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Metrization of Proximities *

Ekaterina Mihaylova[1], *Stoyan Nedev*[2]

The main problem discussed in the talk is the following one: Suppose the proximity δ is generated by a family Ω of entourages of the diagonal. Is then δ generated by a uniformity $\mathcal{U}$ of uniform weight $uw(\mathcal{U}) \leq Card(\Omega)$? In particular, if δ is generated by a countable family of entourages is then δ metrizable?

1. Introduction

The aim of the talk is to outline the known results and the main technics they are obtained with as well as the open questions about metrizability and, more generally, uniformizability of proximities.

Under proximity on a (non-empty) set X we always mean an Efremovich proximity [1]:

Definition 1. *Let X be a non-empty set. A mapping $\delta : \exp(X)\times\exp(X) \to \{0, 1\}$ ($\exp(X)$ stands for the set of all subsets of X) is said to be a **proximity** on X if δ fulfills the following axioms (Efremovich axioms of proximity) :* Prx 1. $\delta(A, B) = \delta(B, A)$ *for every* $A, B \in \exp(X)$; Prx *2. For every* $A, B, C \in \exp(X)$, $\delta(A, B\cup C) = \delta(A, B)\cdot\delta(A, C)$; Prx 3. *For every* $x, y \in X$, $\delta(\{x\}, \{y\}) = 0$ *iff* $x = y$; Prx 4. $\delta(\emptyset, X) = 1$; Prx 5. *For every* $A, B \in \exp(X)$ *if* $\delta(A, B) = 1$ *then there exists* $C \in \exp(X)$ *such that* $A \subset C$, $B \subset X\backslash C$ *and* $\delta(A, X\backslash C) = 1$, $\delta(B, C) = 1$. *Two sets* $A, B \in \exp(X)$ *are said to be* $\delta-$*close if and only if* $\delta(A, B) = 0$. *Otherwise (i.e.* $\delta(A, B) = 1$*) A and B are called* $\delta-$*remote.*

Every proximity δ on the set X generates topology T_δ on X. T_δ is determined by the Kuratovski closure operator $Cl_\delta : \exp(X) \to \exp(X)$ defined as follows: $Cl_\delta(A) = \{x \in X : \delta(\{x\}, A) = 0\}$ for every $A \in \exp(X)$. The so

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defined topology T_δ is known to be a completely regular (Tikhonov) one. And moreover every Tikhonov topology on X is generated in the above mentioned sense by a proximity on X.

Let (X,T) be a Tikhonov topological space and let (cX,cT) be a compactification of (X,T). As it is known (cX, cT) defines a proximity δ_c on X by the formula $\delta_c(A,B)=0$ iff $Cl_{cX}A\cap Cl_{cX}B\neq\emptyset$ for every A, $B\in\exp(X)$. Yet, δ_c generates the topology T i.e. $T=T_{\delta_c}$. The famous Smirnov Theorem establishes that the assignment $(cX, cT)\to\delta_c$ is actually a one to one order preserving correspondence between the lattice of all compactifications of (X,T) and the lattice of all proximities on X generating T.

Definition 2. *Let δ be a proximity on X and $\mathcal{U}$ be a uniformity on X. $\mathcal{U}$ is said to be **generating** δ (δ is said to be **generated** by $\mathcal{U}$) if, for every A, $B\in\exp(X)$, $\delta(A,B)=0$ iff $(A\times B)\cap U\neq\emptyset$ for every $U\in\mathcal{U}$.*

On the other hand every proximity on X is generated in the above mentioned sense by a uniformity and moreover, among the uniformities generating δ there always exists the coarser one. A proximity δ on X is called *regular* if in the lattice of all uniformities generating δ there exists the finest one. For example if δ is generated by a uniformity with countable uniform weight then this last uniformity is the finest in the lattice of all uniformities generating δ so δ is thus regular.

