Mathematica
Balkanica

New Series Vol. 19, 2005, Fasc. 1-2

# On the Small-Time Local Controllability*

*Mikhail I. Krastanov*

Following a geometrical point of view, some known controllability results are discussed. A new sufficient condition for small-time local controllability of a smooth control system at an equilibrium point is stated. Some illustrative examples are also presented.

*AMS Subj. Classification*: 93B03, 93C15

*Key Words*: small-time local controllability, Lie brackets

## 1. Introduction.

Let us consider the following single-input affine control system :

$$\dot{x}(t) \;=\; f^0(x(t)) \;+\; u(t)\; f^1(x(t)), \tag{1}$$

where $f^0$ and $f^1$ are smooth (typically real analytic) vector fields defined on $\mathbb{R}^n$, $f^0(0) = 0$, $f^1(0) \neq 0$, and the control $u$ is a measurable function taking values from a compact interval $U$ containing zero in its interior.

A trajectory of (1) starting from the point $x_0$, corresponding to some admissible control $u(.)$ and defined on $[0,T]$ is an absolutely continuous function $x(t): \; [0,T] \to \mathbb{R}^n$, satisfying (1) for almost every $t$ from $[0,T]$. Let $x_0$ be a fixed point of $\mathbb{R}^n$. The reachable set $\mathcal{R}(x_0,T)$ of (1) at time $T > 0$ starting from the point $x_0$ is defined as the set of all points that can be reached in time $T$ from $x_0$ by means of trajectories of (1). The system (1) is said to be small-time locally controllable (STLC) at $x_0$, if $x_0$ belongs to the interior int $\mathcal{R}(x_0,T)$ of the set $\mathcal{R}(x_0,T)$ for each $T > 0$.

Given two smooth vector fields $h^1$ and $h^2$ defined on $\mathbb{R}^n$, we denote by $[h^1,h^2]$ their Lie product (Lie bracket). Then

$$[h^1,h^2] \;=\; h^1_x(x)h^2(x) - h^2_x(x)h^1(x),$$

*This research was partially supported by the Swiss NSF No. 7 IP 65642 (Program SCOPES) and by the Ministry of Science and Higher Education - National Fund for Science Research under contracts MM-807/98 and MM-1104/01.

where by $h^1_x$ and $h^2_x$ are denoted the corresponding Jacobians. If we view vector fields as operators on smooth functions $\phi : \mathbb{R}^n \to \mathbb{R}$, then $[h^1, h^2]\phi = (h^1h^2 - h^2h^1)\phi$. We also write $[h^1, h^2] = (ad\ h^1, h^2)$ and, inductively $(ad^{k+1}\ h^1\ , h^2) = [h^1, (ad^k h^1, h^2)]$.

The small-time local controllability is a local property for the case of bounded controls. So, we can assume without loss of generality that $x_0 = 0 \in \mathbb{R}^n$ and the vector fields $f^0$ and $f^1$ are defined on some compact neighborhood of the origin. By $\mathcal{L}$ we denote the Lie algebra generated by $f^0$ and $f^1$, i.e. $\mathcal{L}$ is the real linear span generated by all Lie brackets of the vector fields $f^0$ and $f^1$.

There are many possible approaches to study the small-time local controllability, leading to different results and requiring different assumptions. Here we follow a geometrical point of view. The underlying philosophy is that for analytic systems of the form (1) the local properties of the reachable set are determined by the iterated Lie brackets of the vector fields $f^0$, and $f^1$, evaluated at the initial point. These values are in principle easily computable. For that reason, it is very natural to look for conditions for small-time local controllability which can be expressed in terms of elements of the Lie algebra $\mathcal{L}$. Some necessary and sufficient conditions for small-time local controllability are already proved in some special cases (see for example [6], [10], [21], [22], etc.). There are also known some necessary conditions (cf. [11], [14], [17], [19], etc.) as well as some very general sufficient conditions (cf. [2], [4], [7], [9], [20], etc.).

Based on the ideas proposed in [1], [2], [4], [19] and [20], we study the problem of small-time controllability of semi-groups of diffeomorphisms of special kind. The main result is a new sufficient controllability conditions for smooth control systems. This condition is stated in the next section. Some illustrative examples are also presented there.

