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On the Superconvergent Spline Collocation Methods for the Fredholm Integral Equations on Surfaces

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In this paper a special collocation method arising from certain choices of the collocation node points and certain types of triangulation, for the Fredholm integral equations on surfaces is presented. The approximating solution is a special polynomial spline function of two variables on the surfaces and the proposed approximating method has the superconvergent properties.

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1. Introduction

Integral equations are an important subject within applied mathematics. They are used as mathematical models for many and diverse phenomena and processes. Also, integral equations occur as reformulations of other mathematical problems, such as partial differential equations with boundary conditions. Such reformulations were leading to the very efficient approximating approach known as boundary element method.

In this paper, a numerical spline collocation method is presented and analyzed for the solution of Fredholm integral equations of the second kind of the form

$$u(P) - \int_S u(Q)K(P,Q)dS_Q = f(P), \quad P \in S \tag{1}$$

for a smooth or piecewise smooth bounded surface S in $\mathbb{R}^3$, with kernel function $K(P,Q)$ to be absolutely integrable and to satisfy other properties which are sufficient to imply the *Fredholm Alternative Theorem* (see Atkinson [2]) and f continuos function u being the unknown solution.

In operator form this equation is

$$(\mathcal{I} - \mathcal{K})u = f \tag{2}$$

We investigate a certain type of collocation method based on piecewise polynomial spline interpolation of the solution. Begining by triangulating S we shall approximate the unknown function $u(P)$ by polynomial spline functions in a parametrization over the triangulation of S. The approximate form of the solution is substituted into the equation and then the equation is forced to be true at the collocation node points, leading to a system of linear equations for determining the approximate solution.

When the surface S is smooth and the operator $\mathcal{K}$ is compact on $C(S)$ (see Atkinson [4] and Micula [8]), it is relatively easy to do an error analysis of collocation. However, in most applications the surface S will only be piecewise smooth, and in this case the analysis of collocation is more difficult. Also, a lack of smoothness of the kernel function $K(P, Q)$ may imply the $\mathcal{K}$ is no longer compact, nor that any power of it is compact.

Another difficulty in the case where S is not smooth arises in the evaluation of the unit normal to the surface at points located on an edge or at a corner of S. Also, there is a problem in defining the normal at the collocation points which are common to more than one triangular face Δ_k, even for smooth surfaces. To avoid all these problems, we consider only collocation methods for which the collocation points are interior to each triangular face. This also greatly simplifies the programming.

For some approximations of the solution, the function space needs to be changed, namely $C(S)$ must be enlarged to include piecewise polynomial approximants. One way of doing this is by using the space $L^\infty(S)$, the set of all essentially bounded and Lebesgue measurable functions on S, with the essential supremum norm $\|\cdot\|_\infty$.

We shall investigate special collocation method arising from certain choices of the node points, and certain types of triangulation, which leads to superconvergence for some collocation solution u_n at the collocation nodes.

2. Interpolation, numerical integration on surfaces and collocation

We begin by giving some background knowledges of bivariate interpolation theory needed in the description of collocation methods (see Atkinson [1], [2]).

Let σ denote the unit simplex in $\mathbb{R}^2$, $\sigma := \{(s,t) \mid 0 \le s, t, s+t \le 1\}$. Introduce $u = 1-s-t$. The coordinates (s,t,u) are called *barycentric coordinates* on σ.

Let $g \in C(\sigma)$. We will approximate g by a polynomial interpolant p_r :

$\sigma \to \mathbb{R}$, of degree r,

$$p_r(s,t) = \sum_{\substack{i,j\geq 0\\ i+j\leq r}} c_{i,j} s^i t^j \tag{3}$$

Since p_r has $f_r = (r+1)(r+2)/2$ (degrees of freedom), we will determine the coefficients $c_{i,j}$ from f_r interpolation conditions:

$$p_r(q_k) = g(q_k), \quad k = 1, ..., f_r \tag{4}$$

where the interpolation nodes will be chosen in the following way:

$$q_{i,j} = \left(\frac{i+(r-3i)\alpha}{r}, \frac{j+(r-3j)\alpha}{r}\right), \quad i,j \geq 0, \quad i+j \leq r \tag{5}$$

