sequence in $\mathcal{S}$. Then there is a neighborhood $(F^c)^+$ of A such that the set $M = \{m \in \mathbb{N} : D_{n_m} \notin (F^c)^+\}$ is infinite. It means that infinitely many sets U_{n_m} from $\mathcal{U}$ do not contain F. Therefore, $\mathcal{U}$ is not a γ-cover of X and we have a contradiction. ∎

3. γ_k-sets and the space $(2^X, \mathsf{F}^+)$

To get results for the space $(2^X, \mathsf{F}^+)$ similar to Theorem 2.1 we introduce the following two classes of spaces similar to the class of γ-sets. A space X is said to be a *γ_k-set* if each k-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a γ_k-cover of X. Call X a *γ_k'-set* if it satisfies the selection hypothesis $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$.

We begin by a characterization of γ_k'-sets.

Theorem 3.1. *For a space X the following are equivalent:*

(1) *X satisfies $\mathsf{S}_{fin}(\mathcal{K}, \Gamma_k)$;*

(2) *X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$, i.e. X is a γ_k'-set;*

(3) *ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{K}, \Gamma_k)$ played on X.*

Proof. We have to prove only non-trivial cases (1) implies (2) and (2) implies (3).

(1) $\Rightarrow$ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of (countable) k-covers of X; suppose that for each $n \in \mathbb{N}$, $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ denote the family of sets of the form $U_{1,k_1} \cap U_{2,k_2} \cap \cdots \cap U_{n,k_n}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of k-covers of X. Since X satisfies $\mathsf{S}_{fin}(\mathcal{K}, \Gamma_k)$ choose for each $n \in \mathbb{N}$ a finite subset $\mathcal{W}_n$ of $\mathcal{V}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a γ_k-cover of X. (Note that some $\mathcal{W}_n$'s can be empty.)

As $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is infinite and all $\mathcal{W}_n$'s are finite, there exists a sequence $m_1 < m_2 < \cdots < m_p < \cdots$ in $\mathbb{N}$ such that for each $i \in \mathbb{N}$ we have $\mathcal{W}_{m_i} \setminus \bigcup_{j<i} \mathcal{W}_{m_j} \neq \emptyset$. Choose an element $W_{m_i} \in \mathcal{W}_{m_i} \setminus \bigcup_{j<i} \mathcal{W}_{m_j}$, $i \in \mathbb{N}$, and fix its representation $W_{m_i} = U_{1,k_1} \cap U_{2,k_2} \cap \cdots \cap U_{m_i,k_{m_i}}$ as above.

Since each infinite subset of a γ_k-cover is also a γ_k-cover, we have that the set $\{W_{m_i} : i \in \mathbb{N}\}$ is a γ_k-cover of X. For each $n \leq m_1$ let $U_n \in \mathcal{U}_n$ be the n-th coordinate of W_{m_1} in the chosen representation of W_{m_1}, and for each $n \in (m_i, m_{i+1}]$, $i \geq 1$, let $U_n \in \mathcal{U}_n$ be the n-th coordinate of $W_{m_{i+1}}$ in the above representation of $W_{m_{i+1}}$. Observe that each $U_n \supset W_{m_{i+1}}$. Therefore, we obtain a sequence $(U_n : n \in \mathbb{N})$ of elements, one from each $\mathcal{U}_n$, which form a γ_k-cover of X and show that X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$.

(2) $\Rightarrow$ (3): Let σ be a strategy for ONE in $\mathsf{G}_1(\mathcal{K}, \Gamma_k)$ and let the first move of ONE be a k-cover $\sigma(\emptyset) = \{U_{(1)}, U_{(2)}, \cdots, U_{(n)}, \cdots\}$. Suppose that for each finite sequence s of natural numbers of length $\leq m$, U_s has been already defined. Then define $\{U_{(n_1,\cdots,n_m,k)} : k \in \mathbb{N}\}$ to be the set

$$\sigma(U_{(n_1)}, U_{(n_1,n_2)}, \cdots, U_{(n_1,\cdots,n_m)}) \setminus \{U_{(n_1)}, U_{(n_1,n_2)}, \cdots, U_{(n_1,\cdots,n_m)}\}.$$

