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A Classification of the Five-Dimensional Lie Superalgebras, over the Complex Numbers.

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A classification is made of the five-dimensional Lie superalgebras (LS), over the base field C of complex numbers. We distinguish two types of LS, trivial and non trivial, according to if $[L_1, L_1] = \{0\}$ or $[L_1, L_1] \neq \{0\}$.

1. Introduction

The theory of Lie superalgebras (LS), has developed remarkably quickly, both in physics and mathematics, the last thirty years. The main efford has been devoted to the establishment of the theory of simple LS (see [8], [16]) and their representations (see [7], [13], [17]). The major step in the subject, was made by Kac (see [12]) with the classification of simple LS in the finite-dimensional case. The simple LS are used to describe internal symmetries of elementary particles and so there has been an emphasis in the literature on the simple LS. However, we can note and use the non-simple LS, such as the nilpotent Heisenberg LS as supersymmetry algebras for the harmonic oscillator (see [3], [4]). Moreover, since simple LS are a small subset of the set of all possible LS, it is worthwhile paying more attention on non-simple LS. To this direction, it is known the classification of real LS of dimension up to four (see [1]), as well the classification of all nilpotent LS of dimension five (see [10]). In this article, we classify all five-dimensional LS over the field of complex numbers.

We recall that a Lie superalgebra L, is a Z_2-graded vector space over a field K endowed with a bracket operation $[.,.]$, which is *bilinear*, *graded-anticommutative* and satisfies the *graded Jacobi identity.* Thus, if $L = L_0 \oplus L_1$, where L_0 is the even part and L_1 is the odd part of L, then we have:
(*i*) $[L_0, L_0] \subseteq L_0$, $[L_0, L_1] \subseteq L_1$, $[L_1, L_0] \subseteq L_1$, and $[L_1, L_1] \subseteq L_0$,
(*ii*) $[x, y] = -(-1)^{\sigma(x)\sigma(y)}[y, x]$, for all $x, y \in L_0 \bigcup L_1$,

(iii) $(-1)^{\sigma(x)\sigma(z)}[[x,y],z]+(-1)^{\sigma(y)\sigma(x)}[[y,z],x]+(-1)^{\sigma(z)\sigma(y)}[[z,x],y]=0,$
for all $x,y,z\in L_0\bigcup L_1$ (*graded Jacobi identity*),
where to each element of the set $L_0\bigcup L_1$ we attach a sign $\sigma(x)$, which is equal to 0, if $x\in L_0$ (even element) and it is equal to 1, if $x\in L_1$ (odd element). The elements of $L_0\bigcup L_1$, are called homogeneous elements.

It is worth noting that the *graded Jacobi identity* differs from the Jacobi identity for ordinary Lie algebras, only in the case of one even and two odd elements, say a and α,β, respectively. Then it takes the following form:

$$[[a,\alpha],\beta]+[[\alpha,\beta],a]-[[\beta,a],\alpha]=0.$$

Two LS $L=L_0\oplus L_1$ and $L'=L'_0\oplus L'_1$, are said to be equivalent if there exist a graded isomorphism $f:L\rightarrow L'$, which:
(i) preserves the bracket operation,
(ii) is homogeneous of degree 0, i.e. $f(L_0)=L'_0$ and $f(L_1)=L'_1$.
Usually, we determine f, by its restriction on L_0 and L_1.

Coming now to the mechanics of our classification of all five-dimensional complex LS, we begin by considering all trivial complex Lie algebras of dimension m for each of the possible dimension types (m,n), with $m+n=5$ (see [14], [15]).For each such complex Lie algebra, we determine its inequivalent $5-m=n-$dimensional representations. In each case, the complex Lie algebra L_0 and its representation, form a trivial LS of dimension five. We tabulate all indecomposable trivial LS obtained by the described method above. However, the decomposable trivial LS, although not included in the tabulations, are important in that they are needed as building blocks for non trivial LS.

Finally, we consider each pair (L_0,M), where L_0 is a complex Lie algebra, M is a L_0-module and $L_0\bigoplus M$ is a trivial LS of dimension five, indecomposable or decomposable. Then, we determine how many inequivalent LS $L=L_0\bigoplus L_1$ we can construct, so that L_1 and M are equivalent as L_0-modules. In this paper, we tabulate into families of equivalence classes the complex indecomposable LS of dimension five, which are not Lie algebras. We say that $L=L_0\bigoplus L_1$, is an (m,n) LS, if $dimL_0=m$ and $dimL_1=n$.

2. Five-dimensional Lie superalgebras

2.1 (4,1)-dimensional Lie superalgebras

Let $L=L_0\oplus L_1$ be a $(4,1)$-LS and $\{a_1,a_2,a_3,a_4;\alpha\}$ a set of generators. The one-dimensional representation $\rho:L_0\rightarrow L_1$ is defined by:

$$\rho(a_i)(\alpha)\equiv[a_i,\alpha]=\lambda_i\alpha,\ \ i=1,2,3,4.$$

So, we have one of the next two cases:
(1) $\lambda_i = 0, \quad \forall \ i = 1,2,3,4,$ or
(2) $\exists$ at least one $\lambda_i \neq 0,$ for example λ_4.