These f_r nodes form a uniform grid over σ, and because we consider only nodes that are interior to the triangular elements, we will work with $0 < \alpha \leq \frac{1}{3}$. For the given $g \in C(\sigma)$, the formula

$$p_r(s,t) = \sum_{0\leq i+j\leq r} g(q_{i,j}) l_{i,j}(s,t) \tag{6}$$

define the unique polynomial of degree r that interpolates g at the nodes $q_{i,j}$, where $l_{i,j}$ denote the corresponding Lagrange interpolation bases functions of degree r obtained from the conditions $l_{i,j}(q_{i,j}) = 1$, $l_{i,j}(q_{l,k}) = 0$, for $l \neq i$ or $k \neq j$.

Integrating the interpolation formula

$$g(s,t) \approx \sum_{i+j\leq r} g(q_{i,j}) l_{i,j}(s,t) \tag{7}$$

over σ, we obtain the quadrature formula

$$\int_\sigma g(s,t) d\sigma \approx \sum_{0\leq i+j\leq r} \omega_{i,j} g(q_{i,j}) \tag{8}$$

where $\omega_{i,j} = \int_\sigma l_{i,j}(s,t) d\sigma$, the formula (8) has degree of precision at least r.

To construct the interpolation and numerical integration on surface S, we assume S to be a connected piecewise smooth surface in $\mathbb{R}^3$. By this, we mean S can be written as

$$S = S_1 \cup S_2 \cup ... \cup S_J \tag{9}$$

with each S_j the continuous image of a polygonal region in the plane

$$F_j : R_j \xrightarrow[onto]{1-1} S_j, \quad j = 1, ..., J \tag{10}$$

Generally, we will need to assume that the mappings F_j are several times continuously differentiable.

To create triangulations for S, we first triangulate each R_j and then map this triangulation onto S_j.

Let $\{\widehat{\Delta}^j_{n,k} \mid k = 1, ..., n_j\}$ be a triangulation of R_j, and then define

$$\Delta^j_{n,k} = F_j(\widehat{\Delta}^j_{n,k})$$

This yields a triangulation of S, which we refer to collectively as $\mathcal{T}_n = \{\Delta_1, ..., \Delta_n\}$ and we suppose that $\mathcal{T}_n$ is a conforming triangulation (see Micula [8], p.15).

Let Δ_k be some element from $\mathcal{T}_n$, and let it correspond to some $\widehat{\Delta}_k$, say $\widehat{\Delta}_k \subset R_j$ and $\Delta_k = F_j(\widehat{\Delta}_k)$. Let $\{\hat{v}_{k,1}, \hat{v}_{k,2}, \hat{v}_{k,3}\}$ denote the vertices of $\widehat{\Delta}_k$. Define $m_k : \sigma \xrightarrow[onto]{1-1} \Delta_k$ by

$$m_k(s,t) = F_j(u\hat{v}_{k,1} + t\hat{v}_{k,2} + s\hat{v}_{k,3}), \quad (s,t) \in \sigma, \quad u = 1 - s - t \tag{11}$$

Now we can define interpolation and numerical integration over a triangular surface element Δ_k by means of a similar formula over σ. Recall the uniform grid over σ defined in (5), which we refer to collectively as $\{q_1, ..., q_r\}$. For $g \in C(S)$, restrict g to some $\Delta \in \mathcal{T}_n$ and define

$$(\mathcal{P}_n g)(m_k(s,t)) = \sum_{i=1}^{f_r} g(m_k(q_i)) l_i(s,t), \quad P := m_k(s,t) \in \Delta_k \tag{12}$$

Now we present the general framework for the collocation and iterated collocation methods for the equation (2).