Because each compact subset of X belongs to infinitely many elements of a k-cover, we have that for each s a finite sequence of natural numbers, the set $\{U_{s\frown(n)} : n \in \mathbb{N}\}$ is a k-cover. Apply (2) and for each s choose $n_s \in \mathbb{N}$ such that $\{U_{s\frown(n_s)} : s \text{ a finite sequence of natural numbers}\}$ is a γ_k-cover of X. Inductively define a sequence $n_1 = n_\emptyset$, $n_{k+1} = n_{(n_1,\cdots,n_k)}$, for $k \geq 1$. Then

$$U_{(n_1)}, U_{(n_1,n_2)}, \cdots, U_{(n_1,\cdots,n_k)}, \cdots$$

is a γ_k-cover, and because it is actually a sequence of moves of TWO in a play of the game $\mathsf{G}_1(\mathcal{K}, \Gamma_k)$, σ is not a winning strategy for ONE. ∎

Proposition 3.2. *The class $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ is an inverse invariant of perfect irreducible mappings.*

P r o o f. Let $f : X \to Y$ be a perfect irreducible mapping from a space X onto a space Y satisfying $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of k-covers of X. Then $(f^\#(\mathcal{U}_n) : n \in \mathbb{N})$ (where $f^\#(\mathcal{U}_n) = \{f^\#(U) : U \in \mathcal{U}_n\}$ and $f^\#(U) = \{y \in Y : f^\leftarrow(y) \subset U\} \equiv Y \setminus f(X \setminus U)$) is a sequence of k-covers of Y as can be easily verified. Therefore, there is a sequence $(f^\#(U_n) : n \in \mathbb{N})$ such that this collection is a γ_k-cover of Y and for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$. Then $\{U_n : n \in \mathbb{N}\}$ is a γ_k-cover of X. Indeed, if K is a compact subset of X, then there is n_0 such that for all $n > n_0$, $f(K) \subset f^\#(U_n)$. So, for all $n > n_0$ we have $K \subset f^\leftarrow f^\#(U_n) \subset U_n$. ∎

Theorem 3.3. *A space X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ if and only if each finite power of X satisfies $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$.*

P r o o f. Let X belongs to the class of spaces satisfying $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$ and let $(\mathcal{W}_n : n \in \mathbb{N})$ be a sequence of k-covers of X^m for a fixed natural number m. By a lemma in [4] for each $n \in \mathbb{N}$ there is a k-cover $\mathcal{U}_n$ of X such that $\{U^m : U \in \mathcal{U}_n\}$ is a refinement of $\mathcal{W}_n$. Because $X \in \mathsf{S}_1(\mathcal{K}, \Gamma_k)$, choose a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \mathbb{N}\}$ is a γ_k-cover of X. For each $n \in \mathbb{N}$, let W_n be an element in $\mathcal{W}_n$ with $(U_n)^m \subset W_n$. Then the sequence $(W_n :\in \mathbb{N})$ shows that X^m belongs to the class $\mathsf{S}_1(\mathcal{K}, \Gamma_k)$. Let K be a compact subset of X^m. Then the union $\bigcup_{i \leq m} p_i(K) = C$ of its projections onto X is a compact subset of X and thus there is $n_0 \in \mathbb{N}$ such that $C \subset U_n$ for each $n > n_0$. Therefore, for each $n > n_0$ we have $K \subset C^m \subset (U_n)^m \subset W_n$. ∎

Theorem 3.4. *For a space X the following are equivalent:*

(a) $(2^X, \mathsf{F}^+)$ *is a strongly Fréchet-Urysohn space;*

(b) *Each open set $Y \subset X$ is a γ'_k-set.*

Proof. $(a) \Rightarrow (b)$: Let Y be an open subset of X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of k-covers of Y. Then $(\mathcal{A}_n \equiv \mathcal{U}_n^c : n \in \mathbb{N})$ is a sequence of subsets of 2^X and $Y^c \in \bigcap_{n\in\mathbb{N}} Cl_{\mathsf{F}^+}(\mathcal{A}_n)$. As $(2^X, \mathsf{F}^+)$ is strongly Fréchet-Urysohn, for each n choose an element $A_n \in \mathcal{A}_n$ such that the sequence $(A_n : n \in \mathbb{N})$ F^+-converges to Y^c. For each $n \in \mathbb{N}$ put $U_n = A_n^c$. We prove that $\{U_n : n \in \mathbb{N}\}$ is a γ_k-cover of Y.