In case (1), any $(4,1)$-trivial Lie superalgebra is an ordinary Lie algebra of dimension four (for the real case, see [15]).

In case (2), the change of basis in L_0, defined by:
$a'_i = a_i - \frac{\lambda_i}{\lambda_4} a_4$, for $i = 1,2,3$ and $a'_4 = a_4$, gives
$[a'_i, \alpha] = 0$, for $i = 1,2,3$ and $[a'_4, \alpha] = \lambda_4 \alpha$, where $\lambda_4 \neq 0$.
We observe, that:
1) The scalar λ_4 can be reduced to unity by scaling a_4.
2) The kernel of ρ, h, is a three-dimensional ideal of L_0, generated by a_1, a_2, a_3 with a_4 acting on h as an external derivation, namely, the following holds for all $i,j = 1,2,3,\ i \neq j:$ $[a_4,[a_i,a_j]] = [[a_4,a_i],a_j] + [a_i,[a_4,a_j]]$.
3) The quotient Lie algebra $L_0/\ker(\rho)$ is isomorphic to the one-dimensional Abelian Lie algebra. So, we have: $L_0 = h + \mathrm{C}\ \mathrm{a}_4$.
4) Using all inequivalent forms of the three-dimensional complex Lie algebra h and the *graded Jacobi identity*, we finally find 11 pairwise non isomorphic families of trivial $(4,1)$-LS plus 1, namely the LS E^{10}. The latter one does not depend on any parameters, but it is non isomorphic to any other LS from the above families.

For the non trivial case, using the *graded Jacobi identity*, we get:

$$[a_i,[\alpha,\alpha]] = 2[[a_i,\alpha],\alpha], \quad i = 1,2,3,4 \tag{2.1}$$

$$[\alpha,[\alpha,\alpha]] = 0. \tag{2.2}$$

In case (1), L_0 is a four-dimensional Lie algebra with $[L_0, L_1] = 0$. From (2.1) we conclude that the $[\alpha,\alpha]$ belongs to the center of L_0. Thus, we find 3 non isomorphic non trivial $(4,1)$-dimensional LS.

In case (2), we conclude that the element $[\alpha,\alpha]$ belongs to the center of the ideal h. This analysis leads to 4 families of non trivial $(4,1)$-dimensional LS, which are pairwise non isomorphic plus 3 non isomorphic LS, which are not depending on parameters.

2.2 (1,4)-dimensional Lie superalgebras

Let $L = L_0 \oplus L_1$ be a $(1,4)$-LS and $\{a; \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ a set of generators. In the trivial case, we note that the action of a on L_1 is completely determined by a 4×4 complex matrix. Thus, there exist 10 pairwise non isomorphic families of $(1,4)$-trivial Lie superalgebras plus 2 non isomorphic LS not depending on parameters, all arising from the inequivalent Jordan forms of a 4×4 matrix.

Working for the non trivial case, from the *graded Jacobi identity*, we have:

$$[[\alpha_i,\alpha_i],\alpha_i]=0, \quad \text{for all } \ i=1,2,3,4.$$

Let $[\alpha_i,\alpha_j]=S_{ij}\ a, \quad i,j=1,2,3,4$. Then the matrix $S=(S_{ij})$ is a 4×4 complex symmetric matrix and by a linear transformation takes a diagonal form $S=diag\ (s_1,s_2,s_3,s_4)$. We distinguish the following three cases:
(1) If $s_i=0$ for all $i=1,2,3,4$, then the LS is trivial.
(2) If there is at least one $s_i\neq 0$, let $[\alpha_1,\alpha_1]=s_1 a,\ s_1\neq 0$, then $[a,\alpha_1]=0$. Using again the *graded Jacobi identity*, for $i=2,3,4$, we have: $[a,\alpha_i]=0$. So, in order to obtain an indecomposable LS we must have $s_i\neq 0$ for all i.
(3) If $s_i\neq 0$ for all i, then we get only one form (over C) of the matrix S.
This analysis leads to 1 family of non trivial indecomposable $(1,4)$-LS.

2.3 (3,2)-dimensional Lie superalgebras

Let $L=L_0\oplus L_1$ be a $(3,2)$-Lie superalgebra and $\{a_1,a_2,a_3;\alpha_1,\alpha_2\}$ a set of generators. From the classification over C of the 3-dimensional Lie algebras, we can get 7 inequivalent cases for the Lie algebra L_0.We conclude the results, by using all these different forms of L_0 and the following *graded Jacobi identity*, which holds for all $i,j=1,2,3$ and $k=1,2$:

$$[[a_i,a_j],\alpha_k]+[[a_j,\alpha_k],a_i]+[[\alpha_k,a_i],a_j]=0.$$

So, we finally find 18 pairwise non isomorphic families of trivial $(3,2)$-dimensional LS plus 7 more non isomorphic LS, which are not depending on any parameters.

