Mathematica
Balkanica

New Series Vol. 19, 2005, Fasc. 1-2

A Remainder Formula for Hermite Multivariate Interpolation

Dana Simian

The aim of this paper is to analyze a particular case of bivariate Hermite interpolation problem and to derive an integral formula of the remainder. In the end, we obtained a superior bound for the remainder.

AMS Subj. Classification: 65D32

Key Words: Multivariate Hermite interpolation, remainder, Newton bases

1. Introduction

Multivariate polynomial interpolation is more complicated then the univariate case, but in the same time, it is more important by a practical point of view and hence it is studied by many mathematicians. In order to define the general problem of multivariate polynomial interpolation, we will use the following notations. Π^d is the space of all polynomial in d variable, Π_k^0 the space of homogeneous polynomials of total degree k. The expression of a polynomial q, in d variables is: $q = \sum_{|\alpha| \leq deg(q)} c_\alpha ()^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ being a natural multiindex and $|\alpha| = \alpha_1 + \ldots + \alpha_d$. We denote by $\mathbb{N}$ the set of natural number including 0. The set Λ of linear independent functionals represents the set of interpolation conditions.

For a given space of functions $\mathcal{F}$, in d variables, which includes polynomials and a given set of interpolation conditions Λ, the general problem of multivariate polynomial interpolation is to find a polynomial subspace $\mathcal{P}$ such that for an arbitrary function $f \in \mathcal{F}$ there is an unique polynomial $p \in \mathcal{P}$ such that $\lambda(f) = \lambda(p)$, for all functionals $\lambda \in \Lambda$. The polynomial subspace $\mathcal{P}$ is named an interpolation space for the set of conditions Λ. In that case we say that the pair $(\Lambda, \mathcal{P})$ is correct, or the interpolation problem given by Λ is poised.

The generalization of Lagrange univariate interpolation problem is easy to do. We must just consider the set of conditions consisting in evaluation functionals on a set of distinct points: $\Lambda = \{\delta_{\theta_i} | \ \theta_i \in \Theta \subset \mathbb{R}^d; \ \theta_i \neq \theta_j, \ \forall i \neq j; \ i,j = 1,\ldots,n\}$. To generalize the Hermite univariate case we associate, to any polynomial in d variables, the differential operator with constant coefficients: $q(D) = \sum_{|\alpha| \leq deg(q)} c_\alpha D^\alpha$, with $D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}$.

A natural generalization of the Hermite univariate interpolation problem is given by the conditions

$$(1) \qquad \Lambda = \{\delta_{x_j} \circ q_{j,k}(D) | \ x_j \in \mathbb{R}^d, \ x_j \neq x_i, \forall j \neq i, \ q_{j,k} \in \Pi^d\}$$

$j = 1,\ldots,m; \ k = 0\ldots,l_j - 1$, with $Q_j = span\{q_{j,k} | \ k = 0\ldots,l_j - 1\}$ being D-invariant spaces of polynomials and $q_{j,k}$ being linear independent polynomials.

On the other hand, an important class of interpolation problems is given by the set of conditions Λ, satisfying the property that $ker(\Lambda)$ is a polynomial ideal. These types of interpolation schemes are named ideal interpolation schemes. An important result makes the connection between the Hermite multivariate interpolation schemes and the ideal ones.

Theorem 1.1. *(M. Gasca, T. Sauer, [3]) Let Λ be a set of linear independent functionals. Then, $ker(\Lambda)$ is a polynomial ideal if and only if there exists points x_j and polynomial D-invariant spaces $Q_j = span\{q_{j,k} : k = 0,\ldots,l_j - 1\}$, $j = 1,\ldots,m$, such that*

$$span(\Lambda) = span\{(q_{j,k}(D)f)(x_j); \ j = 1,\ldots,m; \ k = 0,\ldots,l_j - 1\}$$

More, we look for minimal interpolation spaces of n order, that is for ideal degree reducing interpolation spaces. It is known from [3] that a polynomial subspace, $\mathcal{P}(\Lambda)$, is a minimal interpolation space of order n with respect to Λ if and only if there exists a Newton basis of order n for $\mathcal{P}(\Lambda)$ with respect to Λ.

A polynomial space $\mathcal{P}(\Lambda)$ admits a Newton basis of order n with respect to the set of functionals Λ, if there exists a nested set of sets of multiindices I_k, $k \in \{-1,0,\ldots,n\}$ such that the functionals in Λ can be reindexed into the blocks: $\Lambda^{(k)} = \{\lambda_\alpha \in \Lambda | \ \alpha \in I_k \setminus I_{k-1}\}$, $k \in \{0,\ldots,n\}$, with $I_0 \subset \ldots \subset I_n$, $I_{-1} = \Phi$, $I_k \setminus I_{k-1} \subset \{\alpha \mid |\alpha| = k\}$, $I_n \setminus I_{n-1} \neq \Phi$, $\#I_n = dim\mathcal{P}(\Lambda)$ and