Let X be a Banach space, let $\{X_m \mid m \geq 1\}$ be a sequence of finite dimensional subspaces. Let $\mathcal{P}_m : X \longrightarrow X_m$ be a linear operator with

$$\mathcal{P}_m u = u, \quad u \in X_m \tag{13}$$

In attempting to solve the problem (2), we will approximate it by solving

$$\mathcal{P}_m(\mathcal{I} - \mathcal{K})u_m = \mathcal{P}_m f, \quad u_m \in X_m \tag{14}$$

This is the form in which the method is implemented as it leads directly to equivalent finite linear systems. To make an error analysis, we rewrite (14) in the equivalent form

$$(\mathcal{I} - \mathcal{P}_m\mathcal{K})u_m = \mathcal{P}_m f, \quad u_m \in X \tag{15}$$

We have the following result:

Theorem 1. *[8, p.16] Let X be a Banach space, $\mathcal{K} : X \longrightarrow X$ a bounded operator with $\mathcal{I} - \mathcal{K} : X \xrightarrow[onto]{1-1} X$. Assume that*

$$\|\mathcal{K} - \mathcal{P}_m\mathcal{K}\| \longrightarrow 0 \text{ as } m \longrightarrow \infty \tag{16}$$

Then for all sufficiently large m, say $m \geq N$, the operator $(\mathcal{I} - \mathcal{P}_m\mathcal{K})^{-1}$ exists as a bounded operator from X to X. Moreover, it is uniformly bounded

$$\sup_{m \geq N} \|(\mathcal{I} - \mathcal{P}_m\mathcal{K})^{-1}\| < \infty. \tag{17}$$

For the solutions of (2) *and* (15)

$$u - u_m = (\mathcal{I} - \mathcal{P}_m\mathcal{K})^{-1}(u - \mathcal{P}_m u) \tag{18}$$

and

$$\frac{1}{\|\mathcal{I} - \mathcal{P}_m\mathcal{K}\|}\|u - \mathcal{P}_m u\| \leq \|u - u_m\| \leq \|(\mathcal{I} - \mathcal{P}_m\mathcal{K})^{-1}\| \cdot \|u - \mathcal{P}_m u\| \tag{19}$$

This leads to $\|u - u_m\|$ converging to zero at exactly the same speed as $\|u - \mathcal{P}_m u\|$.

Of course, because $\mathcal{P}_m u \longrightarrow u$ as $m \longrightarrow \infty$, $u \in X$ and $\mathcal{K} : X \longrightarrow X$ is compact, it follows

$$\|\mathcal{K} - \mathcal{P}_m\mathcal{K}\| \longrightarrow 0 \text{ as } m \longrightarrow \infty \tag{20}$$

For the iterated collocation method, consider the iteration

$$u^{(k+1)} = f + \mathcal{K}u^{(k)}, \quad k = 0, 1, ... \tag{21}$$

If u_m is the solution of the collocation equation (15), define the iterated collocation solution by

$$\hat{u}_m = f + \mathcal{K}u_m \tag{22}$$

Then

$$\mathcal{P}_m\hat{u}_m = \mathcal{P}_m f + \mathcal{P}_m\mathcal{K}u_m = u_m \tag{23}$$

and

$$(\mathcal{I} - \mathcal{K}\mathcal{P}_m)\hat{u}_m = f \tag{24}$$

Combining (15) and (24), we obtain $u - \hat{u}_m = \mathcal{K}(u - u_m)$ and

$$\|u - \hat{u}_m\| \leq \|\mathcal{K}\| \cdot \|u - u_m\| \tag{25}$$

which proves that the convergence of $\hat{u}_m$ to u is at least as rapid as that of u_m to u.

Also, we see that $(\mathcal{I}-\mathcal{P}_m\mathcal{K})^{-1}$ exists if $(\mathcal{I}-\mathcal{K}\mathcal{P}_m)^{-1}$ exists.

In order to solve the equation (2) using a collocation method based on a piecewise polynomial spline interpolation operator (12) we need to use the enlarged space $L^\infty(S)$ which include piecewise polynomial spline approximation $\mathcal{P}_n g$.