For the non trivial case, from the *graded Jacobi identity*, we have:

$$[[a_i,\alpha_j],\alpha_k]+[[\alpha_j,\alpha_k],a_i]-[[\alpha_k,a_i],\alpha_j]=0, \quad \text{for all } \ i=1,2,3,\ j,k=1,2$$

$$\text{and } \ [[\alpha_i,\alpha_j],\alpha_k]+[[\alpha_j,\alpha_k],\alpha_i]+[[\alpha_k,\alpha_i],\alpha_j]=0, \quad \text{for all } \ i,j,k=1,2.$$

Using these two relations and the 25 non isomorphic trivial $(3,2)$-dimensional LS that we found before, we get 9 pairwise non isomorphic families of $(3,2)$-non trivial LS plus 15 LS not depending on parameters.

2.4 (2,3)-dimensional Lie superalgebras

Let $L=L_0\oplus L_1$ be a $(2,3)$-LS and $\{a_1,a_2;\alpha_1,\alpha_2,\alpha_3\}$ a set of generators. Working as in the case of $(3,2)$-dimensional LS, we find 13 pairwise non isomorphic families of trivial $(2,3)$-dimensional LS plus 3 more LS not depending on parameters. For the non trivial case, we find 9 pairwise non isomorphic families plus 16 LS, which are not depending on parameters. It is worthwhile paying some attention to a special case of this type, namely those denoted $(2A_{1,1} + 3A)^i$, for all $i = 1,4,5,6,7,8$. These LS, have the two-dimensional Abelian Lie algebra $2A_{1,1}$ as even part and the zero representation on the odd part. Now, imposing the *graded Jacobi identity* on any triple of generators, does not provide us with any restrictions on anticommutators. Hence, for an arbitrary anticommutator, we can write: $[\alpha_i, \alpha_j] = S_{ij}\ a + T_{ij}\ b$, where S_{ij}, T_{ij} are elements of the complex 3×3 symmetric matrices S, T, respectively, and $a = a_1$, $b = a_2$. For the sequel, the symbols α, β, γ instead of $\alpha_1, \alpha_2, \alpha_3$, respectively, make our computations easier. Thus, our problem is reduced to that of determining all possible inequivalent forms of a pair of complex 3×3 symmetric matrices. In order to apply standard results (see [5], [6], [9]), we have to distinguish two cases:

I. at least one of the matrices S and T is non-singular,

II. both matrices S and T are singular.

In the first case, using transformation of the form:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{2.3}$$

on the set of odd generators, where P is a 3×3 complex invertible matrix and bearing in mind that we can permute, scale or change sign on the set of even generators a and b, we can reduce the pair S, T simultaneously into the forms $S' = P^T\ S\ P$ and $T' = P^T\ T\ P$, where:

(1) $S' = diag(1,1,1)$, $T' = diag(1,q,r)$, $1 \geq |q| \geq |r| \geq 0$

In the first case, we have the LS: $[\alpha,\alpha] = a+b$, $[\beta,\beta] = a+qb$, $[\gamma,\gamma] = a + rb$,
which for $q = 1$ or $q = r$, is decomposable. For $q \neq 1$ and $q \neq r$, we use the invertible linear transformation:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.4}$$

and scaling, if it is necessary, we find the following LS:

$$(2A_{1,1} + 3A)^1 : \quad [\alpha,\alpha] = a, \ [\beta,\beta] = b, \ [\gamma,\gamma] = a + b.$$

In the second case, transformations of the form (2.3), lead to the cases:

(2) $S' = diag(1,1,0)$, $T' = (t'_{ij})$,

(3) $S' = diag(1,0,0)$, $T' = (t'_{ij})$,

where (t_{ij}) is a 3×3 complex symmetric matrix, with $detT' = 0$.
In case (2), we use a linear transformation of the form:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \text{ with } P = \begin{pmatrix} P_1 & O_1 \\ O_2 & P_2 \end{pmatrix},$$

where P_1 is an orthogonal 2×2 matrix diagonalizing $(t'_{ij})_{2\times 2}$ and P_2 is a 1×1 matrix with element $\frac{1}{\sqrt{|t'_{33}|}}$, if $t'_{33} \neq 0$ or 1, if $t'_{33} = 0$. Also, $O_1 = (0\ \ 0)^{T}$ and $O_2 = (0\ \ 0)$. So, we conclude the following pair of complex symmetric matrices:

$$S' = diag(1,1,0), \quad T' = \begin{pmatrix} p & 0 & \kappa \\ 0 & q & \lambda \\ \kappa & \lambda & \varepsilon \end{pmatrix},$$

where $\varepsilon \in \{0, 1, -1\}$ and $detT' = 0$. We distinguish cases as $p \neq q$ or $p = q$, and finally find the following non isomorphic LS, where $\kappa, \lambda \in \mathcal{R}^*$:

$$(2A_{1,1} + 3A)^4 : \ [\alpha,\alpha] = a,\ [\beta,\beta] = b,\ [\alpha,\gamma] = a - b;$$

$$(2A_{1,1} + 3A)^5 : \ [\alpha,\alpha] = a,\ [\beta,\beta] = b,\ [\alpha,\gamma] = a - b,\ [\beta,\gamma] = \lambda(a - b);$$

$$(2A_{1,1} + 3A)^6 : \ [\alpha,\alpha] = a,\ [\beta,\beta] = b,\ [\gamma,\gamma] = a - b,\ [\alpha,\gamma] = \kappa(a - b),$$
$$[\beta,\gamma] = \lambda(a - b);$$

$$(2A_{1,1} + 3A)^7 : \ [\alpha,\alpha] = a,\ [\beta,\beta] = b,\ [\gamma,\gamma] = a - b,\ [\alpha,\beta] = \lambda(a - b).$$