1. There exists a basis $p_\alpha \in \Pi^d_{|\alpha|}$, $\alpha \in I_n$, of $\mathcal{P}(\Lambda)$ having the property:
$$\lambda_\beta(p_\alpha) = \delta_{\alpha,\beta}; \ \forall \ \beta \in I_n; \ |\beta| \leq |\alpha|$$
2. There exists complementary polynomials $p_\alpha^\perp \in \Pi^d_{|\alpha|}$, such that
$$\lambda(p_\alpha^\perp) = 0, \ \forall \ \lambda \in \Lambda, \ \alpha \in I'_n = \{\gamma \in \mathbb{N}^d : \ |\gamma| \leq n\} \setminus I_n \text{ and}$$
$$\Pi^d_n = span\{p_\alpha : \ \alpha \in I_n\} \oplus span\{p_\alpha^\perp : \ \alpha \in I'_n\}.$$

A path, μ in I_n is $\mu = (\mu_0,\ldots,\mu_n)$; $\mu_k \in J_k = I_k \setminus I_{k-1}$; $k = 0,\ldots,n$. We denote by $\mathcal{C}_n$ the set of all paths. The number of path is $\mathbb{N}_c = \prod_{k=0}^n n_k$, $n_k = card \ J_k = card\Lambda^{(k)} = dim\mathcal{P}(\Lambda) \cap \Pi^0_k$.

2. A particular Hermite bivariate interpolation space

The present article is a continuation of paper [5] and its purpose is to give an integral formula for the remainder in a particular Hermite bivariate interpolation problem from a minimal interpolation space. The set of interpolation conditions are obtained by (1) for the particular choice:

$$q_{j,k}(x) = x^{\alpha_{j,k}},\ x \in \mathbb{R}^2,\ \alpha_{j,k} \in L_j \subset \mathbb{N}^2, \tag{2}$$

$k \in \{0, \dots, l_j - 1\}$, $j \in \{1, \dots, m\}$ and L_j is a lower set of indices, that is, if $\alpha \in L_j$ and $\beta < \alpha$, then $\beta \in L_j$.

The number of interpolation conditions is $M = \sum_{j=1}^{m} l_j$. The space $\mathcal{F}$ in definition of general interpolation problem can be replaced by $C^p(\mathbb{R}^2)$, with $p = \max(l_j)$, $j \in \{1, \dots, m\}$.

Proposition 2.1. *The spaces* $Q_j = span\{q_{j,k} \mid k = 0, \dots, l_j - 1\}$, $j \in \{1, \dots, m\}$ *are D- invariant.*

Proof. Let $q(x) = x^\alpha$, $\alpha = (\alpha_1, \alpha_2) \in L_j$ and $p(x) = x^\gamma$, $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$. Then

$$(p(D)q)(x) = \begin{cases} 0, & if\ \gamma_1 > \alpha_1\ or\ \gamma_2 > \alpha_2 \\ (\alpha_1)_{\gamma_1}(\alpha_2)_{\gamma_2} x^{\alpha - \gamma}, & in\ rest \end{cases}$$

with $(a)_k = a \cdot (a-1) \cdot \ldots \cdot (a-k+1)$.

But $\alpha - \gamma \in L_j$, be cause L_j is a lower set and therefore $(p(D)q)(x) \in Q_j$, that is Q_j is D- invariant. ∎

Corollary 2.1. *The set of functionals* Λ*, considered in this section, gives an ideal interpolation scheme.*

To characterize the interpolation conditions in a suitable way, we use the model introduced by T. Sauer in [4]. We associate to each pair (x_j, Q_j), that is to each subset of functionals $\Lambda_j = \{\lambda_{j,k} = \delta_{x_j} \circ q_{j,k}(D); k = 0, \dots, l_j - 1\}$ the following elements:

1. A tree structure $E_j = \{1|\varepsilon_1^1, \varepsilon_2^1|\dots|\varepsilon_1^{l_j-1}, \dots, \varepsilon_{l_j}^{l_j-1}\}$, with $\varepsilon_i^k \in \{0,1\}$, for any $k \in \{1, \dots l_j - 1\}$, $i \in \{1, \dots, k+1\}$ and, for any $\varepsilon_i^k = 1$, $k > 1$ there exists an unique $j = j(i)$ such that $\varepsilon_j^{k-1} = 1$. We say that ε_j^{k-1} is the predecessor of ε_i^k.

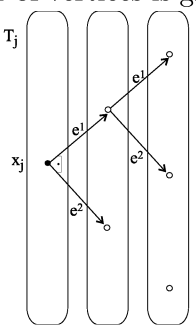
2. A tree T_j according to E_j: $T_j = \{x_j|\ y_1^1,\ y_2^1|\dots|y_1^{l_j-1}, \dots, y_{l_j}^{l_j-1}\}$, $y_i^k \in \{e_1 = (1,0); e_2 = (0,1)\}$ if $\varepsilon_i^k \neq 0$ and $y_i^k = 0$, if $\varepsilon_i^k = 0$.

