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Controllability and Relaxation in Banach Spaces*

Marian Mureşan

The present paper contains three results on controllability. The first one concerns with an approximately locally null-controllability result, while the second one concerns with an approximately controllability result. The third one deals with the connection between the approximately controllability of a given system and the exact controllability of the relaxed control system associated to the original one.

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1. Introduction

In order to present our results we introduce some notations. Let Z be a topological space and $Y \subset Z$. By $\operatorname{int} \mathrm{Y}$ and $\operatorname{cl} \mathrm{Y}$ we denote the set of interior points, and the closure of Y, respectively. Let Z be a linear space and $Y \subset Z$, then by $\operatorname{co} \mathrm{Y}$ we denote the convex hull of Y. If X is a Banach space, then by $\mathcal{L}(X)$ we denote the space of linear and bounded operators from X in X. X^* is the Banach space of the linear and continuous functionals on X. Let F be a multifunction from a σ-algebra to a topological space. By S_F we denote the set of measurable selections from F. Under convenient assumptions, by S_F^1 we denote the set of Bochner integrable selections from F, see [5], [4], and [7].

Consider a real interval $T := [t_0, t_f]$ with $t_0 < t_f$ and λ the Lebesgue measure on T. Let X and Y be separable real Banach spaces. Let $\mathrm{B}_\delta = \{\mathrm{x} \in \mathrm{X} \mid \|\mathrm{x}\| \leq \delta\}$. We denote the closed unit ball by B, too. We further consider

(U) a weakly measurable multifunction $U : T \rightsquigarrow Y$ having nonempty and closed values;

(B) a Carathéodory mapping $B : T \times Y \to X$ (measurable in the first variable and continuous in the second one) satisfying either there exits a positive integrable function m defined on T

$$B(t,u) \subset m(t)\mathrm{B}, \quad \text{for all} \quad t \in T, \quad u \in U(t). \tag{1.1}$$

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or for all $t \in T$ and $u \in U(t)$

$$\|B(t,u)\| \leq a(t) + b\|u\|, \text{ a. e.}, \quad a \in L^1_+, \ b \geq 0. \tag{1.2}$$

(A) a family $\{A(t)\}_{t\in T}$ of linear and densely defined operators generating an evolution operator $S : \Delta = \{(t,s) \in T \times T \mid t_0 \leq s \leq t \leq t_f\} \to \mathcal{L}(X)$, i.e.,
$S(t,t) = I$, $\forall\, t \in T$, I is the identity,
$S(t,\tau)S(\tau,s) = S(t,s)$, $\forall\ t_0 \leq s \leq \tau \leq t \leq t_f$,
$S : \Delta \to \mathcal{L}(X)$ is continuous in the strong operator topology, [10].
Also, $B(t,U(t)) := \{x \in X \mid \exists\ u \in U(t) \text{ with } x = B(t,u)\}$. For $M \subset X$, $M \neq \emptyset$, the support function $\sigma_M(\cdot)$ of M is defined in the usual way by

$$\sigma_M(x^*) = \sup_{x\in M}(x^*,x) = \sup_{x\in M} x^*(x) = \sigma(x^*(M)), \quad x^* \in X^*.$$

Under the above conditions our attention focuses on the following system

$$x'(t) = A(t)x(t) + B(t,u(t)), \quad t \in T, \ u \in S_U. \tag{1.3}$$

Throughout the present paper we are interested in some properties of the mild solutions of the system (1.3), i.e., given $x_0 \in X$ (as initial value) a mild solution of (1.3) is a continuous function $x \in C(T,X)$ which can be written as

$$x(t) = S(t,t_0)x_{t_0} + \int_{t_0}^{t} S(t,s)B(s,u(s))ds, \ t \in T, \tag{1.4}$$

where u is a measurable selection of the multifunction U such that $B(\cdot,u(\cdot)) \in L^1$.

The reachable set from x_0 at time $t \in T$ corresponding to system (1.4) is defined as

$$R(t,x_0) = \{x(t) \in X \mid x(\cdot) \text{ is a mild solution of (1.3)}\}. \tag{1.5}$$

From (1.4) and (1.5) easily follows that

$$R(t,y_0) = S(t,t_0)(y_0 - x_0) + R(t,x_0). \tag{1.6}$$

The latter equality means that the topological properties of the reachable set are invariant under translations.