In the case (3), working as above, we obtain the following LS:

$$(2A_{1,1} + 3A)^8 \ : \ [\alpha,\alpha] \ = \ a,\ [\beta,\beta] = b,\ [\alpha,\gamma] = b.$$

3. Tabulations

The following tabulations are of pairwise non isomorphic families of the five-dimensional complex indecomposable Lie superalgebras, which are not Lie algebras. The relations which describe the structure of every LS, in the next tables, obtained by the non zero commutators only. The trivial and non trivial LS are listed separately. For the the labelling of the five-dimensional LS, we use the letter E. The symbols a, b, c, d instead of a_1, a_2, a_3, a_4 and $\alpha, \beta, \gamma, \delta$ instead of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, make the use of the tables easier.

table I: TRIVIAL LS

	Relations	**Comments**
$(4,1)$		
E^1_{pqr}	$[d,a]=pa,[d,b]=qb,[d,c]=rc;[d,\alpha]=\alpha$	$p,q,r\neq 0$
E^2_{pq}	$[a,b]=c;[d,a]=pa,[d,b]=qb,[d,c]=(p+q)c;[d,\alpha]=\alpha$	$p,q\neq 0$
$E^3_{\mu r}$	$[a,b]=b,[a,c]=\mu c;[d,c]=rc;[d,\alpha]=\alpha$	$r\neq 0,\ \mu\neq 0,1$
E^4_q	$[a,b]=b,[a,c]=b+c;[d,b]=qb,[d,c]=qc;[d,\alpha]=\alpha$	$q\neq 0$
E^5_q	$[a,b]=c;[d,b]=qb,[d,c]=qc;[d,\alpha]=\alpha$	$q\neq 0$
E^8_p	$[d,a]=pa,[d,b]=a+pb,[d,c]=b+pc;[d,\alpha]=\alpha$	$p\neq 0$
E^9_λ	$[a,c]=a,[b,c]=\lambda a+b;[d,b]=a,[d,c]=b;[d,\alpha]=\alpha$	$\lambda\neq 0$
E^{10}	$[a,c]=a,[b,c]=b;[d,b]=a,[d,c]=b;[d,\alpha]=\alpha$	
E^{11}_κ	$[b,c]=\kappa a;[d,b]=a,[d,c]=b;[d,\alpha]=\alpha$	$\kappa\neq 0$
E^{12}_{pq}	$[d,a]=pa,[d,b]=a+pb,[d,c]=qc;[d,\alpha]=\alpha$	$p,q\neq 0$
E^{13}_q	$[a,b]=c;[d,a]=pa,[d,b]=a+pb,[d,c]=2pc;[d,\alpha]=\alpha$	$p\neq 0$
E^{14}_μ	$[b,a]=a+c,[b,c]=\mu c;[d,b]=a;[d,\alpha]=\alpha$	$\mu\neq 0,\ \lvert\mu\rvert\leq 1$
$(1,4)$		
E^{20}_{qrs}	$[a,\alpha]=\alpha,[a,\beta]=q\beta,[a,\gamma]=r\gamma,[a,\delta]=s\delta$	$0\leq\lvert s\rvert\leq\lvert r\rvert$ $\leq\lvert q\rvert\leq 1$
E^{21}_p	$[a,\alpha]=p\alpha+\beta,[a,\beta]=p\beta+\gamma,[a,\gamma]=p\gamma+\delta,[a,\delta]=p\delta$	$p\neq 0$
E^{22}	$[a,\alpha]=\beta,[a,\beta]=\gamma,[a,\gamma]=\delta$	
E^{23}_p	$[a,\alpha]=p\alpha,[a,\beta]=p\beta+\gamma,[a,\gamma]=p\gamma+\delta,[a,\delta]=p\delta$	$p\neq 0$
E^{24}_p	$[a,\alpha]=p\alpha+\beta,[a,\beta]=p\beta,[a,\gamma]=p\gamma+\delta,[a,\delta]=p\delta$	$p\neq 0$
E^{25}	$[a,\alpha]=\beta,[a,\gamma]=\delta$	
E^{26}_p	$[a,\alpha]=p\alpha,[a,\beta]=p\beta,[a,\gamma]=p\gamma+\delta,[a,\delta]=p\delta$	$p\neq 0$
E^{27}_{pq}	$[a,\alpha]=p\alpha,[a,\beta]=q\beta+\gamma,[a,\gamma]=q\gamma+\delta,[a,\delta]=q\delta$	$p,q\neq 0,\ p\neq q$
E^{28}_p	$[a,\alpha]=p\alpha,[a,\beta]=\gamma,[a,\gamma]=\delta$	$p\neq 0$
E^{29}_{pq}	$[a,\alpha]=p\alpha,[a,\beta]=q\beta,[a,\gamma]=q\gamma+\delta,[a,\delta]=q\delta$	$p,q\neq 0,\ p\neq q$
E^{30}_{pq}	$[a,\alpha]=p\alpha+\beta,[a,\beta]=p\beta,[a,\gamma]=q\gamma+\delta,[a,\delta]=q\delta$	$p\neq q$
E^{31}_{pq}	$[a,\alpha]=p\alpha,[a,\beta]=p\beta,[a,\gamma]=q\gamma+\delta,[a,\delta]=q\delta$	$p\neq 0,\ p\neq q$
$(3,2)$		
$E^{35}_{\lambda\mu\nu}$	$[a,b]=2b,[a,c]=-2c,[b,c]=a;[a,\alpha]=(\lambda+2)\alpha,$ $[a,\beta]=\lambda\beta,\ [b,\beta]=\mu\alpha,[c,\alpha]=\nu\beta$	$\mu,\nu\neq 0$
E^{36}	$[a,b]=2b,[a,c]=-2c,[b,c]=a;[a,\alpha]=\alpha,[a,\beta]=-\beta,$ $[b,\beta]=\alpha,[c,\alpha]=\beta$	*full BRS algebra*
E^{37}	$[a,b]=b;[a,\alpha]=\alpha,[b,\beta]=\alpha,[c,\alpha]=\alpha,[c,\beta]=\beta$	
E^{38}_λ	$[a,b]=b;[a,\beta]=\lambda\beta,[c,\alpha]=\alpha,[c,\beta]=\lambda\beta$	$\lambda\neq 0$
$E^{39}_{\lambda\mu}$	$[a,b]=b;[a,\alpha]=\lambda\alpha,[a,\beta]=\lambda\beta,[c,\alpha]=\alpha+\mu\beta,[c,\beta]=\beta$	$\lambda,\mu\neq 0$
E^{40}_λ	$[a,b]=b;[a,\alpha]=\lambda\alpha,[a,\beta]=\lambda\beta,[c,\alpha]=\beta$	$\lambda\neq 0$
E^{41}_λ	$[a,b]=b;[a,\alpha]=(1-\lambda)\beta,[c,\alpha]=\alpha+\lambda\beta,[c,\beta]=\beta$	
$E^{42}_{p\lambda}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=(\lambda+1)\alpha,[a,\beta]=\lambda\beta,[b,\beta]=\alpha$	$0\leq\lvert p\rvert\leq 1,\lambda\in\mathcal{R}$