We say that x_j is the root of the tree and y_i^k for $\varepsilon_i^k = 1$ are vertices of the tree. We denote by $|T_j|$ the length of the tree, that is the number of nonzero vertices, including the root, in T_j. A sequence $\eta = (i_1, \dots, i_k)$ is called a chain in the tree structure E_j if $\varepsilon^1_{i_1} = \ldots = \varepsilon^k_{i_k} = 1$, where $\varepsilon^l_{i_l}$ is the predecessor of $\varepsilon^{l+1}_{i_{l+1}}$, $\forall l = 1, \ldots, k-1$. If $\varepsilon^k_{i_k}$ is not the predecessor of another element in E_j, the chain is called a maximal chain. We denote by $\sigma(\eta) = k$ the length of chain η. A chain $\eta' = (i_1, \ldots, i_j)$ is subordinate to $\eta = (i_1, \ldots, i_k)$, if $j \leq k$. For any chain in E_j, we define the sequence of directions $\mathbf{y}^\eta = (y^1_{i_1}, \ldots, y^k_{i_k})$, with $y_{i_j} \in \{e_1, e_2\}$ and the sequence of directional derivatives $D^{\sigma(\eta)}_{\mathbf{y}^\eta} = D_{y^1_{i_1}} \ldots D_{y^k_{i_k}}$ and $D^0_{\mathbf{y}^0} = I$, where I is the identity operator. We will write $\mathcal{D}_{\mathbf{y}^\eta}$ to denote the derivative $D_{\mathbf{y}^\eta}$ and all derivatives corresponding to the subordinate chains of the chain η.

The following example illustrates the previous notions. More details can be found in [4].

Example 1. Let be the point x_j and the space
$Q_j = span\{1, \xi_1, \xi_2, \xi_1^2, \xi_1\xi_2\}$, with $x = (\xi_1, \xi_2) \in \mathbb{R}^2$.
Then, we obtain $E_j = \{1|1, 1|1, 1, 0\}$, $T_j = \{x_j|e^1, e^2|e^1, e^2, 0\}$ and the maximal chains $\eta_1 = (1, 1)$, $\eta_2 = (1, 2)$, $\eta_3 = (2)$.

In order to make a graphic representation of the tree T_j, we put all the vertices corresponding to the same level on the same vertical. We denoted the vertices of the tree by the symbol "$\circ$", and the root by the symbol "$\bullet$". The vertices on the level k of the tree having $\varepsilon^k_i = 0$ are not used and consequently they will be not connected to another vertices of the tree. In the same level, the number of vertices is given from top to bottom. We obtained the following tree:



The maximal chains correspond to the following derivatives: $D^2_{\mathbf{y}^{\eta_1}} = D_{e^1} D_{e^1} = \dfrac{\partial^2}{\partial \xi_1^2}$; $D^2_{\mathbf{y}^{\eta_2}} = D_{e^2} D_{e^1} = \dfrac{\partial^2}{\partial \xi_2 \partial \xi_1}$; $D^1_{\mathbf{y}^{\eta_3}} = D_{e^2} = \dfrac{\partial}{\partial \xi_2}$. The chain $\eta_4 = (1)$ is subordinates both to η_1 and η_2. $D^1_{\mathbf{y}^{\eta_4}} = D_{e^1} = \dfrac{\partial}{\partial \xi_1}$

The interpolation conditions correspond to maximal chains and to the chains which are subordinated to maximal chains. ■

We can rewrite the set of interpolation conditions as:

$$\Lambda = \{\delta_{x_k} \circ \mathcal{D}^{\sigma(\eta)}_{\mathbf{y}^\eta},\ \forall \eta \in T_k,\ \eta \text{ maximal chain },\ k = 1, \ldots, m\} \tag{3}$$

We observe that l_k interpolation conditions of Hermite type correspond to any point x_k. As a result, we may consider x_k as a point of multiplicity l_k, or as a point having l_k identical copies. We denote the copies of x_k by $x^j_{k,i}$. Thus the point $x^j_{k,i}$ is a copy of the point x_k, situated of level j, that is $\varepsilon^j_{k,i} = 1$ is in the structure of tree E_k, and $\varepsilon^0_{k,1} = 1$. We will use the notation X for the set of all points and theirs copies: $X = \{x^j_{k,i} :\ \varepsilon^j_{k,i} = 1,\ k = 1, \ldots, m\}$.

