Mathematica
Balkanica

New Series Vol. 19, 2005, Fasc. 1-2

A Model of Communication System via Random Walk over Simplex *

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The communication system is considered as finite set of subscribers, establishing temporary pair-wise connections between themselves (for instance, closed telephone network). The possible states of the elements are free, active and paired. The subscribers change their states randomly according specified two dimensional Markov process. Some explicit properties of the system performance are established and compared with results of simulation experiments.

AMS Subj.Classification: 90B15, 60J20, 60K25, 65C05

Key Words: Stochastic network models, Discrete Markov process application, Simulation

1. Introduction

Traffic models are classical problem in stochastics. Queuing theory first steps are Erlang works on telephone traffic model. Many papers are devoted to problems of communication systems and their technical capacity. The popular problem of repeated calls takes into account subscribers' behavioral characteristics ([1], [3], [4], [5]). Usually the system is considered as input flow and server. The proposed here simplified model describes the communication network as a closed Markovian system. Combining technique from Markov chains and processes the existence of stationary state is proved. The mean value or expected number of subscribers in various states is obtained.

Nevertheless, some of the assumptions are too faraway from those which operate in real communication systems. However it is possible to derive some qualitative conclusions of common validity.

Some simulation experiments are also performed. They confirm the theoretical predictions of the considered model.

*Partially supported by funds of IMI - BAS, and contract PRO-ENBIS: GTC1-2001-4303

2. Model formulation.

Define a communication system as a finite set of elements (subscribers) $\{e_1, e_2, \ldots, e_N\}$ establishing temporary connections between themselves.

Assume that only pair-wise connections are possible. Then the system may be presented as an union of three non-overlapping subsets:

$\mathcal{F}$: the subset of free, non busy subscribers;
$\mathcal{A}$: the subset of active subscribers, trying to make connection and
$\mathcal{S}$: the subset of safely connected pairs.

Busy subscribers are those which are connected or active. The triplet $(\mathcal{F}, \mathcal{A}, \mathcal{S})$ is referred as a state of the system.

For the sake of brevity, we note the event "at moment t the element $e_i \in \mathcal{X}$" by $e_i(t) = \mathcal{X}$, where $\mathcal{X}$ is one of $\mathcal{S}, \mathcal{A}$ or $\mathcal{F}$.

Denote the number of elements in the subsets $\mathcal{F}$, $\mathcal{A}$ and $\mathcal{S}$ by F, A and $2S$ respectively (i.e. F is the number of free subscribers, A is the number of active subscribers and $2S$ is the number of subscribers in pairs). All they are non-negative integers, $N = F + A + 2S$ is positive and assumed even without loss of generality.

When the system operates its elements change their states walking through the subsets $\mathcal{F}$, $\mathcal{A}$ and $\mathcal{S}$. The numbers F, A and S also change in time. Thus we denote $F(t), A(t)$ and $S(t)$ these cardinalities at the moment t.

The argument t will be omitted when it is understood. We assume that the system is symmetrical i.e. all elements e_i have equal chance to change its state between the three possibilities.

Consequently, the state of the system is fully determined by the vector $(F(t), A(t))$.

The transitions of the system from $(F(t), A(t))$ to $(F(t+\triangle t), A(t+\triangle t))$ is denoted by $(\triangle F(t), \triangle A(t))$.

For a short time $\triangle t$ only one of the following transitions may occur:

(a) Some free element $e_i(t) = \mathcal{F}$ becomes active: $e_i(t + \triangle t) = \mathcal{A}$. The system changes as: $F(t + \triangle t) = F(t) - 1$ and $A(t + \triangle t) = A(t) + 1$, i.e. $(\triangle F(t), \triangle A(t)) = (-1, +1)$.

(b) Some active element $e_i(t) = \mathcal{A}$ establishes successfully connection with some free element $e_j(t) = \mathcal{F}$ and both become linked: $e_i(t + \triangle t) = \mathcal{S}$ and $e_j(t + \triangle t) = \mathcal{S}$. The system changes as $F(t + \triangle t) = F(t) - 1$ and $A(t + \triangle t) = A(t) - 1$, i.e. $(\triangle F(t), \triangle A(t)) = (-1, -1)$.

(c) Some active element $e_i(t) = \mathcal{A}$ refuses to continue retrial and becomes free: $e_i(t + \triangle t) = \mathcal{F}$. The system changes as $F(t + \triangle t) = F(t) + 1$ and $A(t + \triangle t) = A(t) - 1$, i.e. $(\triangle F(t), \triangle A(t)) = (+1, -1)$.