2. Results

Different notions of controllability are investigated in [12] and [13]. We now recall some of them mentioned in [9], too. System (1.3) is *approximately* *controllable* if for every $x_0 \in X$ we have $\operatorname{int}\operatorname{cl} R(t_f, x_0) \neq \emptyset$. System (1.3) is said to be *approximately locally null-controllable* if there exists an open neighborhood V of the origin such that for all $x_0 \in V$, $0 \in \operatorname{cl}(R(t_f, x_0))$.

Theorem 2.1. ([3]) *Suppose assumptions* (U), (B) *with (1.1), and* (A) *are satisfied. Then*

(a) *if* $S(t_f, t)B(t, U(t)) \neq \{0\}$ *on a set of positive Lebesgue measure and (1.3) is approximately locally null-controllable, then there exists* $x^* \in X^* \setminus \{0\}$ *and* $E \subset T$ *Lebesgue measurable such that*

$$\lambda(E) > 0, \ \ \textit{and} \ \ 0 < \sigma(x^*(S(t_f, t)B(t, U(t)))), \quad \forall\, t \in E;$$

(b) *if* $0 \in B(t, U(t))$ *a.e. and for every* $x^* \in X^* \setminus \{0\}$ *there exists* $E \subset T$ *Lebesgue measurable with* $\lambda(E) > 0$ *such that for all* $t \in E$ $\sigma(x^*(S(t_f, t)B(t, U(t)))) > 0$, *system (1.3) is approximately locally null-controllable.*

Proof. (a) From the definition of approximately locally null-controllability we have that there is a positive δ such that for all $x_0 \in \operatorname{int}(B_\delta)$ it holds that $0 \in \operatorname{cl}(R(t_f, x_0))$. Then $0 \leq \sigma(x^*(\operatorname{cl}(R(t_f, x_0))))$. Also $0 \leq \sigma(x^*(R(t_f, x_0)))$. Using theorem 2.2 in [2], we have

$$\begin{aligned} 0 \leq \sigma(x^*(R(t_f, x_0))) &= \sigma(x^*(S(t_f, t_0)x_0)) + \sigma\left(x^*\Big(\textstyle\int_{t_0}^{t_f} S(t_f, t)B(t, U(t))dt\Big)\right) \\ &= \sigma(x^*(S(t_f, t_0)x_0)) + \textstyle\int_{t_0}^{t_f} \sigma\left(x^*\left(S(t_f, t)B(t, U(t))\right)\right) dt, \end{aligned}$$

for any $x_0 \in \operatorname{int}(B_\delta)$ and $x^* \in X^*$. Therefore we can write

$$0 \leq \int_{t_0}^{t_f} \sigma(x^*(S(t_f, t)B(t, U(t))))dt.$$

Since $S(t_f, t)B(t, U(t)) \neq \{0\}$ on a set of positive Lebesgue measure, we see that there exists $x^* \in X^* \setminus \{0\}$ and $E \subset T$ Lebesgue measurable, with $\lambda(E) > 0$ such that $0 < \sigma(x^*(S(t_f, t)B(t, U(t))))$, for all $t \in E$.

(b) Choose $x^* \in X^* \setminus \{0\}$. Then choose $E \subset T$ Lebesgue measurable with $\lambda(E) > 0$ such that for all $t \in E$ $\sigma(x^*(S(t_f, t)B(t, U(t)))) > 0$. Thus we can define the nonempty multifunction L as

$$E \ni t \rightsquigarrow L(t) := \{u \in U(t) \mid x^*(S(t_f, t)B(t, u)) > 0\}.$$

We consider the following mapping

$$E \times Y \ni (t, u) \mapsto g(t, u) := x^*(S(t_f, t)B(t, u))$$

and remark that it is Carathéodory. Then by theorem 6.5 in [1] the multifunction

$$E \ni t \rightsquigarrow H(t) := x^*(S(t_f,t)B(t,U(t)))$$

is weakly measurable, hence graph measurable. Recalling that g is Carathéodory and using corollary 6.3 in [1], we have that the set

$$\{(t,u) \mid x^*(S(t_f,t)B(t,u)) > 0\}$$

is measurable. Then the multifunction L is graph measurable since

$$graph(L) = graph(H) \cap \{(t,u) \mid x^*(S(t_f,t)B(t,u)) > 0\}.$$

Using the Aumann selection theorem, we get a measurable selection u_1 from L such that $u_1(t) \in L(t)$, a. e. on E.