Let $\mathcal{P}(\Lambda) \subset \Pi^2_n$ be a minimal interpolation space of n order for our conditions. C. de Boor and A. Ron proved, in [1], that always there is a minimal interpolation space with respect to conditions (1) and it is not unique, except the case $\mathcal{P}(\Lambda) = \Pi^2_n$. Then, there exists a Newton basis for this space, or, equivalent, the points in X can be indexed using multiindices $\beta \in I_n$. Obviously, $card(X) = dim\ \mathcal{P}(\Lambda) \leq dim\ \Pi^2_n$. We will use next the notation X_n instead X. If $x_\beta = x^j_{k,i}$ and $\eta = (i_1, \ldots, i_j) \in T_k$ is a chain, we will denote

$$\sigma_\beta = \sigma(\eta) \text{ and } D^{\sigma_\beta}_{\mathbf{y}_\beta} = D^{\sigma(\eta)}_{\mathbf{y}^\eta},\ \text{ that is } \mathbf{y}_\beta = \mathbf{y}^\eta$$

Using the previous notations, the interpolation conditions Λ are given by the pairs $(x_\beta, \mathcal{D}^{\sigma_\beta}_{\mathbf{y}_\beta})$, $\beta \in I_n$.

Definition 2.1. (T. Sauer, Y. Xu, [4]) Let T_x a tree, $\eta = (i_1, \ldots, i_k) \in T_x$ a chain in this tree and $\eta_j = (i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_k)$, $j = 1, \ldots, k$. The tree T_x is regular if for each chain η and each $j = 1, \ldots, k$, there exists a chain $\eta' \in T_x$ such that $\mathbf{y}^{\eta'} = \mathbf{y}^{\eta_j}$

Proposition 2.2. *The tree T_j according to the conditions $\Lambda_j = \{\lambda_{j,k}, k = 0, \ldots, l_j - 1\}$ given in (2) is regular, $\forall\ j = 1, \ldots, m$.*

Proof. The regularity condition consists, in fact, in the commutativity of the interpolation conditions, that is: $\delta_x \circ D^{\sigma(\eta)}_{\mathbf{y}^\eta} = \delta_x \circ (D_{y_{i_1}} \cdots D_{y_{i_k}}) = \delta_x \circ (D_{y_{i_{\tau(1)}}} \cdots D_{y_{i_{\tau(k)}}})$, for any permutation τ of the numbers $1, \ldots, k$.

This property is true in our case be cause the set L_j from (2) is a lower set, $y_i \in \{e_1, e_2\}$ and $\mathcal{F} = C^p(\mathbb{R}^2)$, with $p = \max(l_j)$, $j \in \{1, \ldots, m\}$. ■

Another formulation of the regularity condition of a tree can be given by: for any chain $\eta = (i_1, \ldots, i_k) \in T_x$ and any permutation, τ, of the directions there exists a chain $\eta' \in T_x$ such that $\mathbf{y}^{\eta'} = y_{i_{\tau(1)}}, \ldots, y_{i_{\tau(k-1)}}$.

Corollary 2.1. *Let be $p \in \Pi$, $p \in ker(\Lambda_j)$. Then $D_{\mathbf{y}'}p(x_j) = 0$, for any subset of directions, $\mathbf{y}' \subset \mathbf{y}^\eta$, with $\eta \in T_j$.*

Proof. We know that $D^{\sigma(\eta)}_{\mathbf{y}^\eta}p(x_i) = 0$, $\forall\ \eta \in T_j$, $p \in ker(\Lambda_j)$. The regularity of the tree T_j implies that for any chain η' subordinate to η there is $\eta_1 \in T_j$ such that $\mathbf{y}' = \mathbf{y}^{\eta_1}$ and, consequently, $D_{\mathbf{y}'}p(x_j) = 0$. ■

Definition 2.2. We say that the points set X_n is block minimal derived from Λ, if we can indexed the points and the conditions into blocks

$$X^{(j)} = \{x_\beta :\ \beta \in I_j \setminus I_{j-1}\},\ j = 0, \ldots, n; \tag{4}$$

$card\ X^{(j)} = dim(\mathcal{P}(\Lambda) \cap \Pi_j^0) \leq j+1$, such that for any $k = 1, \ldots, m$ the same level copies of x_k be in the same block, the level order of a point x_k is preserved in X_n blocks order and the interpolation problems with respect to the conditions $\tilde{X}_k = \{(x_\beta, D^{\sigma_\beta}_{\mathbf{y}_\beta}) : \ \beta \in I_k\}$ are poised in $\Pi_k \cap \mathcal{P}(\Lambda)$, $\forall\ k = 0, \ldots, n-1$.

A minimal interpolation space with respect to a set of points block minimal derived from Λ is called block minimal Hermite interpolation space.