	Relations	**Comments**
$E^{43}_{p\lambda\mu}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=\lambda\alpha,[a,\beta]=\mu\beta$	$\lvert p\rvert\leq 1,\ p,\lambda,$ $\mu\neq 0$
$E^{44}_{p\lambda}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=\lambda\alpha,[a,\beta]=\alpha+\lambda\beta$	$0<\lvert p\rvert\leq 1,$ $\lambda\neq 0$
E^{45}_{p}	$[a,b]=b,[a,c]=pc;[a,\beta]=\alpha$	$0<\lvert p\rvert\leq 1$
E^{46}_{λ}	$[a,b]=b,[a,c]=b+c;[a,\alpha]=(\lambda+1)\alpha,[a,\beta]=\lambda\beta,[c,\beta]=\alpha$	$\lambda\in\mathcal{R}$
$E^{47}_{\lambda\mu}$	$[a,b]=b,[a,c]=b+c;[a,\alpha]=\lambda\alpha,[a,\beta]=\mu\beta$	$\lambda,\mu\neq 0$
E^{48}_{λ}	$[a,b]=b,[a,c]=b+c;[a,\alpha]=\lambda\alpha,[a,\beta]=\alpha+\lambda\beta$	$\lambda\neq 0$
E^{49}	$[a,b]=b,[a,c]=b+c;[a,\beta]=\alpha$	
E^{50}_{pq}	$[a,c]=pc,[b,c]=qc;[a,\alpha]=\alpha,[a,\beta]=\beta,[b,\beta]=\alpha$	$p,q\neq 0$
E^{51}_{pq}	$[a,c]=pc,[b,c]=qc;[a,\alpha]=\alpha,[a,\beta]=\beta,$ $[b,\alpha]=-\alpha,[b,\beta]=\beta$	$\lvert p\rvert+\lvert q\rvert\neq 0$
E^{52}_{pq}	$[a,c]=pc,[b,c]=qc;[a,\alpha]=\alpha,[a,\beta]=\beta,$ $[b,\alpha]=-\beta,[b,\beta]=\alpha$	$\lvert p\rvert+\lvert q\rvert\neq 0$
E^{53}	$[a,b]=c;[a,\alpha]=\alpha,[a,\beta]=\beta,[b,\beta]=\alpha$	
E^{54}	$[a,b]=c;[a,\alpha]=\alpha,[b,\beta]=\beta$	
$E^{55}_{\lambda\mu}$	$[a,b]=c;[a,\alpha]=\lambda\alpha,[a,\beta]=\mu\beta$	$\lambda,\mu\neq 0$
E^{56}_{λ}	$[a,b]=c;[a,\alpha]=\lambda\alpha,[a,\beta]=\alpha+\lambda\beta$	$\lambda\neq 0$
E^{57}	$[a,b]=c;[a,\beta]=\alpha$	
E^{58}_{λ}	$[a,b]=c;[a,\alpha]=\lambda\alpha+\beta,[a,\beta]=\lambda\beta,$ $[b,\alpha]=\beta,[c,\alpha]=\alpha,[c,\beta]=\beta$	$\lambda\neq 0$
E^{59}	$[a,b]=c;[a,\alpha]=\beta,[c,\alpha]=\alpha,[c,\beta]=\beta$	
$(2,3)$		
E^{68}_{pqr}	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=q\beta,[a,\gamma]=r\gamma$	$\lvert p\rvert\geq\lvert q\rvert$ $\geq\lvert r\rvert>0$
E^{69}_{pq}	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=(q+1)\beta,[a,\gamma]=q\gamma,[b,\gamma]=\beta$	$p\neq 0$
E^{70}_{p}	$[a,b]=b;[a,\alpha]=(p+2)\alpha,[a,\beta]=(p+1)\beta,[a,\gamma]=p\gamma,$ $[b,\beta]=\alpha,[b,\gamma]=\beta$	$p\in\mathcal{R}$
E^{71}_{pq}	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=\alpha+p\beta,[a,\gamma]=q\gamma$	$q\neq 0,p\in\mathcal{R}$
E^{72}_{p}	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=\alpha+p\beta,[a,\gamma]=(p-1)\gamma,[b,\gamma]=\alpha$	$p\in\mathcal{R}$
E^{73}_{p}	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=\alpha+p\beta,[a,\gamma]=(p+1)\gamma,[b,\beta]=\gamma$	$p\in\mathcal{R}$
E^{74}_{p}	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=\alpha+p\beta,[a,\gamma]=b+p\gamma$	$p\in\mathcal{R}$
E^{75}_{pqr}	$[a,\alpha]=\alpha,[a,\beta]=p\beta,[a,\gamma]=q\gamma,[b,\beta]=\beta,[b,\gamma]=r\gamma$	$1\geq\lvert p\rvert$ $\geq\lvert r\rvert>0$
E^{76}_{p}	$[a,\alpha]=\alpha,[a,\beta]=\beta,[a,\gamma]=p\gamma,[b,\beta]=\alpha,[b,\gamma]=\gamma$	$p\in\mathcal{R}$
E^{77}	$[a,\alpha]=\alpha,[a,\beta]=\beta,[a,\gamma]=\gamma,[b,\beta]=\alpha,[b,\gamma]=\beta$	
E^{78}_{pq}	$[a,\alpha]=\alpha,[a,\beta]=p\alpha+\beta,[a,\gamma]=q\gamma,[b,\beta]=\alpha,[b,\gamma]=\gamma$	$p\neq 0,q\in\mathcal{R}$
E^{79}_{pq}	$[a,\alpha]=\alpha,[a,\beta]=p\alpha+\beta,[a,\gamma]=q\gamma,[b,\alpha]=\gamma$	$p\neq 0,q\in\mathcal{R}$
E^{80}	$[a,\beta]=\alpha,[b,\alpha]=\gamma$	
E^{81}_{pq}	$[a,\alpha]=\alpha,[a,\beta]=p\alpha+\beta,[a,\gamma]=p\beta+\gamma,[b,\beta]=\alpha,[b,\gamma]=q\alpha+\beta$	$p,q\neq 0$
E^{82}_{p}	$[a,\alpha]=\alpha,[a,\beta]=p\alpha+\beta,[a,\gamma]=p\beta+\gamma,[b,\gamma]=\alpha$	$p\neq 0$
E^{83}	$[a,\beta]=\alpha,[a,\gamma]=\beta,[b,\gamma]=\beta$	