(d) Some connected element $e_i(t) = \mathcal{S}$ interrupts the link with its party $e_j(t) = \mathcal{S}$ and both become free: $e_i(t + \triangle t) = \mathcal{F}$ and $e_j(t + \triangle t) = \mathcal{F}$. The system changes as $F(t + \triangle t) = F(t) + 2$ and $A(t + \triangle t) = A(t)$, i.e. $(\triangle F(t), \triangle A(t)) = (+2, 0)$.

The possible transitions are illustrated on *Figure 1.*

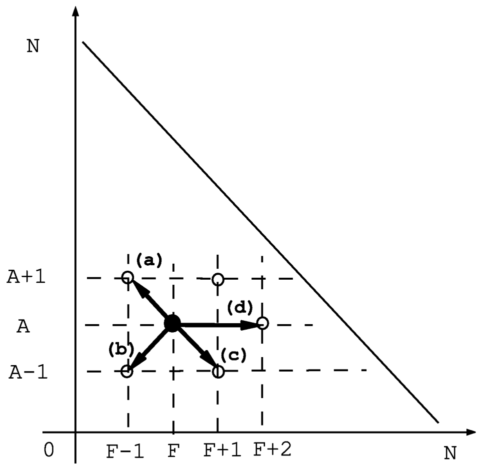

Figure 1. Explicit random walk over the simplex.

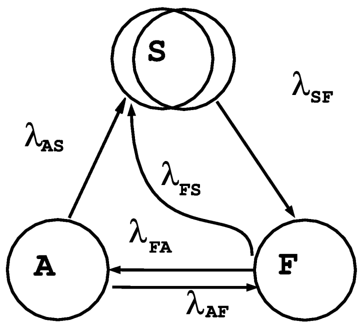

Figure 2. The aggregated states of the respective Markov chain.

The natural model of the system is then represented by a two dimensional Markov process $(F(t), A(t))$ in continuous time.

The joint distribution $\mathbf{P}(F(t) = i, A(t) = j) \equiv p_{(i,j)}(t)$, is determined by an initial distribution $p_{(i,j)}(0)$ and the transition probabilities

$$\mathbf{P}(F(t + \triangle t) = k, A(t + \triangle t) = l | F(t) = i, A(t) = j) \equiv q_{(k,l)}(i, j).$$

We will consider our system homogeneous, i.e. $q_{(k,l)}(i, j)$ does not depend on time t. We derive the transition probabilities $q_{(k,l)}(i, j)$ on the base of the behavioral characteristic of a single subscriber e_i.

The sequence of the states of a single element e_i over the time is described by a simple Markov chain as follows: The element $e(t)$ may change its state between three states with transition intensities λ_{XY} (see *Figure 2*), i.e.

$$\mathbf{P}(e(t + 1) = \mathcal{Y} |\ e(t) = \mathcal{X}) = \lambda_{XY}, \quad \mathcal{X}, \mathcal{Y} \in \{\mathcal{F}, \mathcal{A}, \mathcal{S}\}.$$

The transition intensities depend on the current state of the system or pair (F, A) and on fixed parameters of subscribers' behavior described below.

So the model is homogeneous since it does not depend on the time. The time scale may be appropriately chosen, so that only one jump is possible for a small unit of time.

A single change of the system state is made in one of the following manners.

An element in the state $\mathcal{A}$ may realize connection if its choice is a free subscriber. The probability is F/N if there are no preferences at all. A simple correction is possible by introducing a *handicap parameter* γ as conditional probability of success if a free subscriber is chosen. Thus, the probability an active element to make a connection is:

$$(i) \qquad \mathbf{P}(e(t+1)=\mathcal{S}|\, e(t)=\mathcal{A}) = \lambda_{AS} = \gamma\frac{F}{N}, \qquad 0<\gamma\le 1.$$

There are two other possibilities for the active element, if the jump to $\mathcal{S}$ is not realized. The element may continue the trials (staying in $\mathcal{A}$) or may stop trials and jump to $\mathcal{F}$. The *refusing parameter* φ is the probability an active subscriber to refuse keeping retrial. The probability of a jump from $\mathcal{A}$ to $\mathcal{F}$ is

$$(ii) \qquad \mathbf{P}(e(t+1)=\mathcal{F}|\, e(t)=\mathcal{A}) = \lambda_{AF} = \varphi(1-\gamma\frac{F}{N}), \qquad 0\le\varphi\le 1.$$