We proved in [5] the following proposition:

Proposition 2.3. *If X_n is block minimal derived from Λ, then there exists a Newton basis, (p_α), $\alpha \in I_n$, such that $D^{\sigma_\beta}_{\mathbf{y}_\beta} p_\alpha(x_\beta) = \delta_{\beta,\alpha}$, $\forall \alpha, \beta \in I_n$ and $|\beta| \leq |\alpha|$*

Using corollary 2.1. and the notations from proposition 2.2., we obtain the following corollary:

Corollary 2.2. *The following equality holds, for any $\beta \in \mathbb{N}^2$, with $|\beta| \leq n$ and any $\gamma \in \mathbb{N}^2$, $|\gamma| \leq |\beta|$: $\left(D^{\sigma_\gamma - 1}_{\hat{\mathbf{y}}^k_\gamma} p_\beta\right)(x_\gamma) = 0$, $\forall k = 1, \ldots, \sigma_\gamma$, with $\hat{\mathbf{y}}^k_\gamma = (y_{\gamma,1}, \ldots, y_{\gamma,k-1}, y_{\gamma,k+1}, \ldots, y_{\gamma,\sigma_\gamma})$*

Proposition 2.3. *There exists the unique defined directions, $d_{\gamma,\beta} \in \mathbb{R}^2$,such that, for any $\gamma \in J_{k-1}$, $\beta \in I'_k$, $k = 1, \ldots, n$ the following equality holds:*

$$(5) \quad p_\gamma(x)(x - x_\gamma) - \sum_{\alpha \in J_k} D^{\sigma_\alpha}_{\mathbf{y}^\alpha}(p_\gamma(\cdot)(\cdot - x_\gamma))(x_\alpha) \cdot p_\alpha(x) = \sum_{\beta \in I'_k} d_{\gamma,\beta} \cdot p^\perp_\beta(x)$$

Proof. Let $Q_\gamma(x)$, $\gamma \in J_{k-1}$ be the left term in (5). We will prove that $Q_\gamma(x) \in span\{p^\perp_\beta : \ \beta \in I'_k\}$. First, we consider $\beta \in J_k$. In that case

$$D^{\sigma_\beta}_{\mathbf{y}_\beta}(Q_\gamma)(x_\beta) = D^{\sigma_\beta}_{\mathbf{y}_\beta}(p_\gamma(x)(x - x_\gamma))(x_\beta) - \sum_{\alpha \in J_k} D^{\sigma_\alpha}_{\mathbf{y}_\alpha}(p_\gamma(x)(x - x_\gamma))(x_\alpha) \cdot D^{\sigma_\beta}_{\mathbf{y}_\beta} p_\alpha(x_\beta)$$

We have $D^{\sigma_\beta}_{\mathbf{y}_\beta} p_\alpha(x_\beta) = \delta_{\beta,\alpha}$, if $|\beta| = |\alpha|$ and consequently we obtain $D^{\sigma_\beta}_{\mathbf{y}_\beta} Q_\gamma(x_\beta) = 0$, $\forall \beta \in J_k$.

Let be, next, $\beta \in I_{k-1}$. To obtain the derivatives, we use Leibnitz formula (see [4]):

$$\left(D^{\sigma(\mu)}_{\mathbf{y}^\mu}(pq)\right)(x_j) = \sum_{\mathbf{y}_1 + \mathbf{y}_2 = \mathbf{y}^\mu} D_{\mathbf{y}_1} p(x_j) \cdot D_{\mathbf{y}_2} q(x_j) = 0, \ \forall \mu \in T_j, \ j = 1, \ldots, m,$$

and obtain

$$D^{\sigma_\beta}_{\mathbf{y}^\beta}(p_\gamma(x)(x - x_\gamma))(x_\beta) = (x_\beta - x_\gamma) D^{\sigma_\beta}_{\mathbf{y}^\beta} p_\gamma(x_\beta) + \sum_{j=1}^{\sigma_\beta} y_{\beta,j} D^{\sigma_\beta - 1}_{\hat{\mathbf{y}}^j_\beta} p_\gamma(x_\beta)$$

The corollary 2.2 and the definition of Newton polynomials involve: $D^{\sigma_\beta}_{\mathbf{y}^\beta}(p_\gamma(x)(x-x_\gamma))(x_\beta)=0,\ \forall\beta\in I_{k-1}$. More, be cause $D^{\sigma_\beta}_{\mathbf{y}^\beta}p_\alpha(x_\beta)=0,\ \forall\ \alpha\in J_k,\ \beta\in I_{k-1}$, we have

$$D^{\sigma_\beta}_{\mathbf{y}^\beta}\left(\sum_{\alpha\in J_k} D^{\sigma_\alpha}_{\mathbf{y}_\alpha}(p_\gamma(x)(x-x_\gamma))(x_\alpha)\cdot p_\alpha(x)\right)(x_\beta)=0,\ \forall\beta\in I_{k-1},\ \alpha\in J_k,$$

Therefore $D^{\sigma_\beta}_{\mathbf{y}_\beta}Q_\gamma(x_\beta)=0,\ \forall\beta\in I_k,\ \gamma\in J_{k-1}$. Taking into account that $deg\ Q_\gamma(x)=k$, the proof is complete. ∎

Using a generalization of the divided difference introduced in [4], we defined, recursively, in [5], the k order block divided difference $b_k,\ k\in\{0,\dots,n+1\}$:

$b_0[x;f]=f(x)$

$b_{n+1}[X^{(0)},\dots,X^{(n)},x;f]=$

$$=b_n[X^{(0)},\dots,X^{(n-1)},x;f]-\sum_{\alpha\in J_n} D^{\sigma_\alpha}_{\mathbf{y}_\alpha}b_n[X^{(0)},\dots,X^{(n-1)},x_\alpha;f]\cdot p_\alpha(x),$$

with $D^{\sigma_\alpha}_{\mathbf{y}_\alpha}b_n[X^{(0)},\dots,X^{(n-1)},x_\alpha;f]=\left(D^{\sigma_\alpha}_{\mathbf{y}_\alpha}b_n[X^{(0)},\dots,X^{(n-1)},x;f]\right)(x_\alpha)$.