The following theorem holds:

Theorem 2. [8, p.20] *Assume S is a smooth surface in $\mathbb{R}^3$ satisfying (9) and (10) with each $F_j \in C^{r+2}$. Assume that equation (1) is uniquely solvable for all functions $f \in C(S)$. Assume $\mathcal{K} : L^\infty(S) \longrightarrow C(S)$ is compact and $u \in C^{r+1}(S)$. Then for all sufficiently large n, say $n \geq n_0$, the operators $\mathcal{I}-\mathcal{P}_n\mathcal{K}$ are invertible on $C(S)$ and have uniformly bounded inverses. Moreover, for the true solution u of (1) and the solution u_n of (15)*

$$\|u-u_n\|_\infty \leq \left\|(\mathcal{I}-\mathcal{P}_n\mathcal{K})^{-1}\right\| \cdot \|u-\mathcal{P}_n u\|_\infty \tag{26}$$

Furthermore, if $f \in C^{r+1}(S)$, then

$$\|u-u_n\|_\infty \leq O(h^{r+1}), \quad n \geq n_0 \tag{27}$$

Remarks.

1. *This theorem can be easly generalized to piecewise smooth surfaces.*

2. *It is clear that the accurasy of collocation method based on piecewise polynomial spline interpolation depends on the degree of precision of the interpolation formula.*

3. Superconvergent collocation methods

Consider the integral equation (1).

Let $\mathcal{T}_n = \{\Delta_1, ..., \Delta_k\}$ be a triangulation of S and $m_k : \sigma \longrightarrow \Delta_k$ be defined as in (11). Recall the interpolation formula

$$g(s,t) \approx \sum_{j=1}^{f_r} g(q_j) l_j(s,t), \quad g \in C(S) \tag{28}$$

Let

$$\mathcal{P}_n g(m_k(s,t)) = \sum_{j=1}^{f_r} g(m_k(q_j)) l_j(s,t), \quad P = m_k(s,t) \in \Delta_k \tag{29}$$

with the nodes $\{q_1, ..., q_{f_r}\}$ and $\{l_1, ..., l_{f_r}\}$ given by (5) and (6).

Define a collocation method using (29). Substitute

$$\begin{aligned} u_n(P) &= \sum_{j=1}^{f_r} u_n(v_{k,j}) l_j(s,t), \quad P \in m_k(s,t) \in \Delta_k \\ v_{k,j} &= m_k(q_j), \quad k = 1, ..., n \end{aligned} \tag{30}$$

into (1). This leads to the linear system

$$\begin{aligned} u_n(v_i) \quad - &\sum_{k=1}^{n} \sum_{j=1}^{f_r} u_n(v_{k,j}) \int_{\sigma} K(v_i, m_k(s,t)) l_j(s,t) \cdot \\ &|(D_S m_k \times D_t m_k)(s,t)| d\sigma = f(v_i), \quad i = 1, ..., n f_r \end{aligned} \tag{31}$$

We have shown in Theorem 2. that under suitable assumptions this method has the error

$$\|u - u_n\|_\infty \leq O(h^{r+1}) \tag{32}$$

where h is the mesh size of the triangulation $\mathcal{T}_n$. Sometimes at the collocation node points, the collocation method converges more rapidly than over all S, in which case

$$\lim_{n \to \infty} \frac{\max\limits_{1 \leq i \leq n f_r} |u(v_i) - \hat{u}_n(v_i)|}{\|u - u_n\|_\infty} = 0 \tag{33}$$

Such methods are *superconvergent* at the collocation node points.

Let us examine more carefully the terms in (33). For simplicity, we work with the solution $\hat{u}_n$ of the iterated collocation equation (24). This should cause no problems, since we know that the convergence of $\hat{u}_n$ to u is at least as rapid as that of the solution of the collocation equation (15) to u, and the inverses for the collocation equation and iterated collocation equation are related by the identities

$$\begin{aligned} (\mathcal{I} - \mathcal{K}\mathcal{P}_n)^{-1} &= \mathcal{I} + \mathcal{K}(\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1}\mathcal{P}_n \\ (\mathcal{I} - \mathcal{P}_n\mathcal{K})^{-1} &= \mathcal{I} + \mathcal{P}_n(\mathcal{I} - \mathcal{K}\mathcal{P}_n)^{-1}\mathcal{K} \end{aligned} \tag{34}$$

Moreover, $\hat{u}(v_i) = u_n(v_i)$ at all collocation nodes.