table II: NON TRIVIAL LS

	Relations	**Comments**
$(4,1)$		
E^1_{2qr}	$[d,a]=2a,[d,b]=qb,[d,c]=rc;[d,\alpha]=\alpha;[\alpha,\alpha]=a$	$q,r\neq 2$
$E^2_{p(2-p)}$	$[a,b]=c;[d,a]=pa,[d,b]=(2-p)b,[d,c]=2c;[d,\alpha]=\alpha;$ $[\alpha,\alpha]=c$	$p\neq 0,\ q\neq 2$
E^5_2	$[a,b]=c;[d,b]=2b,[d,c]=2c;[d,\alpha]=\alpha;[\alpha,\alpha]=c$	
E^8_2	$[d,a]=2a,[d,b]=a+2b,[d,c]=b+2c;[d,\alpha]=\alpha;$ $[\alpha,\alpha]=a$	
E^{12}_{2q}	$[d,a]=2a,[d,b]=a+2b,[d,c]=qc;[d,\alpha]=\alpha;[\alpha,\alpha]=c$	$q\neq 0,2$
E^{12}_{p2}	$[d,a]=pa,[d,b]=a+pb,[d,c]=2c;[d,\alpha]=\alpha;[\alpha,\alpha]=c$	$p\neq 0,2$
$(E^{13}_1)^1$	$[a,b]=c;[d,a]=a,[d,b]=a+b,[d,c]=2c;[d,\alpha]=\alpha;$ $[\alpha,\alpha]=c$	
$(A_{4,3}+A)$	$[d,a]=a,[d,c]=b;[\alpha,\alpha]=b$	
$(A_{4,1}+A)$	$[d,b]=a,[d,c]=b;[\alpha,\alpha]=a$	
$(A_{4,8}+A)^1$	$[a,b]=c;[d,a]=a,[d,b]=-b;[\alpha,\alpha]=c$	
$(1,4)$		
$(A_{1,1}+4A)^1$	$[\alpha,\alpha]=a,[\beta,\beta]=a,[\gamma,\gamma]=a,[\delta,\delta]=a$	
$(3,2)$		
E^{36}	$[a,b]=b,[a,c]=-c,[b,c]=2a;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=-\frac{1}{2}\beta,$ $[b,\beta]=\frac{1}{2}\alpha,[c,\alpha]=\frac{1}{2}\beta;[\alpha,\alpha]=b,[\beta,\beta]=-c,[\alpha,\beta]=-a$	*di*-spin algebra *or* $osp(1,2)$
D^{10}_0+A	$[a,b]=b;[a,\alpha]=\alpha,[b,\beta]=\alpha;[\beta,\beta]=c$	
$D^{10}_{\frac{1}{2}}+A$	$[a,b]=b;[a,\alpha]=\frac{3}{2}\alpha,[a,\beta]=\frac{1}{2}\beta,[b,\beta]=\alpha;[\beta,\beta]=c$	
$E^{42}_{p\frac{p}{2}}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=(\frac{p}{2}+1)\alpha,[a,\beta]=\frac{p}{2}\beta,$ $[b,\beta]=\alpha;[\beta,\beta]=c$	$0\leq\lvert p\rvert\leq 1$
$(E^{43}_{p\frac{1}{2}\frac{1}{2}})^1$	$[a,b]=b,[a,c]=pc;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\frac{1}{2}\beta;[\alpha,\alpha]=b$	$p\neq 0$
$(E^{43}_{p\frac{1}{2}\frac{1}{2}})^2$	$[a,b]=b,[a,c]=pc;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\frac{1}{2}\beta;$ $[\alpha,\alpha]=b,[\beta,\beta]=b$	$p\neq 0$
$E^{43}_{p\frac{1}{2}\frac{p}{2}}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\frac{p}{2}\beta;$ $[\alpha,\alpha]=b,[\beta,\beta]=c$	$p\neq 0$
$E^{43}_{p\frac{1}{2},p-\frac{1}{2}}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=(p-\frac{1}{2})\beta;$ $[\alpha,\alpha]=b,[\alpha,\beta]=c$	
$E^{43}_{1\frac{1}{2}r}$	$[a,b]=b,[a,c]=c;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=r\beta;[\alpha,\alpha]=b$	$r\neq 0,\frac{1}{2}$
$E^{43}_{1q,1-q}$	$[a,b]=b,[a,c]=c;[a,\alpha]=q\alpha,[a,\beta]=(1-q)\beta;[\alpha,\beta]=b$	$q\neq\frac{1}{2}$
$E^{44}_{p\frac{1}{2}}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta;[\beta,\beta]=b$	$p\neq 0$