The following equality holds (see [5]):

$$(\mathbb{R}_n(f))(x)=(f-H_n(f))(x)=b_{n+1}[X^{(0)},\dots,X^{(n)},x;f]$$

3. The remainder formula

To obtain an integral form of the remainder in the interpolation formula from a minimal block Hermite interpolation space, we introduce, like in [4], for every pair of multiindices, $\alpha,\ \beta$, with $|\alpha|=|\beta|-1<n$ and $\mathbf{y}_\beta=(y_{\beta,1},\dots,y_{\beta,\sigma_\beta});\ y_{\beta,i}\in\mathbb{R}^2$:

1. A differential operator $\mathcal{D}_{\alpha,\beta}=\begin{cases} D^{\sigma_\beta}_{\mathbf{y}_\beta} & \text{if}\quad x_\alpha\neq x_\beta\\ \sum_{i=1}^{\sigma_\beta} D^{\sigma_\beta-1}_{\hat{\mathbf{y}}^i_\beta} & \text{if}\quad x_\alpha=x_\beta,\end{cases}$

2. A direction $z_{\alpha,\beta}=\begin{cases} x_\beta-x_\alpha, & \text{if}\quad x_\alpha\neq x_\beta\\ y_{\beta,\sigma_\beta}, & \text{if}\quad x_\alpha=x_\beta\end{cases}$

3. A directional derivative of n order, associate to every path $\mu\in\mathcal{C}_n$
$$D^n_{z^\mu}=D_{z_{\mu_{n-1},\mu_n}}\dots D_{z_{\mu_0,\mu_1}}$$

4. A number $\Pi_\mu(X^\mu,\mathbf{y}^\mu)=\mathcal{D}_{\mu_{n-1},\mu_n}p_{\mu_{n-1}}(x_{\mu_n})\dots\mathcal{D}_{\mu_0,\mu_1}p_{\mu_0}(x_{\mu_1})$, where $X^\mu=\{x_{\mu_0},\dots,x_{\mu_n}\}$.

The following two theorems represent the main result of this paper.

Theorem 3.1. *Let $\mathcal{P}(\Lambda)$ be a minimal block Hermite interpolation space, of n order. Then, the expression of the remainder is:*

$$(6)\ (R_n(f,\Lambda))(x) = \sum_{\mu\in\mathcal{C}_n} p_{\mu_n}(x)\cdot\Pi_\mu(X^\mu,\mathbf{y}^\mu)\int_{\mathbb{R}^2} D_{x-x_{\mu_n}} D^n_{z^\mu} f(t)M(t|X^\mu,x)dt +$$

$$+\sum_{j=1}^{n}\sum_{\mu\in\mathcal{C}_{j-1}}\sum_{\beta\in I'_j} p^\perp_\beta(x)\Pi_\mu(X^\mu,\mathbf{y}^\mu)\int_{\mathbb{R}^2} D_{d_{\mu_{j-1},\beta}} D^{j-1}_{z^\mu} f(t)M(t|X^\mu,x)dt;$$

$x\in\mathbb{R}^2$, where $M(t|V)$ is the normalized simplex spline distribution, given by

$$\int_{\mathbb{R}^2} f(t)M(t|v^0,\dots,v^n)dt = (n-2)!\int_{S_n} f(\sigma_0 v^0+\cdots+\sigma_n v^n)d\sigma,\ \ with$$

$f\in C(\mathbb{R}^2)$ and $S_n=\{\sigma=(\sigma_0,\dots,\sigma_n):\ \sigma_i\geq 0;\ \sigma_0+\cdots+\sigma_n=1\}$

Proof. We know that

$$(7)\qquad D_y M(t|v^0,\dots,v^n) = \sum_{j=0}^{n}\mu_j M(t|v^0,\dots,v^{j-1},v^{j+1},\dots,v^n),$$

where $y=\sum_{j=0}^{n}\mu_j v^j$ and $\sum_{j=0}^{n}\mu_j=0$, $\mu_j\in\mathbb{R}$. Consequently,

$$(8)\qquad D_{v^i-v^j}M(t|v^0,\dots,v^n) =$$
$$= M(t|v^0,\dots,v^{i-1},v^{i+1},\dots,v^n) - M(t|v^0,\dots,v^{j-1},v^{j+1},\dots,v^n),$$

$\forall\ i,\ j=0,\dots,n$

On the other hand, from [2], we have: $\int_{[v^0,\dots,v^n]} f = \int_{\mathbb{R}^2} f(t)M(t|v^0,\dots,v^n)dt$.