## 2. Preliminaries and statement of the main result.

The traditional approach towards proving sufficient STLC conditions is to construct "high order control variations". Heuristically, if one can construct control variations in all possible directions, then the reachable set ought to be a full neighborhood of the starting point. This argument can be usually made rigorous by some topological considerations (cf. [2], [7], [8], [19], [21], [22], etc.). A possible definition of "high order control variations" is the following (cf. for example [8], [15], [21] and [22]):

**Definition 1.** Let $\Omega$ be a compact neighbourhood of the origin of $\mathbb{R}^n$ and $A : \Omega \to \mathbb{R}^n$ be a continuous function. It is said that $A$ is a variation of order $\alpha \geq 1$ of the reachable set of the control system (1) at the origin iff there exist positive real numbers $\beta > \alpha$, $T$, $M$, $Q$, $\theta$, $p_i$, $i = 1, \ldots, k$, and

$1 = q_1 < q_2 < \cdots < q_k$, such that for each $t \in [0, T]$ and for each point $x \in \Omega$ the following inclusion holds true

$$x + t^{\alpha} A(x) + a(t, x) + O(t^{\beta}, x) \in \mathcal{R}(x, p(t)), \tag{2}$$

where $p(t) = \sum_{i=1}^{k} p_i t^{q_i}$, and the continuous functions $a(\cdot, \cdot) : [0, T] \times \Omega \to \mathbb{R}^n$ and $O(\cdot, \cdot) : [0, T] \times \Omega \to \mathbb{R}^n$ satisfy the following estimates

$$\|a(t, x)\| \le M \, t^{\theta} \, \|x\| \quad \text{and} \quad O(t^{\beta}, x) \le Q \, t^{\beta}, \; t \in [0, T], \; x \in \Omega.$$

By $E^{\alpha}$ we denote the set of all variations of order $\alpha$ of the attainable set of (1) at the origin. We set $E^{+} := \cup_{\alpha \ge 1} E^{\alpha}$.

Remark 1. An open question is how to construct all elements of the set $E^{+}$. Partial answers can be found in papers where the local properties of the attainable set are studied (cf. for example, [2], [4], [7], [8], [20], [21], etc.).

Traditional proofs of sufficient STLC conditions (cf. [2], [3], [6], [8], [20], [21], etc.) are based on using piece-wise constant admissible controls with bounded number of switchings. Let $N$ be an arbitrary positive integer and $\mathcal{U}_N$ consist of all piece-wise constant admissible controls whose number of switchings is not greater than $N$. The next example shows that the class $\mathcal{U}_N$ is not large enough when we study the local properties of the reachable sets of smooth affine systems:

Example 1. Let us consider the following control system:

$$\left| \begin{array}{lll} \dot{x}_1 & = & u, \quad u \in [-1, 1], \qquad x_1(0) = 0, \\ \dot{x}_2 & = & x_1, \qquad\qquad\qquad\quad\; x_2(0) = 0, \\ \dot{x}_3 & = & x_1^3, \qquad\qquad\qquad\quad\; x_3(0) = 0, \\ \dot{x}_4 & = & x_3^2 - x_2^7, \qquad\qquad\;\;\; x_4(0) = 0. \end{array} \right.$$

It is proved in [12] the following: 1) This control system is STLC at the origin by means of fast switching controls, i.e. when we use controls whose number of switchings tends to infinity as the time $T > 0$ decreases to zero; 2) Let $N$ be an arbitrary positive integer. If we consider only controls from the class $\mathcal{U}_N$, then this control system is not STLC at the origin.

This remarkable example motivates the paper [2], where a general geometrical approach is introduced to handle this "fast switching" phenomenon. To formulate a result from [2] pertaining to the STLC property of Example 1, we shall introduce some notations: Let $y_0$ and $y_1$ be two symbols (called "indeterminates") and let Lie $(y_0, y_1)$ be the free Lie algebra generated by $y_0$ and $y_1$.

This implies the following: a) $y_0,\ y_1 \in \text{Lie }(y_0, y_1)$; b) if $\pi_1,\ \pi_2 \in \text{Lie }(y_0, y_1)$, then $c_1\pi_1 + c_2\pi_2 \in \text{Lie }(y_0, y_1)$, where $c_1$ and $c_2$ are arbitrary real numbers; c) if $\pi_1,\ \pi_2 \in \text{Lie }(y_0, y_1)$, then their Lie bracket $[\pi_1,\ \pi_2]$ belongs to Lie $(y_0, y_1)$, where $[\pi_1,\ \pi_2] := \pi_1\pi_2 - \pi_2\pi_1$. If $\pi$ is a Lie bracket belonging to Lie $(y_0, y_1)$, we denote by $\pi(f^0, f^1)(x)$ that Lie bracket from $\mathcal{L}$ which is obtained from $\pi$ by substituting for each $y_i$ the operator $f^i$, $i = 0, 1$. Also, we denote by deg $(\pi)$ its length, i.e. by deg $(\pi)$ is denoted the number of times that $y_0$ and $y_1$ appear in $\pi$. Let $\Pi_1$ and $\Pi_2$ be sets of Lie brackets from Lie $(y_0, y_1)$. We set

$$[\Pi_1,\ \Pi_2] := \{[\pi_1,\ \pi_2] \mid \pi_1 \in \Pi_1,\ \pi_2 \in \Pi_2\}.$$