An element in the state $\mathcal{F}$ jumps to $\mathcal{S}$ with probability

$$(iii) \qquad \mathbf{P}(e(t+1)=\mathcal{S}|\, e(t)=\mathcal{F}) = \lambda_{FS} = \gamma\frac{A}{N}.$$

The probability to choose (or to be chosen from) subscriber from $\mathcal{A}$ is A/N and the handicap parameter γ is the same. A non-chosen subscriber becomes active with probability α (*activity parameter*), $0<\alpha<1$. Therefore

$$(iv) \qquad \mathbf{P}(e(t+1)=\mathcal{A}|\, e(t)=\mathcal{F}) = \lambda_{FA} = \alpha(1-\gamma\frac{A}{N}).$$

An element in the state $\mathcal{S}$ ends its contact and jumps into $\mathcal{F}$ with probability

$$(v) \qquad \mathbf{P}(e(t+1)=\mathcal{F}|\, e(t)=\mathcal{S}) = \lambda_{SF} = \beta, \quad 0<\beta<1.$$

The parameter β is related to the mean duration of a connection which is supposed equal to $(1-\beta)/\beta$.

Note that transitions are not independent since an element move from $\mathcal{A}$ to $\mathcal{S}$ implies an additional move from $\mathcal{F}$ to $\mathcal{S}$, and an element move from $\mathcal{S}$ to $\mathcal{F}$ causes another move of the same type.

3. The Stationary state

Now our purpose is to prove that the system $(\mathcal{F}, \mathcal{A}, \mathcal{S})$ is ergodic and a stationary distribution exists. First we perform conversion from discrete time to continuous following Poisson process scheme, i.e. if some event occurs in the unique time interval with intensity λ, then the probability of its occurrence in a short time $\triangle t$ is $\lambda\triangle t + o(\triangle t)$. If M independent experiments with the same distribution are performed, the probability of just one occurrence is $M\lambda\triangle t + o(\triangle t) = \lambda'\triangle t + o(\triangle t)$. The probability of zero occurrences is $1 - \lambda'\triangle t + o(\triangle t)$.

Using the probabilities λ_{XY} for a jump from $\mathcal{X}$ to $\mathcal{Y}$ of a single element given by $(i) - (v)$ and the state of the whole system (F, A), then the intensities λ'_{XY} for a jump in a short time $\triangle t$ are as follow:

$$
\begin{aligned}
\lambda'_{AS} &= A\lambda_{AS} = \gamma\frac{AF}{N}, \\
\lambda'_{AF} &= A\lambda_{AF} = \varphi(A - \gamma\frac{AF}{N}), \\
\lambda'_{FS} &= F\lambda_{FS} = \gamma\frac{AF}{N}, \\
\lambda'_{FA} &= F\lambda_{FA} = \alpha(F - \gamma\frac{AF}{N}), \\
\lambda'_{SF} &= 2S\lambda_{SF} = 2\beta S = \beta(N - F - A).
\end{aligned}
$$

Remember that the distribution of the Markov process was denoted as $\mathbf{P}(F(t) = F, A(t) = A) = p_{(F,A)}(t)$, $(F = 0, 1, 2, \ldots, N,\ A = 0, 1, \ldots, (N - F))$ The possible transitions (a), (b), (c) and (d) correspond to λ, indexed with subscripts FA, AS (together with FS), AF and SF respectively. Hence, the basic equations come out to be

$$
\begin{aligned}
\frac{dp_{(F,A)}(t)}{dt} &= p_{(F-2,A)}(t)\ \beta(N-(F-2)-A) + p_{(F-1,A-1)}(t)\ \gamma\frac{(F-1)(A-1)}{N} + \\
&+ p_{(F-1,A+1)}(t)\ \alpha(F-1-\gamma\frac{(F-1)(A+1)}{N}) + \\
&+ p_{(F+1,A-1)}(t)\ \varphi(A-1-\gamma\frac{(F+1)(A-1)}{N}) - \\
&- p_{(F,A)}(t)\ \Big(\beta(N-F-A) + \gamma\frac{FA}{N} + \alpha(F-\gamma\frac{FA}{N}) + \varphi(A - \gamma\frac{FA}{N})\Big).
\end{aligned}
$$