We use induction by n and we will prove that the right side of relation (6) is in fact $b_{n+1}[X^{(0)},\dots,X^{(n)},x;f]$.

For $n=0$, $\mathcal{C}_0=\mu_0=\{(0,0)\}$, $X^\mu=\{x_{\mu_0}\}$, $X^{(0)}=\{x_{\mu_0}\}$, $\Pi_\mu(X^{\mu_0},\mathbf{y}^{\mu_0})=1$, $I'_0=\Phi$, $D^0_{z^\mu}=I$ and we obtain the true equality

$$b_1[X^{(0)},x;f] = p_{\mu_0}(x)\int_{[x_{\mu_0},x]} D_{x-x_0}f + 0 = f(x)-f(x_0).$$

We suppose relation (6) true for any $1\leq k\leq n$ and we will prove it for $k=n+1$. The induction hypothesis gives:

$$(9)\qquad b_n[X^{(0)},\dots,X^{(n-1)},x;f] =$$

$$= \sum_{\mu\in\mathcal{C}_{n-1}} p_{\mu_{n-1}}(x)\cdot\Pi_\mu(X^\mu,\mathbf{y}^\mu)\int_{\mathbb{R}^2} D_{x-x_{\mu_{n-1}}} D_{z^\mu}^{n-1} f(t) M(t|X^\mu,x)dt +$$

$$+\sum_{j=1}^{n-1}\sum_{\mu\in\mathcal{C}_{j-1}}\sum_{\beta\in I'_j} p_\beta^{\perp}(x)\Pi_\mu(X^\mu,\mathbf{y}^\mu)\int_{\mathbb{R}^2} D_{d_{\mu_{j-1},\beta}} D_{z^\mu}^{j-1} f(t)M(t|X^\mu,x)dt =$$

$$= T_1(x)+T_2(x)$$

Using the directional derivative of simplex spline (see [4]) and Leibnitz rule, we obtain:

$$D_{\mathbf{y}}^{\sigma}(\partial x)(T_1(x)) = \sum_{\mu\in\mathcal{C}_{n-1}}\sum_{\mathbf{y}_1+\mathbf{y}_2=\mathbf{y}} D_{\mathbf{y}_1}^{\sigma_1}\, p_{\mu_{n-1}}(x)\Pi_\mu(X^\mu,\mathbf{y}^\mu)\sigma_2!\cdot$$

$$\Big(\int_{\mathbb{R}^2} D_{\mathbf{y}_2}^{\sigma_2} D_{x-x_{\mu_{n-1}}} D_{z^\mu}^{n-1} f(t) M(t|X^\mu,\underbrace{x,\dots,x}_{\sigma_2+1\ times})dt + \int_{\mathbb{R}^2} D_{\mathbf{y}_2}^{\sigma_2} D_{z^\mu}^{n-1} f(t)M(t|X^\mu,\underbrace{x,\dots,x}_{\sigma_2\ times})dt\Big)$$

where, for $\sigma_2=0$, the last integral is zero.

The property of Newton polynomials and the regularity property of the tree imply
$D_{\mathbf{y}_1}^{\sigma_1} p_{\mu_{n-1}}(x_\alpha)=0$ for any $\sigma_1<\sigma_\alpha-1$. Thus $\left(D_{\mathbf{y}_\alpha}^{\sigma_\alpha}(\partial x)(T_1(x)\right)(x_\alpha) =$

$$= \sum_{\mu\in\mathcal{C}_{n-1}}\Bigg(D_{\mathbf{y}_\alpha}^{\sigma_\alpha} p_{\mu_{n-1}}(x_\alpha)\Pi_\mu(X^\mu,\mathbf{y}^\mu)\int_{\mathbb{R}^2} D_{x_\alpha-x_{\mu_{n-1}}} D_{z^\mu}^{n-1} f(t)M(t|X^\mu,x_\alpha)dt+$$

$$+\sum_{i=1}^{\sigma_\alpha} D_{\hat{\mathbf{y}}_\alpha^i}^{\sigma_\alpha-1} p_{\mu_{n-1}}(x_\alpha)\Pi_\mu(X^\mu,\mathbf{y}^\mu)\int_{\mathbb{R}^2} D_{y_{\alpha,i}} D_{x_\alpha-x_{\mu_{n-1}}} D_{z^\mu}^{n-1} f(t)M(t|X^\mu,x_\alpha,x_\alpha)dt+$$

$$+\sum_{i=1}^{\sigma_\alpha} D_{\hat{\mathbf{y}}_\alpha^i}^{\sigma_\alpha-1} p_{\mu_{n-1}}(x_\alpha)\Pi_\mu(X^\mu,\mathbf{y}^\mu)\int_{\mathbb{R}^2} D_{y_{\alpha,i}} D_{z^\mu}^{n-1} f(t)M(t|X^\mu,x_\alpha)dt\Bigg)$$