	Relations	**Comments**
$E^{46}_{\frac{1}{2}}$	$[a,b]=b,[a,c]=b+c;[a,\alpha]=\frac{3}{2}\alpha,[a,\beta]=\frac{1}{2}\beta,[c,\beta]=\alpha;$ $[\beta,\beta]=b$	
$(E^{47}_{\frac{1}{2}\frac{1}{2}})^1$	$[a,b]=b,[a,c]=b+c;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\frac{1}{2}\beta;$ $[\alpha,\alpha]=b,[\beta,\beta]=b$	
$E^{47}_{\frac{1}{2}r}$	$[a,b]=b,[a,c]=b+c;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=r\beta;[\alpha,\alpha]=b$	$r\neq 0$
$(E^{48}_{\frac{1}{2}})^1$	$[a,b]=b,[a,c]=b+c;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta;[\beta,\beta]=b$	
$(E^{48}_{\frac{1}{2}})^2$	$[a,b]=b,[a,c]=b+c;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta;$ $[\alpha,\beta]=\frac{1}{2}b,[\beta,\beta]=c$	
E^{50}_{20}	$[a,c]=2c;[a,\alpha]=\alpha,[a,\beta]=\beta,[b,\beta]=\alpha;[\beta,\beta]=c$	
E^{51}_{22}	$[a,c]=2c,[b,c]=2c;[a,\alpha]=\alpha,[a,\beta]=\beta,$ $[b,\alpha]=-\alpha,[b,\beta]=\beta;[\beta,\beta]=c$	
E^{51}_{20}	$[a,c]=2c;[a,\alpha]=\alpha,[a,\beta]=\beta,[b,\alpha]=-\alpha,[b,\beta]=\beta;$ $[\alpha,\beta]=c$	
$E^{55}_{1,-1}$	$[a,b]=c;[a,\alpha]=\alpha,[a,\beta]=-\beta;[\alpha,\beta]=c$	
E^{55}_{10}	$[a,b]=c;[a,\alpha]=\alpha;[\beta,\beta]=c$	
$(E^{55}_{00})^1$	$[a,b]=c;[\alpha,\alpha]=c,[\beta,\beta]=c$	Heisenberg LS
E^{57}_{00}	$[a,b]=c;[a,\beta]=\alpha;[\beta,\beta]=c$	
$3A_{1,1}+2A$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\alpha,\beta]=c$	
$(2,3)$		
$E^{68}_{\frac{1}{2}qr}$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=q\beta,[a,\gamma]=r\gamma;[\alpha,\alpha]=b$	$q,r\neq 0$
$E^{68}_{\frac{1}{2}\frac{1}{2}r}$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\frac{1}{2}\beta,[a,\gamma]=r\gamma;$ $[\alpha,\alpha]=b,[\beta,\beta]=b$	$r\neq 0$
$E^{68}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\frac{1}{2}\beta,[a,\gamma]=\frac{1}{2}\gamma;$ $[\alpha,\alpha]=b,[\beta,\beta]=b,[\gamma,\gamma]=b$	
$E^{68}_{\frac{1}{2}q,1-q}$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=q\beta,[a,\gamma]=(1-q)\gamma;$ $[\alpha,\alpha]=b,[\beta,\gamma]=b$	$q\neq 0$
$(E^{69}_{00})^1$	$[a,b]=b;[a,\beta]=\beta,[b,\gamma]=\beta;$ $[\gamma,\gamma]=-2a,[\beta,\gamma]=b,[\alpha,\gamma]=a,[\alpha,\beta]=-b$	
$(E^{69}_{00})^2$	$[a,b]=b;[a,\beta]=\beta,[b,\gamma]=\beta;[\alpha,\gamma]=a,[\alpha,\beta]=-b$	
$E^{71}_{p,1-p}$	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=\alpha+p\beta,[a,\gamma]=(1-p)\gamma,$ $[\beta,\gamma]=b$	$p\in\mathcal{R}$