In the above definitions a uniformity on X is obviously meant as a family of entourages of the diagonal in $X\times X$. This motivates the following definition:

Definition 3. *Let δ be a proximity on X and Ω be any family (not necessarily a uniformity) of entourages of the diagonal in $X\times X$. Ω **generates** δ (δ is said to be **generated** by Ω) if, for every A, $B\in\exp(X)$, $\delta(A,B)=0$ iff $(A\times B)\cap U\neq\emptyset$ for every $U\in\Omega$.*

Recall, under *entourage* of the diagonal it is meant any subset $U\subset X\times X$ containing the diagonal $\Delta=\{(x,x)\in X\times X\}$. In fact this is one way of defining *generation* of δ by an arbitrary family of entourages. Later on we will give some other possible definitions of this term.

Obviously, if δ is a proximity on X generated by uniformity $\mathcal{U}$ on X (in the sense of Definition 2) then δ is generated by any base of $\mathcal{U}$ (in the sense of definition 3).

Convention: For local use let's agree to say that a proximity δ on X is *τ-generated* if δ is generated by a family Ω of entourages with $Card(\Omega)\leq\tau$ (τ

stands for an infinite cardinal number). Yet, δ will be said τ-*uniformizable* when δ is generated by a uniformity $\mathcal{U}$ with $uw(\mathcal{U}) \leq \tau$. For $\aleph_0$-*generated* proximity we will also use *countably-generated* proximity, as well as for $\aleph_0$-*uniformizable* proximity we will use *countably-uniformizable* proximity. Obviously countably-uniformizable proximities are actually the metrizable ones.

The main problem under discussion in this talk can now be formulated as follows:

Question 1. Is every τ-generated proximity τ-uniformizable?

A special attention is paid to the following particular case of question 1:

Question 2. Is every countably-generated proximity metrizable?

2. Countably-generated proximities

The interest to the class of countably-generated proximities was provoked by the following two Theorems and especially by the second among them, that was announced for the first time in a joint note of N. Hadjiivanov and St. Nedev [5]:

Theorem 4. *[4] If δ is countably-generated proximity on X, then the topology T_δ is metrizable.*

Theorem 5. *A proximity δ on X is metrizable if (and, obviously, only if) δ is generated by a decreasing (by inclusion) sequence $\Omega = \{U_1 \supseteq U_2 \supseteq ... \supseteq U_n \supseteq ...\}$ of entourages.*

It's now clear that Question 2 is actually a variant of the question about essentiality of the requirement on Ω to be decreasing in Theorem 5..

The following Theorem marks one of the first steps in the study of countably-generated proximities:

Theorem 6. *[10] Every countably-generated proximity is regular.*

The most recent progress in the topic is given by the following result of Sv. Ivanov:

Theorem 7. *[7] A proximity δ is metrizable if (and only if) δ is generated by a countable family of symmetric entourages.*

The formal comparison of Theorem 5. and Theorem 7. might be misleading for one to conclude that the progress is not so significant since one requirement on Ω (namely to be decreasing) is merely replaced by another one (the entourages in Ω to be symmetric). Let's show that such a conclusion is pretty hasty. To this end recall the following definition:

Definition 8. *Let δ be a proximity on X and $\mathcal{C}$ be a family of coverings of X. δ is said to be* ***generated*** *by $\mathcal{C}$ if, for every A, $B \in \exp(X)$, $\delta(A,B)=0$ iff for every $c \in \mathcal{C}$ there is an element of c meeting both A and B.*

It's now easy to realize that Theorem 7. implies the following:

Corollary 9. *[7]A proximity δ on X is metrizable if (and only if) δ is generated by a countable family of coverings of X.*

On the other hand we can't see a way to infer Corollary 9. from Theorem 5..

A key role in the proof of Theorem 7. is played by the following result of Sv. Ivanov, St. Nedev, J. Pelant [8] :

Theorem 10. *Every countably-generated proximity is sequential.*

Definition 11. *[4]The proximity δ on X is called* ***sequential*** *if, for every A, $B \in \exp(X)$, such that $\delta(A,B)=0$ there are sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in A and B respectively with the property for every infinite $F \subset \mathbb{N}$ it follows that $\delta(\{x_n : n \in F\}, \{y_n : n \in F\}) = 0$. For short we will call such two sequences* ***δ-equivalent***.

