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Discrete Functions with a Given Range and a Given C-Spectrum

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In this paper, we generalize some of the results published in [3]. We consider k-valued functions of n variables. We have found the number of functions, which have a given range and a given C-spectrum with respect to a given set of variables of these functions.

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1. Introduction

Let $P_n^k = \{f : E^n \to E \mid E = \{0, 1, \dots, k-1\}, n \geq 1, k \geq 2\}$ be the set of all functions of n variables, and let f take values from the set E.

Definition 1.1. [1] The number of different values of the function $f \in P_n^k$ is called *range* of f.

The range of the function f is denoted by $Rng(f)$.

Let $X_f = \{x_1, x_2, \dots, x_n\}$ and let $\mu_n^k(q)$ be the number of the functions (see [1]) from P_n^k with *range* that is equal to q, where $q \in \{1, 2, \dots, k\}$, and

$$\mu_n^k(q) = \binom{k}{q} \sum_{\substack{r_1 + r_2 + \cdots + r_q = k^n \\ r_i \geq 1, \; i = 1, 2, \dots, q}} \frac{k^n!}{r_1! r_2! \dots r_q!} = \binom{k}{q} \sum_{j=1}^{q} (-1)^{q-j} \binom{q}{j} j^{k^n} \quad (1)$$

Definition 1.2. A function h is called a subfunction of $f \in P_n^k$ with respect to R, $R = \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} \subseteq X_f$, if h is obtained from f by replacing the variables of the set R with constants $c_1, c_2, \dots, c_r$, and we write

$$h \overset{R}{\prec} f,$$

or

$$h = f(\ x_{j_1} = c_1,\ x_{j_2} = c_2, \ldots,\ x_{j_r} = c_r).$$

By $\widehat{K}$:=$[1^{p_1},\ 2^{p_2}, \ldots,\ k^{p_k}]$ we will denote the multiset generated by the set $K = \{1,\ 2, \ldots,\ k\}$, where every element i, $i \in K$, occurs p_i times in $\widehat{K}$.

Definition 1.3. [2] The multiset $[1^{p_1},\ 2^{p_2}, \ldots,\ k^{p_k}]$ is called *C-spectrum* of M for f, where $M \subseteq X_f$, $p_t \geq 0$, and p_t, $t = 1, \ldots, k$, is the number of the different sets of values for the variables of the set $X_f \setminus M$, by which from f we obtain subfunctions with a range equal to t, and

$$p_1 + p_2 + \cdots + p_k = k^{n-|M|} \quad \text{for } f \in P_n^k.$$

The *C-spectrum* of M for f is denoted by *C-Spr*(M, f), where

$$C - Spr(M, f) = [1^{p_1},\ 2^{p_2}, \ldots,\ k^{p_k}].$$

2. Main Results

The following question is interesting: What is the number of the functions of P_n^k with an apriori given range of the function and given C-spectrum? The answer to this question is given by *Theorem 2.1.* below.

Let $b,\ t,\ p_1, \ldots, p_s,\ q_1, \ldots, q_s,\ s \geq 1$, are natural numbers and let $\Im(b, t)$ denote the set of matrices of order $b \times t$ and elements zeroes and ones such that:
a) there is at least one 1 in each row;
b) if we partition columns of the matrices into s, $s \geq 1$, sets of columns, there are p_j columns in the j-th set and there are q_j 1's in each column of this j-th set, $j = 1,\ 2, \ldots,\ s$.

Since there is at least one 1 in each row, the number of 1's in the matrix must be greater than or equal to the number of rows, that is,

$$(i) \quad p_1 q_1 + p_2 q_2 + \cdots + p_s q_s \geq b.$$

Also, the number of columns of all sets of columns is necessary equal to the number of columns of the matrix, that is,

$$(ii) \quad p_1 + p_2 + \cdots + p_s = t$$

and the number of 1's in each set of columns must be less than or equal to the number of rows of the matrix, that is,

$$(iii) \quad b \geq q = max\{\ q_1,\ q_2, \ldots,\ q_s\}.$$

Lemma 2.1. *The number of the matrices from the set $\Im(b,t)$ is equal to*

$$|\Im(b,t)| = \sum_{i=0}^{b-q} (-1)^i \binom{b}{i} \; |\Psi(b-i,t)|,$$

where

$$|\Psi(v,t)| = t! \prod_{i=1}^{s} \frac{\binom{v}{q_i}^{p_i}}{p_i!}, \;\; v \geq q.$$