Some of the equations above have terms with indexes out of the simplex boundaries: $F \geq 0,\ A \geq 0,\ F + A \leq N$. Those terms are to be considered as zeros. This fact implies that the boundaries are reflecting barriers and there are no absorbing states. Consequently, the following proposition holds:

Proposition *Considered Markov process* $(F(t), A(t))$ *possesses the ergodic property.*

Corollary *There exists unique stationary distribution*

$$\lim_{t\to\infty} p_{(i,j)}(t) = p_{(i,j)}.$$

The stationary distribution $\mathbf{P}(F = i, A = j) = p_{(i,j)}$ is a solution of the basic equations with left-hand sides replaced by zero, i.e. $\frac{dp_{(F,A)}(t)}{dt} = 0$. We can solve it using the model of a single subscriber, i.e. the Markov chain $e(t)$ with transition probabilities (1)-(5): $\lambda_{A,S}, \lambda_{A,F}, \lambda_{F,S}, \lambda_{F,A}$ and $\lambda_{S,F}$. Denote $\mathbf{P}(e(t) = \mathcal{F}) = f(t)$, $\mathbf{P}(e(t) = \mathcal{A}) = a(t)$ and $\mathbf{P}(e(t) = \mathcal{S}) = 2s(t)$, $f(t) + a(t) + 2s(t) = 1$. The Chapman-Kolmogorov equations have the form

$$\begin{aligned}
f(t+1) &= (1-\alpha)f(t)\Big(1-\gamma\frac{A(t)}{N}\Big) + \varphi a(t)\Big(1-\gamma\frac{F(t)}{N}\Big) + 2\beta s(t),\\
a(t+1) &= \alpha f(t)\Big(1-\gamma\frac{A(t)}{N}\Big) + (1-\varphi)a(t)\Big(1-\gamma\frac{F(t)}{N}\Big),\\
s(t+1) &= (1-\beta)s(t) + \gamma\frac{F(t)}{N}a(t).
\end{aligned}$$

Since the subscribers are alike, we can replace $F(t)/N$ and $A(t)/N$ by $f(t)$ and $a(t)$, respectively. Denote limits

$$\lim_{t\to\infty} f(t) = f,\ \lim_{t\to\infty} a(t) = a\ \lim_{t\to\infty} s(t) = s.$$

Setting $t \to \infty$ in left-hand and right-hand sides of Chapman-Kolmogorov equations, we obtain:

$$\begin{aligned}
f &= (1-\alpha)f(1-\gamma a) + \varphi a(1-\gamma f) + 2\beta s,\\
a &= \alpha f(1-\gamma a) + (1-\varphi)a(1-\gamma f),\\
s &= (1-\beta)s + \gamma f a.
\end{aligned}$$

After simple algebra we get the following quadratic equation for a:

$$a^2\Big(\gamma + \alpha\gamma - \varphi\gamma - \varphi\frac{2\gamma}{\beta}\Big) - a\Big(\alpha + \varphi + \gamma + \alpha\gamma - \varphi\gamma)\Big) + \alpha = 0,$$

with exactly one root in the interval $(0, 1)$. This is consequence from the fact that the left-side of the equation, say $\psi(a)$, has opposite sign over two ends of the interval $(0, 1)$:

$\psi(0) = \alpha > 0$ and $\psi(1) = -\varphi - \varphi\frac{2\gamma}{\beta} < 0$.
The solution depends on the behaviour parameters α, β, γ and φ.

4. Simulations

Simulations are performed using software system "MATLAB"[6]. The program code (the "m-file") is available "Copyleft" on request via e-mail.

The input parameters of the program include the number of subscribers N, the behaviour parameters $\alpha, \varphi, \gamma, \beta$ and the duration of simulation as a number of time steps n. The theoretical mean values of F and A are calculated and they are used as initial state of the system.

The single step of simulation follows exactly the Markov process model of the described communication system. Using generated random numbers, the direction of transition (a), (b), (c) or (d) is chosen with probability proportionally to $\lambda'_{FA}, \lambda'_{AS}, \lambda'_{AF}$ and λ'_{SF}, respectively. The walking trajectory is presented graphically in different projections and simple statistics are calculated.

The experiments are showing satisfactory behaviour of (F, A), i.e. the mean values of generated distributions are near to theoretical expectations. Two examples are presented as illustration. The simulations are performed using the parameter values: $\alpha = 0.1$ in the first example and $\alpha = 0.4$ in the second. The rest parameters are fixed $\varphi = 0.1, \gamma = 0.9$ $\beta = 0.98$. The number of steps is $n = 10000$ in the both experiments.