Going back to Definitions 2. and 3. one immediately realizes that Definition 3. results from Definition 2. by a simple replacement of the term *uniformity* by the term *family of entourages.* On the other hand in the case of uniformity Definition 2. has many variants that are formally different but logically equivalent to each other. For instance:

Definition 12. *Let δ be a proximity on X and $\mathcal{U}$ be a uniformity on X. $\mathcal{U}$ is said to be* ***generating*** *δ (δ is said to be* ***generated*** *by $\mathcal{U}$) if, for every A, $B \in \exp(X)$, $\delta(A,B)=1$ iff $(A\times B)\cap U \neq \emptyset$ for at most finitely many $U \in \mathcal{U}$.*

Starting now with Definition 12. and applying the procedure of "replacement of terms" we obtain:

Definition 13. *Let δ be a proximity on X and Ω be a family of entourages of the diagonal. Ω is said to be* ****-generating*** *δ (δ is* ****-generated*** *by Ω) if, for every A, $B \in \exp(X)$, $\delta(A,B) = 1$ iff $(A \times B) \cap U \neq \emptyset$ for at most finitely many elements $U \in \Omega$.*

(The symbol *- is added for distinction.)

A proximity δ on X will be called *countably-*-generated* if δ is *-generated by a countable family Ω of entourages of the diagonal.

While the Definitions 2. and 12. are equivalent the question about the equivalence of Definitions 3. and 13. is open for us. Moreover we have the following theorem obtained in a discussion with M. Choban:

Theorem 14. *A proximity is metrizable iff it's countably-*-generated .*

3. τ-generated proximities

Theorem 15. *[6] If δ is τ-generated proximity on X, then the topology T_δ is τ-uniformizable.*

Theorem 16. *[2][4] If a proximity δ on X is generated by linearly ordered (by inclusion) family Ω of entourages of the diagonal in $X \times X$ then δ is generated by uniformity $\mathcal{U}$ on X such that $\mathcal{U}$ has a uniform base B which is linearly ordered by inclusion and $Card(B) \leq Card(\Omega)$.*

Remark. Obviously Theorem 5. can be regarded as a particular case (namely the case $Card(\Omega) = \aleph_0$) of the last Theorem 16..

Theorem 17. *[3] Let δ be a proximity on X generated by a directed (by inclusion) family Ω of entourages. Then δ is generated by a uniformity $\mathcal{U}$ on X with $uw(\mathcal{U}) \leq Card(\Omega)^{\aleph_0}$.*

Recall, the family Ω of entourages is said to be *directed* by inclusion if for every $U, V \in \Omega$ there is $W \in \Omega$ with $W \subset U \cap V$.

Obviously our Question 1 can be regarded as a version of the question about essentiality of the requirement on Ω to be directed (by inclusion) and the appearance of the power $\aleph_0$ on $Card(\Omega)$ in the last Theorem 17..

4. Methods and proofs

We start this section by sketching the proof of Theorem 16. (see [4]). As it was already mentioned this proof covers also Theorem 5.. Yet, Theorem 14. in turn can easily be reduced to Theorem 5..

So let now δ be a proximity on the set X and Ω be a linearly ordered (by inclusion) family of entourages of the diagonal in $X \times X$, generating δ. We construct a generating δ uniformity $\mathcal{U}$ on $X, \mathcal{U}$ having a linearly ordered uniform base B with $Card(B) \leq Card(\Omega)$ passing through the following steps:

Step 1. By a straight forward verification we realize that the family of entourages $\Omega \circ \Omega^{-1} = \{U \circ U^{-1} : U \in \Omega\}$ (here for every couple U, $V$of entourages, $U \circ V = \{(x, y) \in X \times X$: there exists $z \in X$ such that $(x, z) \in V$, $(z, y) \in U\})$ is generating δ as well. This observation is giving us that the family $(\Omega \circ \Omega^{-1})^3 = \{U \circ U^{-1} \circ U \circ U^{-1} \circ U \circ U^{-1} : U \in \Omega\}$ also generates δ.