Let K be a compact subset of Y. The set $(K^c)^+$ is an F^+-neighborhood of Y^c and thus $A_n \in (K^c)^+$ for all but finitely many n. So, for all but finitely many n we have $K \subset A_n^c = U_n$, i.e. $\{U_n : n \in \mathbb{N}\}$ is indeed a γ_k-cover of Y.

$(b) \Rightarrow (a)$: Let $(\mathcal{A}_n : n \in \mathbb{N})$ be a sequence of subset of 2^X and let $S \in \bigcap_{n\in\mathbb{N}} Cl_{\mathsf{F}^+}(\mathcal{A}_n)$. Then for each $n \in \mathbb{N}$, $\mathcal{U}_n := \mathcal{A}_n^c$ is a k-cover of the set S^c open in X. Let us show the last assertion. Take a compact set C in S^c. Then $(C^c)^+$ is an F^+-neighborhood of S so that $(C^c)^+$ meets $\mathcal{A}_n$ in some B_n. It implies $C \subset B_n^c \in \mathcal{U}_n$, i.e. $\mathcal{U}_n$ is a k-cover of S^c.

By assumption there are sets $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_n : n \in \mathbb{N}\}$ is a γ_k-cover of S^c. We prove that the sets $A_n = U_n^c$, $n \in \mathbb{N}$, form a sequence F^+-converging to S and showing that $(2^X, \mathsf{F}^+)$ is strongly Fréchet-Urysohn. Suppose $(K^c)^+$ is an F^+-neighborhood of S. Then K is a compact subset of S^c and since $\{U_n : n \in \mathbb{N}\}$ is a γ_k-cover of Y^c, there is $n_0 \in \mathbb{N}$ such that $K \subset U_n$ for $n > n_0$. It follows that for all $n > n_0$ we have $A_n \in (K^c)^+$, i.e. that $(A_n : n \in \mathbb{N})$ F^+-converges to S. Therefore, $(2^X, \mathsf{F}^+)$ is a strongly Fréchet-Urysohn space. ■

Similarly we can prove the following result.

Theorem 3.5. *For a space X the following are equivalent:*

(a) $(2^X, \mathsf{F}^+)$ *is a Fréchet-Urysohn space;*

(b) *Each open set $Y \subset X$ is a γ_k-set.*

We do not know the answer to the following question.

Problem 3.6. *Is the γ'_k-set property equivalent to the γ_k-set property?*

4. $\mathsf{U}_{fin}(\Gamma_k, \mathcal{K})$

In [10] (see also [11]), Hurewicz introduced a covering property of a space X, nowadays called the *Hurewicz property* in this way: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\cup\mathcal{V}_n$ for all but finitely many n. In [14] it was shown that this property is actually an S_{fin}-type property. In [18] a property weaker than the Hurewicz property and denoted by $\mathsf{U}_{fin}(\Gamma, \Omega)$ was introduced (see also [12]), and in [2] it was studied in some details. In this section we shall consider the following class of spaces.

The symbol $\mathsf{U}_{fin}(\Gamma_k, \mathcal{K})$ denotes the selection principle:
For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of γ_k-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that each $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and either $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is a k-cover for X, or for some $n \in \mathbb{N}$, $X = \cup\mathcal{V}_n$.

We shall show now that this selection principle is of the S_{fin} type.

An open cover $\mathcal{U}$ of a space X is said to be *large* if each point of X belongs to infinitely many elements of $\mathcal{U}$. A countable large cover $\mathcal{U}$ of X is *k-weakly groupable* (see [4], [5]) if there is a partition of $\mathcal{U}$ into infinitely many finite, pairwise disjoint subsets $\mathcal{U}_n$ such that each compact subset of X is contained in $\cup\mathcal{U}_n$ for some $n \in \mathbb{N}$. Let Λ^{k-wgp} denote the family of k-weakly groupable large covers of a space.