The basic descriptive statistics (empirical means, standard deviations and correlation matrix) of F, A and S as well the mathematical expectations are given in the table below.

Experiment	States	Expected value	Obs. mean	(st. dev.)	Correlations
α=0.1	F	76.64	77.34	(4.96)	1.00 -0.56 -0.77
	A	9.70	9.93	(3.20)	-0.56 1.00 -0.10
	S	6.83	6.36	(2.07)	-0.77 -0.10 1.00
α=0.4	F	46.47	47.41	(6.88)	1.00 -0.52 -0.79
	A	28.88	28.22	(4.23)	-0.52 1.00 -0.11
	S	12.32	12.17	(2.95)	-0.79 -0.11 1.00

Table 1. Statistics of simulation examples results.

It seems that mean and variance are more sensitive than correlations when behavioral parameters are changing.

Figures 3 and 4 illustrate the behaviour of the simulated system in the both examples. The three dimensional walking $(F(t), A(t), S(t))$, $t = 1, 2, \ldots, n$

is presented as four diagrams. The two dimensional projection on the simplex (F, A) is shown in the upper-right corner. The other plots are one dimensional projections of $A(t)$, $F(t)$ and $S(t)$ vs time t.

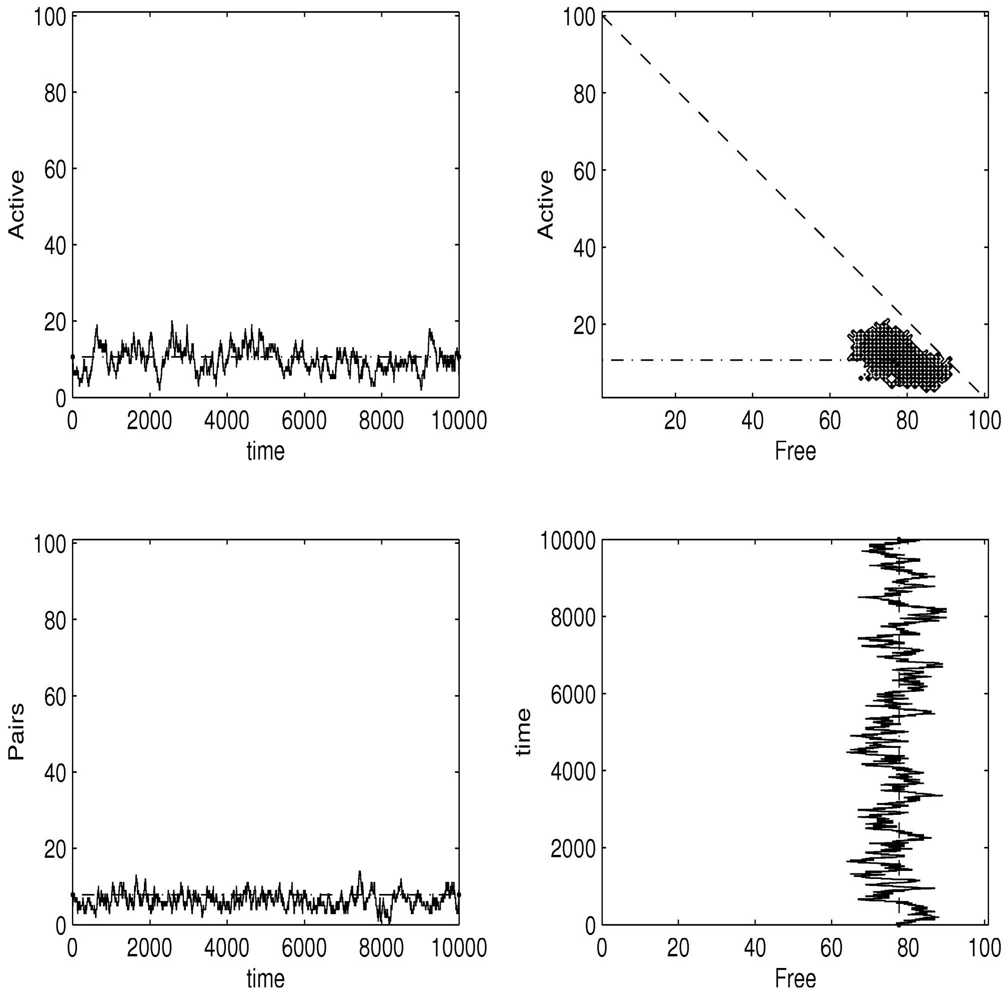

Figure 3. Simulation results for $\alpha = 0.1$ $\varphi = 0.1, \gamma = 0.9$ $\beta = 0.98$ and $n = 10000$ number of steps.