The mapping $T \times Y \ni (t,u) \mapsto S(t_f,t)B(t,u)$ is Carathéodory. U has closed values. Then by theorem 6.5 in [1] the multifunction $T \ni t \rightsquigarrow S(t_f,t)B(t,U(t))$ is weakly measurable. Thus it is graph measurable. By hypothesis $0 \in S(t_f,t)B(t,U(t))$, for all $t \in T$. Then by theorem 7.2 in [1], we get a measurable selection $u_2(t) \in U(t)$, $t \in T$, such that $0 = S(t_f,t)B(t,u_2(t))$, a. e. The selections u_1 and u_2 are integrable, too. Thus we can define $\hat{u} = \chi_E u_1 + \chi_{T\setminus E} u_2 \in S^1_U$.

Let $\hat{x} \in C(T,X)$ be the (unique) mild solution generated by $\hat{u}$ and starting from the origin, i.e., $x_0 = 0$. Then we have

$$\begin{aligned} x^*(\hat{x}(t_f)) &= \textstyle\int_{t_0}^{t_f} \sigma(x^*(S(t_f,t)B(t,\hat{u}(t))))dt \\ &= \textstyle\int_E \sigma(x^*(S(t_f,t)B(t,u_1(t))))dt > 0. \end{aligned}$$

Thus $\sigma(x^*(R(t_f,0))) > 0$. Since $x \mapsto \sigma(x^*(R(t_f,x)))$ is continuous, we can find $\delta > 0$ such that for all $x \in \operatorname{int} \mathrm{B}_\delta$ we have $\sigma(x^*(R(t_f,x))) > 0$. Then $0 \in \operatorname{cl}\operatorname{co} R(t_f,x) = \operatorname{cl} R(t_f,x)$ for all $x \in \operatorname{int} \mathrm{B}_\delta$ and thus system (1.3) is approximately locally null-controllable.

Now the proof is complete. ∎

Remark 1.

(a) Our theorem 2.1 is related to theorem 2.2 in [9].

(b) In theorem 2.2 in [9] the multifunction U is considered having convex values and being on a weakly compact subset of Y. We need not such an assumption of convexity of U. Regarding the second assumption, we have required instead that U is integrably bounded.

(c) In theorem 2.2 in [9] the Carathéodory mapping B has linear growth. We need not such an assumption.

Theorem 2.2. ([8]) *Admit* (U), (B) *with (1.2) and* (A) *hypotheses and consider system (1.3). Moreover, suppose that*

(i) $\mu\{t \in T \mid S(t_f,t)B(t,U(t))$ *is not a singleton*$\} > 0$,

(ii) *the multifunction* $T \ni t \mapsto S(t_f,t)B(t,U(t))$ *is graph measurable.*

Then system (1.3) is approximately controllable on T *if and only if there exists no* $x^* \in X^* \setminus \{0\}$ *so that* $x^*(S(t_f,t)B(t,U(t))) = constant$, *a. e. on* T.

Proof. *Necessity.* Suppose that there exists $x^* \in X^* \setminus \{0\}$ with $x^*(S(t_f,t)B(t,U(t))) = constant$, a. e. on T. Then there exists $u \in S^1_U$ so that if $c(t) := x^*(S(t_f,t)B(t,u(t)))$, it follows $c(\cdot) \in L^1$ and $R(t_f,0) \neq \emptyset$. Let $x \in R(t_f,0)$. Then there exists $u \in S^1_U$ such that

$$x(t_f) = \int_0^{t_f} S(t_f,t)B(t,u(t))\,dt.$$

Taking into account Corollary V.5.2, page 134, in [14] it follows that

$$\begin{aligned} x^*(x) &= x^*\left(\int_{t_0}^{t_f} S(t_f,t)B(t,u(t))dt\right) = \int_{t_0}^{t_f} x^*(S(t_f,t)B(t,u(t))dt \\ &= \int_{t_0}^{t_f} c(t)dt = k \in R. \end{aligned}$$

Let $z \in V := \{z \in X \mid x^*(z) = k\}$. V is a closed hyperplane and $\operatorname{cl} R(t_f,0) \subset V$. Hence $\operatorname{int}\operatorname{cl} R(t_f,0) \subset \operatorname{int} V$, so $\operatorname{int}\operatorname{cl} R(t_f,0) = \emptyset$, i.e. our (1.3) system is not controllable.