Step 2. We show that this last family $(\Omega \circ \Omega^{-1})^3$ is actually a base of a uniformity on X. To this end we have only to show that for every $V \in (\Omega \circ \Omega^{-1})^3$ there is $W \in (\Omega \circ \Omega^{-1})^3$ such that $W \circ W \subset V$. This fact is achieved by applying the following useful Lemma:

Lemma 18. *(A. Effremovich, N. Ramm, A. Schwartz)[11] Let U be a symmetric entourage of the diagonal in $X \times X$, Λ be a linearly ordered set and $f : \Lambda \to X$ and $g : \Lambda \to X$ be such that $(f(\lambda), g(\lambda)) \notin U^3$ for every $\lambda \in \Lambda$. Then there exists a cofinal in Λ subset $\Lambda' \subset \Lambda$ such that $(f(\Lambda') \times g(\Lambda')) \cap U = \emptyset$.*

The proofs of Theorem 10. and Theorem 7. we have in mind are passing through a particular case of Ramsey's Theorem (see [9]). For instance Ramsey's Theorem is used to obtain the following Lemma that, in turn, plays the main role in the proof of Theorem 7. :

Lemma 19. *[7]Let δ be a proximity on X and U_1, U_2, U_3, U_4 and U_5 be 5 $\delta-$wide symmetric entourages of the diagonal in $X \times X$. Then the composition $U_1 \circ U_2 \circ U_3 \circ U_4 \circ U_5$ is $\delta-$sequentially-wide.*

Here $\delta-$*wide* means that the entourage U_i has the property:
$(A \times B) \cap U_i \neq \emptyset$ whenever A, B are δ-close subset of X.

Next, the entourage U is called δ-*sequentially-wide* if for every couple $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ of δ-equivalent sequences in X (see Definition 11.) there exists $n \in \mathbb{N}$ with $(x_n, y_n) \in U$.

Question 3. Is the requirement on U_i to be symmetric in the above lemma essential?

Question 4. Is the composition of 2 (3, 4) δ-wide (symmetric) entourages δ-sequentially-wide?

Few words about the proof of Theorem 17. (see [3]). For the proof we have in mind it is more convenient to regard uniformities as families of pseudometrics. Under the assumptions of Theorem 17. we denote by P_Ω the family of all pseudometrics on X with the properties : $p(x, y) \leq 1$ for every $x, y \in X$ and for every $\varepsilon > 0$ there is $U \in \Omega$ with $p(x, y) < \varepsilon$ whenever $(x, y) \in U$. Next, Σ_Ω stands for the set of all decreasing by inclusion sequences of elements of Ω. For a particular $\sigma = (U_1 \supseteq U_2 \supseteq ... \supseteq U_n \supseteq ...) \in \Sigma_\Omega$ we denote by p_σ the supremum of the set of all $p \in P_\Omega$ with $p(x, y) < n^{-1}$whenever $(x, y) \in U_n$. At last let $P_{\Sigma_\Omega} = \{p_\sigma : \sigma \in \Sigma_\Omega\}$. Obviously, $Card(P_{\Sigma_\Omega}) \leq Card(\Sigma_\Omega) \leq Card(\Omega)^{\aleph_0}$.

Yet, it's not so hard to realize that P_{Σ_Ω} is a uniform base of a uniformity generating δ.

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Institute of Mathematics and Informatics,
Bulgarian Academy of Sci.,
bl. 8, Acad. G. Bonchev str.,
1113 Sofia, Bulgaria,
E-mail: [1] *katiamih@yahoo.com,* [2] *nedev@math.bas.bg*

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