By looking at the linear system associated with

$$(\mathcal{I} - \mathcal{K}\mathcal{P}_n)(u - \hat{u}_n) = \mathcal{K}(u - \mathcal{P}_n u) \tag{35}$$

we have

$$\max_{1 \leq i \leq n f_r} |u(v_i) - \hat{u}_n(v_i)| \leq c \max_{1 \leq i \leq n f_r} |\mathcal{K}(\mathcal{I} - \mathcal{P}_n) u(v_i)| \tag{36}$$

(see [4, p. 449]). So, now we can focus on finding errors for $\mathcal{K}(\mathcal{I}-\mathcal{P}_n)u(v_i)$.

First, we need some assumptions for the interpolation over σ. Recall that for $g \in C(\sigma)$, we are considering interpolation of degree r over σ:

$$g(s,t) \approx (\mathcal{L}_\sigma g)(s,t) \equiv \sum_{j=1}^{f_r} g(q_j) l_j(s,t) \tag{37}$$

This leads to the numerical integration formula

$$\int_\sigma g(s,t)d\sigma \approx \int_\sigma \mathcal{L}_\sigma g(s,t)d\sigma \tag{38}$$

which has a degree of precision of at least r. Assume there is a value $0 < \alpha_0 < \frac{1}{3}$ such that for q_j and l_j defined with $\alpha = \alpha_0$, the formula (38) is exact for all polynomials in σ_1, σ_2, where $\sigma_1 = s+t-s^2-st-t^2$, $\sigma_2 = st-s^2t-st^2$, of degree $\leq r+1$, i. e. has degree of precision $r+1$. For the remainder of this section, we will assume $\alpha = \alpha_0$.

Now, let $\tau \subset \mathbb{R}^2$ be a planar triangle with vertices $\{v_1, v_2, v_3\}$ and define the mapping $m_\tau : \sigma \longrightarrow \tau$ as in (11). For $g \in C(\tau)$, define

$$\mathcal{L}_\tau g(x,y) = \sum_{j=1}^{f_r} g(m_\tau(q_j)) l_j(s,t) \tag{39}$$

which is a polynomial of degree r in the parametrization variables s and t, interpolating g at the nodes $\{m_\tau(q_1), ..., m_\tau(q_{f_r})\}$.

Define a numerical integration formula over τ by

$$\int_\tau g(x,y)d\tau \approx \int_\tau \mathcal{L}_\tau g(x,y)d\tau \tag{40}$$

By our earlier assumption on α_0, this has degree of precision at least $r+1$. In what follows, for differentiable functions g, we will use the notation

$$|D^k g(x,y)| = \max_{0\leq i\leq k} \left| \frac{\partial^k g(x,y)}{\partial x^i \partial y^{k-i}} \right| \tag{41}$$

We have the following result.

Lemma 1. *Let τ be a planar right triangle and assume the two sides which form the right angle have length h. Assume $\alpha = \alpha_0$. Let $g \in C^{r+2}(\tau), \Phi \in C^1(\tau)$. Then*

$$\left| \int_\tau \Phi(x,y)(\mathcal{I}-\mathcal{L}_\tau)g(x,y)d\tau \right| \leq ch^{r+2} \left[\int_\tau (|\Phi| + |D\Phi|)d\tau \right] \max_\tau \{|D^{r+1}g|, |D^{r+2}g|\} \tag{42}$$

where c denotes a generic constant.