Also, by span $\Pi_i$ we denote the minimal linear subspace of Lie $(y_0, y_1)$ containing the elements of $\Pi_i$, $i = 1, 2$. At last, by Bad $(y_0, y_1)$ we denote the set of all Lie brackets of $y_0$ and $y_1$ which are of even degree with respect to $y_1$ and of odd degree with respect to $y_0$. According to the general sufficient condition proved in [20], the possible obstructions for the STLC property of (1) are contained in the set

$$\left\{\pi(f^0, f^1) \mid \pi(f^0, f^1)(0) \neq 0,\ \pi \in \text{Bad }(y_0, y_1)\right\}.$$

The following result shows how these "bad Lie brackets" can be neutralized in order not to be an obstruction for small-time local controllability:

**Theorem 1 (Theorem 4, [2]).** *Let $\nu \in [0, 1]$, $r \geq 0$ and $\Pi^1$ be a subset of Lie $(y_0, y_1)$ such that the elements of $\Pi^1$ generate the Lie algebra Lie $(\Pi^1)$ with the followings properties:*

*a) $\Pi^1$ is a set of free generators of the algebra Lie $(\Pi^1)$;*

*b) Bad $(y_0, y_1) \subseteq$ Lie $(\Pi^1)$.*

*Denote recurrently*

$$\Pi^{k+1} := \left[\Pi^1,\ \Pi^k\right],\quad \Pi := \bigcup_{k=1}^{\infty} \Pi^k;$$

$$d_\nu\ (\pi) := \ deg\ (\pi) - \nu.k,\ \ for\ every\ \pi \in \Pi^k,\ k = 1, 2, \ldots.$$

*Suppose that for all $l \geq 0$ and every $\pi \in \Pi^{2l+1} \cap$ Bad $(y_0, y_1)$ which satisfies the relation $d_\nu(\pi) \leq r$, the following inclusion*

$$\pi(f^0, f^1)(0) \ \in\ \ span\ \left\{\pi^1(f^0, f^1)(0) \mid \pi^1 \in \Pi,\ d_\nu(\pi^1) < d_\nu(\pi)\right\}$$

*holds true. Then the linear subspace of $\mathbb{R}^n$ generated by the set*

$$\left\{\pi(f^0, f^1)(0) \mid \pi \in \Pi,\ d_\nu(\pi) \leq r\right\}$$

*is a subset of the set $E^+$ of the reachable set of (1) at the origin.*

Before to state the main result of this note, we define the set Good $(\Pi^1)$ as follows:

$$\text{Good }(\Pi^1) \; := \; \Pi^1 \; \setminus \; \text{Bad }(y_0, y_1)$$

where $\Pi^1$ is an arbitrary set of Lie brackets from Lie $(y_0, y_1)$.

**Theorem 2.** *Let $\nu \in [0,1]$, $r \geq 0$ and $\Pi^1$ be a subset of Lie $(y_0, y_1)$ such that the elements of $\Pi^1$ generate the Lie algebra Lie $(\Pi^1)$ with the followings properties:*

*a) $\Pi^1$ is a set of free generators of the algebra Lie $(\Pi^1)$;*

*b) $\{\pi |\; \pi \in Bad\; (y_0, y_1),\; \pi(f_0, f_1) \not\equiv 0\} \subseteq Lie\; (\Pi^1)$.*

*Denote recurrently*

$$\Pi^{k+1} \; := \; \left[\Pi^1, \; \Pi^k\right], \quad \Pi \; := \; \bigcup_{k=1}^{\infty} \; \Pi^k,$$

$$d_\nu(\pi) \; := deg(\pi) \; - \; \nu.k \text{ for every } \pi \in \Pi^k, \; k = 1, 2, \dots$$

*Suppose that for every Lie bracket $\pi_0 \in \Pi^{2l+1}$ which is of even degree with respect to each element of Good $(\Pi^1)$ and satisfies the relation $d_\nu(\pi) \leq r$, the following inclusion*

$$\pi_0(f^0, f^1)(0) \; \in \; span \; \left\{\pi^1(f^0, f^1)(0) | \; \pi^1 \in \Pi, \; d_\nu(\pi^1) < d_\nu(\pi)\right\}$$

*holds true. Then the linear subspace of $\mathbb{R}^n$ generated by the set*

$$\left\{\pi(f^0, f^1)(0) | \; \pi \in \Pi, \; d_\nu(\pi) \leq r\right\}$$

*is a subset of the set $E^+$ of the reachable set of (1) at the origin.*

Remark 2. A detailed proof of Theorem 2 will be given in a forthcoming paper for the case of multi-input affine control systems:

$$\dot{x}(t) \; = \; f^0(x(t)) \; + \; \sum_{i=1}^{m} \; u_i(t) \; f^i(x(t)). \tag{3}$$