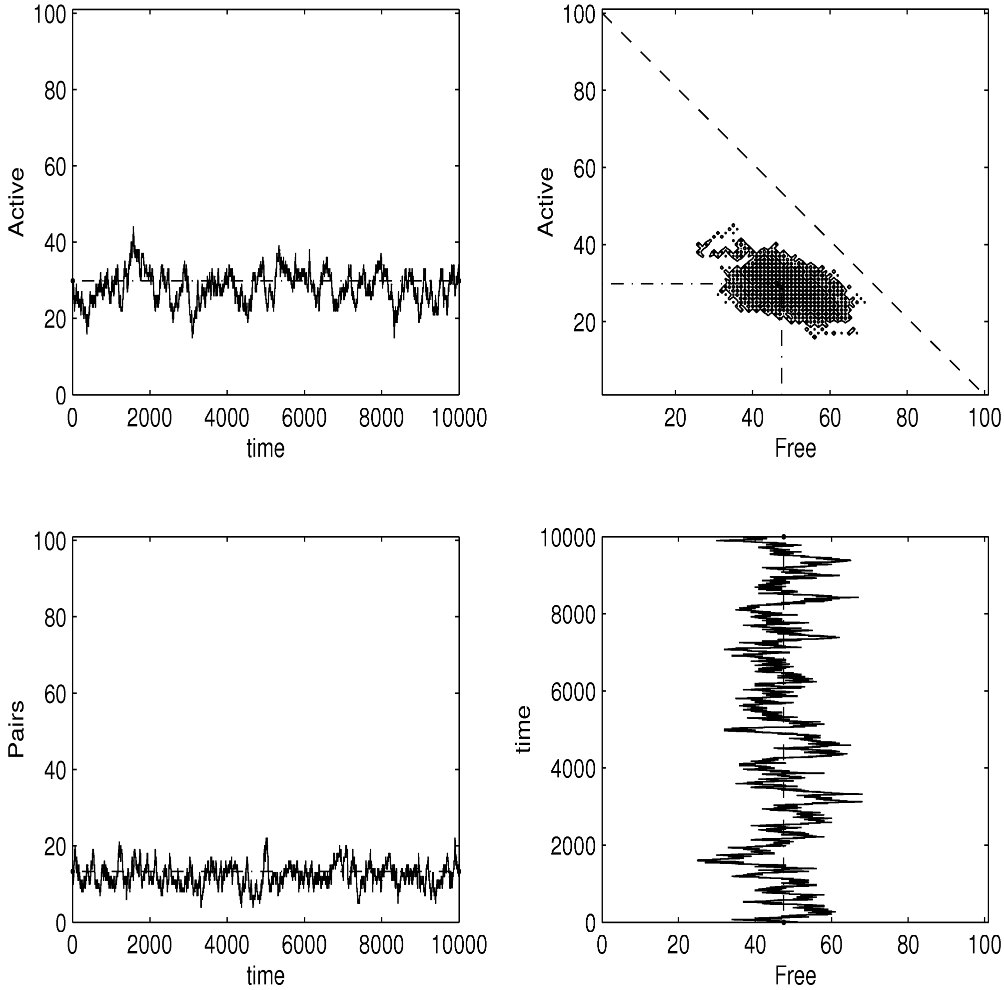

Figure 4. Simulation results for $\alpha = 0.4$ $\varphi = 0.1, \gamma = 0.9$ $\beta = 0.98$ and $n = 10000$ number of steps.

Comparing the results we note that four time increasing activity parameter α leads to doubling the mean value of connected pairs S and triplicate the mean value of the active (in vain!) subscribers. Comparing the standard deviations, one may conclude that the system becomes significantly more unstable.

5. Conclusions

It is shown that a simple stochastic model can describe the influence of some behavioral characteristics of the subscribers on the performance of a communication system. The used scheme is close to one discussed in [2], where more complicated - grouped in classes mobile subscribers approach is considered. The accent of our considerations is on subscribers' behaviour. Simplicity of the model offers some new perspectives of development: randomization of the behavioral parameters and refinement of the service schemes.

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Received 30.09.2003