P r o o f. Let $p_i(x,y)$ denote Taylor expansions of g around a suitable point in τ, of degree i, for $i = r,\ r+1$. Then, since $g \in C^{r+2}(\tau)$, we have that

$$\|g - p_i\|_\infty \le ch^{i+1}\|D^{i+1}g\|_\infty, \quad i = r, r+1 \tag{43}$$

with $\|\cdot\|_\infty$ denoting the uniform norm on $C(\tau)$. From (43) it follows that

$$\begin{aligned}\|p_{r+1} - p_r\|_\infty &\le \|g - p_{r+1}\|_\infty + \|g - p_r\|_\infty \le ch^{r+2}\|D^{r+2}g\|_\infty + ch^{r+2}\|D^{r+2}g\|_\infty \\ &+ch^{r+1}\|D^k g\|_\infty = ch^{r+1}\left(h\|D^{r+2}g\|_\infty + \|D^{r+1}g\|_\infty\right)\end{aligned} \tag{44}$$

Since $\Phi \in C^1(\tau)$, there is a constant Φ_0 such that

$$\|\Phi - \Phi_0\|_\infty \le ch\,|D\Phi| \tag{45}$$

To shorten the notation, let $\mathcal{L}_\tau^{'} = \mathcal{I} - \mathcal{L}_\tau$. We can write

$$\int_\tau \Phi\mathcal{L}_\tau^{'} g d\tau = \int_\tau \Phi\mathcal{L}_\tau^{'}(g - p_{r+1})d\tau + \int_\tau (\Phi - \Phi_0)\mathcal{L}_\tau^{'}(p_{r+1} - p_r)d\tau \tag{46}$$

To see why (46) is true, note first that

$$\mathcal{L}_\tau^{'} p_r = 0 \tag{47}$$

since formula (11) has degree of precision r. Also, by our assumption that for $\alpha = \alpha_0$, formula (38) has degree of precision $r+1$, we have that

$$\int_\tau \Phi_0\mathcal{L}_\tau^{'} p_{r+1} d\tau = 0 \tag{48}$$

Then, using these facts, (46) follows from expanding the right side and simplifying. Taking norms in (46) and using the bounds in (43), (44), and (45), we have

$$\begin{aligned}\left|\int_\tau \Phi\mathcal{L}_\tau^{'} g d\tau\right| \le\ & ch^{r+2}\|\mathcal{L}_\tau^{'}\| \cdot \int_\tau |\Phi| d\tau + ch\|\mathcal{L}_\tau^{'}\| \cdot ch^{r+1} \cdot \\ & \Big(h\|D^{r+2}g\|_\infty + \|D^{r+1}g\|_\infty\Big) \cdot \int_\tau |D\Phi| d\tau\end{aligned} \tag{49}$$

The term on the right of (49) is bounded by

$$ch^{r+2}\left[\int_\tau (|\Phi| + |D\Phi|)d\tau\right] \cdot \max_\tau \Big\{|D^{r+1}g|, |D^{r+2}g|\Big\}$$

which proves (42). ∎

This result can be extended to general triangles, but then the derivatives of g and Φ will involve the mapping m_τ from (11). Let $h(\tau)$ denote the diameter of τ and $h^*(\tau)$ the radius of the circle inscribed in τ and tangent to its sides. Define

$$r(\tau) = \frac{h(\tau)}{h^*(\tau)} \tag{50}$$

Assume that for our triangulations $\mathcal{T}_n = \{\Delta_{n,k}\}$, $n \geq 1$, we have

$$\sup_n \left[\max_{\Delta_{n,k} \in \mathcal{T}_n} r(\Delta_{n,k}) \right] < \infty \tag{51}$$

Condition (51) prevents the triangles $\Delta_{n,k}$ from having angles which approach 0 as $n \longrightarrow \infty$. Then, Lemma 1. can be generalized to arbitrary triangles as follows

Corollary 1. *Let τ be a planar triangle of diameter h, let $g \in C^{r+2}(\tau)$ and $\Phi \in C^1(\tau)$. Assume $\alpha = \alpha_0$. Then*

$$\left| \int_\tau \Phi(x,y)(\mathcal{I} - \mathcal{L}_\tau) g(x,y) d\tau \right| \leq c(r(\tau)) h^{r+2} \left[\int_\tau (|\Phi| + |D\Phi|) d\tau \right] \max_\tau \{|D^{r+1} g|, |D^{r+2} g|\} \tag{52}$$

where $c(r(\tau))$ is some function of $r(\tau)$, with $r(\tau)$ from (50).