The next example shows a control system of the type (1) for which the set $\Pi^{2l+1} \cap$ Bad $(y_0, y_1)$ is strictly greater than the set of all Lie brackets $\pi \in \Pi^{2l+1}$ that are of even degree with respect to each element of Good $(\Pi^1)$:

E x a m p l e 2. Let us consider the following control system

$$\left| \begin{array}{rcll} \dot{x}_1 & = & u, \quad u \in [-1,1], & x_1(0) = 0 \\ \dot{x}_2 & = & x_1, & x_2(0) = 0 \\ \dot{x}_3 & = & x_1^3/6, & x_3(0) = 0 \\ \dot{x}_4 & = & x_2 x_3, & x_4(0) = 0 \end{array} \right.$$

where $|u| \leq 1$, and $x(0) = (0,0,0,0,0)^T$. For this system we have that

$$f^0(x) = (0, x_1, \frac{1}{6}x_1^3, x_2 x_3) \text{ and } f^1(x) = (1,0,0,0).$$

One easily calculates all nonvanishing at the origin Lie brackets:

$$f^1(0) = \frac{\partial}{\partial x_1}, \ \left[f^1, f^0\right](0) = \frac{\partial}{\partial x_2}, \ \left(ad^3\ f_1, f_0\right)(0) = \frac{\partial}{\partial x_3},$$

$$\text{and } \left[\left[f^1, f^0\right], \left[\left(ad^3\ f^1, f^0\right), f^0\right]\right](0) = \frac{\partial}{\partial x_4}.$$

The last Lie bracket is of even degree (4-th degree) with respect to $f_1$ and of odd degree (3-rd degree) with respect to $f_0$. Moreover, the value of this Lie bracket at the origin can be not represented as a linear combination of the remainder Lie brackets evaluated also at the origin. Thus, we can not apply Theorem 1 in order to obtain that this control system is STLC at the origin. To the author knowledge, this system is studied at first in [13] where it is proved directly its small-time local controllability at the origin.

Let $S$ be an arbitrary set and let $s_0$ belong to $S$. The so called elimination theorem (cf. [5]) implies that the linear hull of all Lie brackets of the elements of $S$, except $s_0$, is a Lie algebra freely generated by the Lie brackets:

$$\left\{\left(ad^i\ s_0, s\right) \mid s \in S \setminus \{s_0\},\ i = 1,2,\ldots\right\}.$$

According to the elimination theorem, the only Lie subalgebra of Lie $(y_0, y_1)$ that does not contain $y_1$ is freely generated by the set

$$\left\{\left(ad^i\ y_1, y_0\right) \mid i = 0,1,2,\ldots\right\}.$$

We set

$$\Pi^1 \ := \ \left\{\left(ad^i\ y_1, y_0\right) \mid i = 0,1,2,3\right\}.$$

Clearly,

$$\text{Good } (\Pi^1) \ = \ \left\{\left[f^1,\ f^0\right],\ \left(ad^3\ f^1,\ f^0\right)\right\}.$$

We set $\nu = 1$. It can be directly checked that all Lie brackets $\pi$ of Bad $(y^0, y^1)$, for which $\pi(f^0, f^1)$ is nonvanishing, belong to Lie $(\Pi^1)$. Moreover, all Lie brackets $\pi$ from $\Pi^{2l+1}$ that are even degree with respect to each element of Good $(\Pi^1)$ and satisfy the relation $d_1(\pi) \le 4$ vanish at the origin. Hence, all assumptions of Theorem 2 hold true and we can conclude that $g_i^\pm \in E^+$, $i = 1, 2, 3$, where

$$g_1^\pm := \pm \left[f^1, f^0\right], \; g_2^\pm := \pm \left(ad^3 \; f^1, f^0\right), \; g_3^\pm := \pm \left[\left[f^1, f^0\right], \left[\left(ad^3 \; f^1, f^0\right), f^0\right]\right].$$

Clearly, the vector fields $g_4^\pm := \pm f^1$ also belong to $E^+$. Since the convex hull of the set

$$\left\{g_i^\pm(0) | \; i = 1, 2, 3, 4\right\}$$

has nonempty interior with respect to $\mathbb{R}^4$, we obtain (cf. for example [8], [16], [21], [22]) that this control system is STLC at the origin.

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*Institute of Mathematics and Informatics*
*Bulgarian Academy of Sciences*
*Sofia 1090, BULGARIA*
*e-mail: krast@math.bas.bg*

*Received 30.09.2003*