Sufficiency. The idea is simple: to choose two integrable selections from $S(b,\cdot)B(\cdot,U(\cdot))$ far away one from the other such that the corresponding solutions of system (1.3) to be also sufficiently far one from the other.

From the Castaing representation theorem, theorem 5.6 in [1] or theorem 4.2.3 in [5], it follows that there exists $\{u_n\}_{n\geq 1}$ a countable family of measurable functions such that $U(t) = \operatorname{cl}\{u_n(t) \mid n \geq 1\}$, for all $t \in T$.

We choose an arbitrary, but fixed $x^* \in X^* \setminus \{0\}$. Then for t in a subset of T having a strictly positive measure there exist $v_1, v_2 \in S(t_f,\cdot)B(t,U(\cdot))$ with $x^*(v_1 - v_2) \neq 0$ on that set.

For a while we admit that U has bounded values, too. Later on we will remove this extra assumption. Define the following mappings

$$\begin{aligned} M(t) &= \sup\{x^*(S(t_f,t)B(t,U(t)))\} = \sup_n\{x^*(S(t_f,t)B(t,u_n(t)))\}, \\ m(t) &= \inf\{x^*(S(t_f,t)B(t,U(t)))\} = \inf_n\{x^*(S(t_f,t)B(t,u_n(t)))\}. \end{aligned}$$

From the hypotheses it follows that

$$\|x^*(S(t_f,t)B(t,U(t)))\| \le \|x^*\| \cdot \|S(t_f,t)\|(a(t)+b\|u(t)\|),$$

and from the boundedness of U we may write $-\infty < m(t) \le M(t) < +\infty$, a. e. on T. Also we have that the mappings m and M are measurable on T and, at the same time, $\eta(t) := [M(t)-m(t)]/2$, $t \in T$, is measurable. From (i) it follows that if $C := \{t \in T \mid \eta(t) > 0\}$, then $\mu(C) > 0$. Define $\varepsilon : C \to R_+$ as $\varepsilon(t) := \eta(t)/2$, $t \in C$. Since the differences $M(t) - \varepsilon(t)$, respectively $m(t) - \varepsilon(t)$ are well defined for all $t \in C$ we may consider the multifunctions $L_i : C \rightsquigarrow Y$, $i = 1, 2$ defined by

$$(2.1) \qquad \begin{cases} L_1(t) & := \{u \in U(t) \mid x^*(S(t_f,t)B(t,u)) \ge M(t) - \varepsilon(t)\}, \\ L_2(t) & := \{u \in U(t) \mid x^*(S(t_f,t)B(t,u)) \le m(t) + \varepsilon(t)\}. \end{cases}$$

We check that L_1 and L_2 are graph measurable. Note that

$$graph\, L_1 = graph\, U \cap graph\, F_1, \quad graph\, L_2 = graph\, U \cap graph\, F_2,$$

where

$$C \ni t \mapsto F_1(t) = \{x \in Y \mid f_1(t,x) \ge 0\},\ C \ni t \mapsto F_2(t) = \{x \in Y \mid f_2(t,x) \le 0\},$$

and

$$(2.2) \qquad \begin{cases} f_1(t,x) & := x^*(S(a,t)B(t,x)) - (M(t) - \varepsilon(t)), \quad t \in C,\ x \in Y, \\ f_2(t,x) & := x^*(S(a,t)B(t,x)) - (m(t) + \varepsilon(t)), \quad t \in C,\ x \in Y. \end{cases}$$

Invoking theorem 6.4 in [1], we infer the measurability of F_1 and F_2. Hence L_1 and L_2 are graph measurable. By the Aumann selection theorem, theorem 5.2 in [1], we can choose two measurable functions u_1 and u_2 such that $u_i : C \to Y$, $u_i(t) \in L_i(t)$, a. e., $i = 1, 2$. Obviously

$$(2.3) \qquad x^*(S(t_f,t)B(t,u_2(t))) < x^*(S(t_f,t)B(t,u_1(t)), \quad t \in C.$$