Proof. Let $\bar{\tau}$ be a right triangle. Then using a mapping of the form (11) , $m_{\bar{\tau}} : \bar{\tau} \longrightarrow \tau$, we can write

$$\int_\tau \Phi(x,y)(\mathcal{I} - \mathcal{L}_\tau) g(x,y) d\tau = |(D_s m_k \times D_t m_k)| \int_{\bar{\tau}} \Phi(m_{\bar{\tau}}(s,t))(\mathcal{I} - \mathcal{L}_\tau) g(m_{\bar{\tau}}(s,t)) d\bar{\tau} \tag{53}$$

which shows that this case can be reduced to the case where τ is a right triangle whose two sides which form the right angle have length h, keeping in mind that the derivatives of Φ and g will depend on $r(\tau)$. Note that in this case $D_s m_k \times D_t m_k$ is a constant. ∎

We want to apply the above results to the individual subintegrals in

$$\mathcal{K}u(v_i) = \sum_{k=1}^{n} \int_\sigma K(v_i, m_k(s,t)) u(m_k(s,t)) \left| (D_s m_k \times D_t m_k)(s,t) \right| d\tau \tag{54}$$

Let

$$\begin{aligned} g(x,y) &= u(m_k(s,t)) \left| (D_s m_k \times D_t m_k)(s,t) \right| \\ \Phi(x,y) &= K(v_i, m_k(s,t)) \end{aligned} \tag{55}$$

Then, with the definition of $\mathcal{L}_\tau$ given in (39), the term in the right side of (36), $|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)u(v_i)|$ can be bounded by

$$\sum_{k=1}^{n}\left|\int_{\Delta_k}\Phi(x,y)(\mathcal{I}-\mathcal{L}_\tau)g(x,y)d\tau\right| \tag{56}$$

In the following, by $g\in C^k(S)$ we mean $g\in C(S)$ and $g\in C^k(S_j)$ (i.e $g\circ F_j\in C^k(R_j)$), $j=1,...,J$, for R_j and F_j as in (9) and (10).

Theorem 3. *Assume the hypotheses of Theorem* 2. *with each parametrization function* $F_j\in C^{r+3}$, *assume* $u\in C^{r+2}(S)$ *and* $K\in C^1(S)$ *with respect to* Q. *Assume the triangulation* $\mathcal{T}_n$ *of* S *satisfies* (51). *Then*

$$\max_{1\le i\le nf_r}|u(v_i)-\hat{u}_n(v_i)|\le ch^{r+2} \tag{57}$$

Proof. Following (36), we will bound

$$\max_{1\le i\le nf_r}|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)u(v_i)|$$

using (56). On each triangle Δ_k, apply Lemma 1. or Corollary 1.. ($c(r(\tau))$ of Corollary 1. will be denoted c to simplify the notation.) Since $u\in C^{r+2}(S)$ and $K\in C^1(S)$ with respect to Q, we have that

$$|D_QK|,\left|D^iu\right|,\quad i=r+1,r+2 \tag{58}$$

are bounded.

Then, by (56)

$$\max_{1\le i\le nf_r}|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)u(v_i)|\le\sum_{k=1}^{n}ch^{r+2}\int_{\Delta_k}d\tau \tag{59}$$

Since there are $n=O(h^{-2})$ triangles, and the integral in the right side of (59) is the area of Δ_k, which is $O(h^2)$, (59) leads to

$$\max_{1\le i\le nf_r}|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)u(v_i)|\le ch^{r+2} \tag{60}$$

By (36), this proves (57). ■

So, for $\alpha=\alpha_0$, the collocation method defined by (29) is superconvergent. These results can still be improved, sometimes, using symmetric triangles.

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