$E^{71}_{p\frac{1}{2}}$	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=\alpha+p\beta,[a,\gamma]=\frac{1}{2}\gamma,[\gamma,\gamma]=b$	$p\in\mathcal{R}$
$(E^{71}_{\frac{1}{2}\frac{1}{2}})^1$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\frac{1}{2}\gamma,[\beta,\beta]=b$	
$(E^{71}_{\frac{1}{2}\frac{1}{2}})^2$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\frac{1}{2}\gamma;$ $[\beta,\beta]=b,[\gamma,\gamma]=b$	

	Relations	Comments
$(E^{74}_{\frac{1}{2}})^1$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\beta+\frac{1}{2}\gamma;$ $[\beta,\beta]=b,[\alpha,\gamma]=-b,[\gamma,\gamma]=\kappa b$	$\kappa\neq 0$
$(E^{74}_{\frac{1}{2}})^2$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\beta+\frac{1}{2}\gamma;$ $[\beta,\beta]=b,[\alpha,\gamma]=-b$	
$(E^{74}_{\frac{1}{2}})^3$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\beta+\frac{1}{2}\gamma;$ $[\gamma,\gamma]=b$	
$D^{11}_{-1,0}+A$	$[a,\alpha]=\alpha,[a,\beta]=-\beta;[\alpha,\beta]=b,[\gamma,\gamma]=b$	
$D^{11}_{-1,q}+A$	$[a,\alpha]=\alpha,[a,\beta]=-\beta,[a,\gamma]=q\gamma;[\alpha,\beta]=b$	$q\neq 0,\pm 1$
$(C^3+A)^1$	$[a,\beta]=\alpha;[\beta,\beta]=b,[\gamma,\gamma]=b$	
$(D^{15}+A_{1,1})^1$	$[a,\beta]=\alpha,[a,\gamma]=\beta;[\gamma,\gamma]=b$	
$(D^{15}+A_{1,1})^2$	$[a,\beta]=\alpha,[a,\gamma]=\beta;[\beta,\beta]=b,[\alpha,\gamma]=-b$	
$(D^{15}+A_{1,1})^3$	$[a,\beta]=\alpha,[a,\gamma]=\beta;[\beta,\beta]=b,[\gamma,\gamma]=b,[\alpha,\gamma]=-b$	
$(2A_{1,1}+3A)^1$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\gamma,\gamma]=a+b$	
$(2A_{1,1}+3A)^4$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\alpha,\gamma]=a-b$	
$(2A_{1,1}+3A)^5$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\alpha,\gamma]=a-b,[\beta,\gamma]=\lambda(a-b)$	$\lambda\neq 0$
$(2A_{1,1}+3A)^6$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\gamma,\gamma]=a-b,[\alpha,\gamma]=\kappa(a-b),$ $[\beta,\gamma]=\lambda(a-b)$	$\kappa,\lambda\neq 0$
$(2A_{1,1}+3A)^7$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\gamma,\gamma]=a-b,[\alpha,\beta]=\lambda(a-b)$	$\lambda\neq 0$
$(2A_{1,1}+3A)^8$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\alpha,\gamma]=b$	

Remark 3.1. *The reader may notice the discontinuity in the numbering of the LS in table I. This is due to the presence of real LS in each (m,n)-dimension ($m,n=1,..4$), which of course cannot be included in this classification.*

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