P r o o f. Define a map which maps each column of the j-th set on the number j, $j = 1,\ 2, \dots,\ s$. The number of different maps of this type is equal to

$$\frac{(p_1 + p_2 + \cdots + p_s)!}{p_1! p_2! \dots p_s!} = \frac{t!}{p_1! p_2! \dots p_s!}.$$

This number gives all possible different configurations of columns of different sets in the matrix, and it does not represent the location of ones in columns.

It is clear that each column of j-th set can be chosen in $\binom{b}{q_j}$ ways since q_j ones at b positions can be located in $\binom{b}{q_j}$ ways, $j = 1,\ 2, \dots,\ s$.

Let $\Psi(b,t)$ denote the set of matrices of order $b \times t$ that have p_j columns with q_j ones, $j = 1,\ 2, \dots,\ s$, that is, only condition b) is satisfied, whence it follows that $\Im(b,t) \subseteq \Psi(b,t)$.

According to the above considerations, the number of matrices in the set $\Psi(b,t)$ is equal to

$$|\Psi(b,t)| = \frac{t!}{p_1! p_2! \dots p_s!} \binom{b}{q_1}^{p_1} \binom{b}{q_2}^{p_2} \cdots \binom{b}{q_s}^{p_s} = t! \prod_{i=1}^{s} \frac{\binom{b}{q_i}^{p_i}}{p_i!}.$$

In order to obtain the cardinality of the set $\Im(b,t)$, from the number of matrices of the set $\Psi(b,t)$ we have to subtract the number of those matrices of the last set which have rows of zeroes only.

In this way we comply with the requirement each row of considered matrices to contain at least one 1.

If we remove matrices with a row of a zero elements, matrices with two rows of zeroes, and so on, finally the matrices with $(b-q)$ rows of zeroes from the set of matrices $\Psi(b,t)$, we obtain the set of matrices $\Im(b,t)$. For finding $|\Im(b,t)|$, we use the principle of inclusion and exclusion. ■

Theorem 2.1. *If $M \subset X_f, |M| = m \neq 0$, then the number of functions $f \in P_n^k$, with range b, $b \leq k$, for which*

$$C - Spr(M, f) = [{q_1}^{p_1},\ {q_2}^{p_2}, \dots,\ {q_s}^{p_s}],\ 1 \leq s \leq k,$$

is equal to

$$\binom{k}{b} \prod_{i=1}^{s} \left[\frac{\mu_m^k(q_i)}{\binom{k}{q_i}} \right]^{p_i} |\Im(b, t)|,$$

where

$$t = k^{n-m}.$$

P r o o f. From $Rng(f) = b$ it follows that f must take b different values. Fix b values of the possible k values, and let the chosen b values be $u_1, u_2, \dots, u_b$.

Let $X_f \setminus M = \{x_{j_1}, x_{j_2}, \dots, x_{j_{n-m}}\}$. Denote all sets of constants for the variables of $X_f \setminus M$ by

$$\{ c_1^i, c_2^i, \dots, c_{n-m}^i\},\ i = 1,\ 2, \dots,\ k^{n-m}\ .$$

From $g_i = f(\ x_{j_1} = c_1^i,\ x_{j_2} = c_2^i, \dots,\ x_{j_{n-m}} = c_{n-m}^i)$, it follows that

$$g_i \in\ P_m^k,\ i = 1,\ 2, \dots,\ k^{n-m}\ .$$

In its tabular representation, function f can be represented by $t = k^{n-m}$ subfunctions $g_i,\ i = 1,\ 2, \dots,\ k^{n-m}$.

From $C\text{-}Spr(M, f) = [{q_1}^{p_1},\ {q_2}^{p_2}, \dots,\ {q_s}^{p_s}]$ it follows that among the above $t = k^{n-m}$ subfunctions $g_i,\ i = 1,\ 2, \dots,\ k^{n-m}$ there are p_j subfunctions which take exactly q_j values of the fixed b values, $j = 1,\ 2, \dots,\ s$.