Now our desire is to substitute the measurable functions u_1 and u_2 by integrable ones, [9]. The substitution is realized in such a way keeping valid an inequality of the form (2.1). For $p > 0$ and $u \in S^1_U$ define

$$u_{1,p}(t) = \begin{cases} u_1(t), & \text{if } t \in C \text{ and } \|u_1(t)\| \le p, \\ u(t), & \text{otherwise,} \end{cases}$$

$$u_{2,p}(t) = \begin{cases} u_2(t), & \text{if } t \in C \text{ and } \|u_2(t)\| \le p, \\ u(t), & \text{otherwise.} \end{cases}$$

Then $u_{1,p}, u_{2,p} \in S^1_U$ and

$$x^*(S(t_f,t)B(t,u_{2,p}(t))) \le x^*(S(t_f,t)B(t,u_{1,p}(t))), \quad t \in T. \tag{2.4}$$

For p sufficiently large the above inequality is strictly on a measurable subset having strictly positive measure. Let $x_{1,p}, x_{2,p}$ be the trajectories of system (1.3) corresponding to $u_{1,p}$, respectively $u_{2,p}$. Then for p sufficiently large we have

$$0 < \int_{t_0}^{t_f} x^*(S(t_f,t)[B(t,u_{1,p}(t)) - B(t,u_{2,p}(t))])\,dt = x^*(x_{1,p}(t_f) - x_{2,p}(t_f)).$$

Since the functional $x^* \in X^* \setminus \{0\}$ has been chosen arbitrary, we infer that the reachable set $R(t_f, 0)$ is not included in any closed hyperplane in X.

Hereafter the proof goes identically as the last part of proof to theorem 2.1 in [9].

Now let us remove the assumption on the boundedness of the values of U. It means that M or m or both may be unbounded on some $t \in T$. Let us introduce the following functions

$$\overline{M}(t) = \begin{cases} M(t), & \text{if } M(t) < +\infty, \\ 1, & \text{otherwise,} \end{cases} \quad \text{and} \quad \overline{m}(t) = \begin{cases} m(t), & \text{if } m(t) > -\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then we define $\eta(t) := [\overline{M}(t) - \overline{m}(t)]/2$ and in (2.1) and (2.2) we consider $\overline{M}$ and $\overline{m}$ instead of M, respectively m. Then we repeat the last part of the proof of the previous case.

At the end we get the same conclusion on the reachable set as before. Thus the proof is complete. ■

When X is a separable real Hilbert space the above result has been obtained in [6].

In this the last part of the paper we consider that there are given

(U′) a weakly measurable multifunction $U : T \rightsquigarrow Y$ having nonempty, closed, and convex values such that $U(t) \subset W$ a. e., W being a weakly compact subset of Y;

(B′) a mapping $B : T \times Y \to X$ measurable in the first variable and sequentially weakly continuous in the second one satisfying (1.2).

To system (1.3) we associate the convexified system

$$\begin{cases} x'(t) = A(t)x(t) + \int_W B(t,u))\mu(t)(du), & t \in T, \\ \mu(t) \in \sum(t) \text{ a.e.}, \quad \mu(\cdot) \text{ is measurable} \end{cases} \tag{2.5}$$

with $\sum(t) = \{\theta \in M^1_+(W_w) \mid \theta(U(t)) = 1\}$, where $M^1_+(W_w)$ denotes the space of probability measures on the compact, metrizable space W_w (the set W endowed with the relative weak topology). Taking $\eta(t)(\cdot) = \delta_{u(t)}(\cdot)$, where u is the original control and $\delta_{u(t)}(\cdot)$ is the Dirac probability concentrated on $u(t)$, we can see that system (1.3) embeds in the relaxed one, (2.5). The reachable set from x_0 at time $t \in T$ corresponding to the relaxed system (2.5) is denoted by $R_c(t, x_0)$.

Theorem 2.3. *Suppose assumptions* (U′), (B′) *and* (A) *are satisfied. Then system (1.3) is approximately controllable if and only if system (2.6) is exactly controllable.*

P r o o f. From (1.6) it follows that we may suppose that the initial value x_0 is the origin.

We prove that

$$R_c(t_f, 0) = \operatorname{cl} R(t_f, 0). \tag{2.6}$$

From [2] we have that $\operatorname{cl} R(t_f, 0)$ is a convex set. From Mazur's theorem it follows that it is enough to prove

$$R_c(t_f, 0) = \operatorname{cl}(R(t_f, 0))_w, \tag{2.7}$$

the closure in respect to the weak topology. Then instead of (2.6) we prove (2.7).