Theorem 4.1. *For a space X the following assertions are equivalent:*

(1) *X has property $\mathsf{U}_{fin}(\Gamma_k, \mathcal{K})$;*

(2) *X has property $\mathsf{S}_{fin}(\Gamma_k, \Lambda^{k-wgp})$;*

(3) *For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of γ_k-covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, $\mathcal{V}_n$'s are pairwise disjoint and for each compact set $K \subset X$ there exists some $n \in \mathbb{N}$ with $K \subset \cup\mathcal{V}_n$.*

Proof. (1) $\Rightarrow$ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of γ_k-covers of X. Because each infinite subset of a γ_k-cover is also a γ_k-cover one can suppose that $\mathcal{U}_n$'s are pairwise disjoint. Also, without loss of generality, one can assume that for each $n \in \mathbb{N}$, no finite subset of $\mathcal{U}_n$ is a cover of X.

Apply (1) to find a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subset \mathcal{U}_n$ and $\{\cup\mathcal{V}_n : n \in \mathbb{N}\}$ is a k-cover of X. This implies that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a large cover of X. Its partition $\{\mathcal{V}_n : n \in \mathbb{N}\}$ shows that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is actually k-weakly groupable.

(2) $\Rightarrow$ (3): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of γ_k–covers of X. As in (1) $\Rightarrow$ (2) one can suppose that $\mathcal{U}_n$ and $\mathcal{U}_m$ are disjoint for distinct n and m, and else that all covers are countable. For $n \in \mathbb{N}$ let $\mathcal{U}_n = \{U_{n,k} : k \in \mathbb{N}\}$. Define new γ_k-covers $\mathcal{V}_n$, $n \in \mathbb{N}$, in the following way:

$$\mathcal{V}_n = \{U_{1,k} \cap U_{2,k} \cap \cdots \cap U_{n,k} : k \in \mathbb{N}\} \setminus \{\emptyset\}.$$

We again may suppose that $\mathcal{V}_n \cap \mathcal{V}_m = \emptyset$ for $m \neq n$.

Apply (2) to this new sequence $(\mathcal{V}_n : n \in \mathbb{N})$ to choose a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{W}_n \subset \mathcal{V}_n$ (and so $\mathcal{W}_n$'s are pairwise disjoint) and $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a k-weakly groupable large cover of X. This means that $\bigcup_{n\in\mathbb{N}} \mathcal{W}_n = \bigcup_{n\in\mathbb{N}} \mathcal{H}_n$, where $\mathcal{H}_n$'s are finite, pairwise disjoint and each compact set K in X is contained in $\cup\mathcal{H}_n$ for some $n \in \mathbb{N}$.

Since sets $\mathcal{W}_n$ and $\mathcal{H}_n$ are finite, there are $i \in \mathbb{N}$ such that $\mathcal{W}_1 \cap \mathcal{H}_j = \emptyset$ for $j > i$. Let i_1 be the smallest such natural number. Define $\mathcal{C}_1$ to be the set of all $U_{1,k}$ such that $U_{1,k}$ is a term in the above representation of a member of $\mathcal{H}_j$ for some $j \leq i_1$. Let $i_2 > i_1$ be the minimal natural number such that $\mathcal{W}_2 \cap \mathcal{H}_j = \emptyset$ whenever $j > i_2$. Define $\mathcal{C}_2$ to be the set of all $U_{2,k}$, where $U_{2,k}$ is a term in the representation of an element of $\mathcal{H}_j$ for some $j \leq i_2$. And so on.

We get the sequence $(\mathcal{C}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$ $\mathcal{C}_n$ is a finite subset of $\mathcal{U}_n$, and consequently $\mathcal{C}_n \cap \mathcal{C}_m = \emptyset$ for $m \neq n$. We claim that the sequence $(\mathcal{C}_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X satisfies (3).

Let $K \subset X$ be a compact set. There exists some $n \in \mathbb{N}$ such that $K \subset \cup\mathcal{H}_n$. Pick the smallest k such that $n \leq i_k$. Then $(\mathcal{W}_1 \cup \cdots \cup \mathcal{W}_{k-1}) \cap \mathcal{H}_n = \emptyset$. That implies that each element H in $\mathcal{H}_n$ has in its (above chosen) representation a set of the form $U_{k,j}$ (such elements are in $\mathcal{C}_k$) and therefore we have $\cup\mathcal{H}_n \subset \cup\mathcal{C}_k$. It follows $K \subset \cup\mathcal{C}_k$ and (3) holds.

(3) $\Rightarrow$ (1): It is evident by the definition of $\mathsf{U}_{fin}(\Gamma_k, \mathcal{K})$. ■

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Received 30.09.2003