Let $x_{lr} = \begin{cases} 1, & \text{if } u_l \text{ is a value of } g_r, \\ 0, & \text{if } u_l \text{ is not a value of } g_r,\ l = 1, 2, \dots, b;\quad r = 1, 2, \dots, t. \end{cases}$

The matrix $\{x_{lr}\}$ is of order $b \times t$ and its elements are zeroes and ones. Columns of this matrix can be partitioned into s sets of columns, such that in j-th set there are p_j columns and in each column of this j-th set there are q_j ones, $j = 1,\ 2, ...,\ s$, which guarantees that

$$C - Spr(M, f) = [{q_1}^{p_1},\ {q_2}^{p_2}, \dots,\ {q_s}^{p_s}],\ q_j \in \{1,\ 2, \dots,\ k\}.$$

From the definition of *C-spectrum* it follows that

$$(ii) \quad p_1 + p_2 + \dots + p_s = t = k^{n-m}$$

and if we set $q = max\{ q_1,\ q_2, \ldots,\ q_s\}$, by the assumption we have

$$(iii) \quad q \leq\ b \leq\ k.$$

Since $q_j \in \{1,\ 2, \ldots,\ k\}$, it follows that $q_j \geq\ 1$ for $j = 1,\ 2, \ldots,\ s$.

Then for $p_1 q_1 + p_2 q_2 + \cdots + p_s q_s$ we have

$$(i) \quad p_1 q_1 + p_2 q_2 + \cdots + p_s q_s \geq\ p_1 + p_2 + \cdots + p_s = t = k^{n-m} \geq\ k \geq\ b,$$

that is, the number of 1's in the matrix is greater than or equal to the number of its rows.

In order to have $Rng(f) = b$, we must use as values of subfunctions $g_r,\ r = 1,\ 2, \ldots,\ k^{n-m} = t$, all b values, which is equivalent to having the least one 1 in each row of the matrix $\{x_{lr}\}$. (If v-th row of matrix $\{x_{lr}\}$ contains only zeroes, this means that u_v is not a value of any subfunction $g_r,\ r = 1,\ 2, \ldots,\ t$, and therefore it is not a value of the function f as well, that is, $Rng(f) < b$.)

Obviously the matrix $\{x_{lr}\}$ belongs to the set $\Im(b,t)$ and according to *Lemma 2.1,* the number of all various matrices $\{x_{lr}\}$ is equal to

$$|\Im(b,t)| = \sum_{i=0}^{b-q} (-1)^i \binom{b}{i}\ |\Psi(b-i,t)|,$$

where

$$|\Psi(b,t)| = t! \prod_{i=1}^{s} \frac{\binom{b}{q_i}^{p_i}}{p_i!}.$$

Each matrix $\{x_{lr}\}$ has p_j columns that take exactly q_j of the fixed b values, and the number of matrices $\{x_{lr}\}$ takes into account only different q_j-tuples of values, which are used in the subfunctions generating the function $f,\ j = 1, 2, \ldots, s$.

We associate to each q_j-tuple of values, and therefore to each column of the matrix with q_j 1's, using (1)

$$\frac{\mu_m^k(q_j)}{\binom{k}{q_j}},\ j = 1,\ 2, \ldots,\ s,$$

different functions, corresponding to subfunctions of the function f. Since there are p_j columns with q_j 1's, $j = 1, 2, \ldots, s$, taking into account the number of matrices $\{x_{lr}\}$ and the fact that we can fix b values among k values in $\binom{k}{b}$ ways, we complete the proof of *Theorem 2.1.* ∎

Remark 2.1. *In the special case when $s = 1$, $q_1 = a$, Lemma 2.1. implies Lemma 1 in [3].*

Remark 2.2. *In the special case when $s = 1$, $q_1 = a$, Theorem 2.1. implies Theorem 1 in [3].*

The proof of *Theorem 2.1.* can be also used for the "construction" of the functions under consideration.

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