First we show that $R_c(t_f, 0)$ is convex and closed in X.

The convexity follows in the following way. Consider two arbitrary points $x_1, x_2 \in R_c(t_f, 0)$, and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then we write

$$x_1 = \int_0^{t_f} \int_W S(t_f, s)B(s, u)\mu_1(s)(du)ds, \ \mu_1 \in S_{\sum},$$
$$x_2 = \int_0^{t_f} \int_W S(t_f, s)B(s, u)\mu_2(s)(du)ds, \ \mu_1 \in S_{\sum}.$$

Then we have

$$\alpha x_1 + \beta x_2 = \int_0^{t_f} \int_W S(t_f, s)B(s, u)[\alpha\mu_1(s) + \beta\mu_2(s)](du)ds,$$

and $\alpha\mu_1 + \beta\mu_2 \in S_{\sum}$.

To prove the closedness of $R_c(t_f, 0)$ we consider a sequence $(x_n)_n$ such that for all n, $x_n \in R_c(t_f, 0)$. Then

$$x_n = \int_0^{t_f} \int_W S(t_f, s)B(s, u)\mu_n(s)(du)ds, \ \mu_n \in S_{\sum}.$$

From [11] it follows that $S_{\sum}$ is w^*−compact in $L^\infty(T, M(W_w))$ $= (L^1(T, C(W_w))^*$, where $C(W_w)$ is the space of real valued continuous functions defined on the compact metrizable space W_w, and $M(W_w)$ is the space of all Radon measures on W_w. Thus $M(W_w) = C(W_w)^*$. Since $C(W_w)$ is a separable Banach, $L^1(T, C(W_w))$ is separable. Then the w^*−topology on $S_{\sum}$ is metrizable. Thus passing to a subsequence if necessary, we suppose that $\mu_n \xrightarrow{w^*} \mu \in S_{\sum}$. Then since by hypothesis (B′), $u \to S(t_f, s)B(s, u)$ is sequentially continuous, we have that $s \to x^*(S(t_f, s)B(s, \cdot))$ belongs to $L^1(T, C(W_w)$ for every $x^* \in X^*$, and so

$$\int_0^{t_f} \int_W S(t_f, s)B(s, u)\mu_n(s)(du)ds \xrightarrow{w} \int_0^{t_f} \int_W S(t_f, s)B(s, u)\mu(s)(du)ds.$$

Thus

$$x_n \xrightarrow{w} \int_0^{t_f} \int_W S(t_f, s)B(s, u)\mu(s)(du)ds \in R_c(t_f, 0).$$

Hence $R_c(t_f, 0)$ is weakly closed in X.

Next we show that $R_c(t_f, 0) = \operatorname{cl} R(t_f, 0)_w$ in X. We have that

$$R(t_f, 0) \subset R_c(t_f, 0) \Longrightarrow \operatorname{cl} R(t_f, 0)_w \subset \operatorname{cl} R_c(t_f, 0)_w = R_c(t_f, 0).$$

So one inclusion is obvious. To prove the other inclusion we consider an arbitrary $x \in R_c(t_f, 0)$. Then there exists a $\mu \in S_{\sum}$ such that

$$x = \int_0^{t_f} \int_W S(t_f, s)B(s, u)\mu(s)(du)ds.$$

By [11] we can find a sequence $(u_n)_n$ of integrable selections in U such that $\delta_{u_n} \xrightarrow{w^*} \mu$. Set

$$x_n = \int_0^{t_f} S(t_f, s)B(s, u_n(s))ds = \int_0^{t_f} \int_W S(t_f, s)B(s, u)\delta_{u_n(s)}(du)ds, \ n \in N.$$

Then $x_n \in R(t_f, 0)$, for all $n \in N$ and $x_n \xrightarrow{w} x$. Further we have $R_c(t_f, 0) \subset \operatorname{cl} R(t_f, 0)_w$.

Hence $R_c(t_f, 0) = \operatorname{cl} R(t_f, 0)_w$ and the theorem is proved. ■

When X and Y are separable real Hilbert spaces the above result has been proved in [9].

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Faculty of Mathematics and Computer Science
Babes-Bolyai University
M. Kogalniceanu, 1
Cluj-Napoca 3400, ROMÂNIA
e-mail: mmarian@math.ubbcluj.ro

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