The second sum vanishes for any $\mu\in\mathcal{C}_{n-1}$. On the other hand

$$(10)\qquad D_{\mathbf{y}_\alpha}^{\sigma_\alpha}(\partial x)(T_2(x)) = \sum_{j=1}^{n}\sum_{\mu\in\mathcal{C}_{n-1}}\sum_{\beta\in I'_j}\sum_{\mathbf{y}_1+\mathbf{y}_2=\mathbf{y}_\alpha} D_{y_1}^{\sigma_1} p_\beta^{\perp}(x)\Pi_\mu(X^\mu,\mathbf{y}^\mu)\cdot$$

$$\cdot(-1)^{\sigma_2}\cdot\sigma_2!\int_{\mathbb{R}^2} D_{\mathbf{y}_\alpha}^{\sigma_\alpha} D_{d_{\mu_{j-1},\beta}} D_{z^\mu}^{j-1} f(t)M(t|X^\mu,\underbrace{x,\dots,x}_{\sigma_2+1\ times})$$

$$\left(D_{\mathbf{y}_\alpha}^{\sigma_\alpha}(\partial x)(T_2(\cdot))\right)(x_\alpha)=0,\ \alpha\in J_n$$

The equality from (5) can be reformulated, taking $\gamma=\mu_{n-1}\in J_{n-1}$, $k=n$ and using the linearity property of the directional derivatives:
$p_{\mu_{n-1}}(x)D_{x-x_{\mu_{n-1}}} = \sum_{\alpha\in J_n} p_\alpha(x)\left(D_{\mathbf{y}_\alpha}^{\sigma_\alpha} p_{\mu_{n-1}}(x_\alpha)D_{x_\alpha-x_{\mu_{n-1}}}+\right.$

$$+\sum_{i=1}^{\sigma_\alpha} D^{\sigma_\alpha-1}_{\hat{\mathbf{y}}^i_\alpha} p_{\mu_{n-1}}(x_\alpha) D_{y_{\alpha,i}}\Bigg) + \sum_{\beta\in I'_n} p^{\perp}_{\beta}(x) D_{d_{\mu_{n-1},\beta}}$$

Hence,

$$b_n[X^{(0)},\dots,X^{(n-1)},x;f] = \sum_{\mu\in\mathcal{C}_{n-1}} \Pi_\mu(X^\mu,\mathbf{y}^\mu) \sum_{\alpha\in J_n} p_\alpha(x)\cdot$$

$$\cdot(D^{\sigma_\alpha}_{\mathbf{y}_\alpha} p_{\mu_{n-1}}(x_\alpha) \int_{\mathbb{R}^2} D_{x_\alpha - x_{\mu_{n-1}}} D^{n-1}_{z^\mu} f(t) M(t|X^\mu,x)dt +$$

$$+\sum_{i=1}^{\sigma_\alpha} D^{\sigma_\alpha-1}_{\hat{\mathbf{y}}^i_\alpha} p_{\mu_{n-1}}(x_\alpha) \int_{\mathbb{R}^2} D_{y_{\alpha,i}} D^{n-1}_{z^\mu} f(t) M(t|X^\mu,x)dt) +$$

$$+\sum_{\mu\in\mathcal{C}_{n-1}} \sum_{\beta\in I'_n} \Pi_\mu(X^\mu,\mathbf{y}^\mu) p^{\perp}_{\beta}(x) \int_{\mathbb{R}^2} D_{d_{\mu_{n-1},\beta}} D^{n-1}_{z^\mu} f(t) M(t|X^\mu,x)dt +$$

$$+\sum_{j=1}^{n-1} \sum_{\mu\in\mathcal{C}_{j-1}} \sum_{\beta\in I'_j} p^{\perp}_{\beta}(x) \Pi_\mu(X^\mu,\mathbf{y}^\mu) \int_{\mathbb{R}^2} D_{d_{\mu_{j-1},\beta}} D^{j-1}_{z^\mu} f(t) M(t|X^\mu,x)dt$$

From the definition of block divided difference results:

$$(11)\; b_{n+1}[X^{(0)},\dots,X^{(n)},x;f] = \sum_{\mu\in\mathcal{C}_{n-1}} \Pi_\mu(X^\mu,\mathbf{y}^\mu) \sum_{\alpha\in J_n} p_\alpha(x)\cdot$$

$$\cdot\Bigg(D^{\sigma_\alpha}_{\mathbf{y}_\alpha} p_{\mu_{n-1}}(x_\alpha) \int_{\mathbb{R}^2} D_{x_\alpha - x_{\mu_{n-1}}} D^{n-1}_{z^\mu} f(t) (M(t|X^\mu,x) - M(t|X^\mu,x_\alpha))dt +$$

$$+\sum_{i=1}^{\sigma_\alpha} D^{\sigma_\alpha-1}_{\hat{\mathbf{y}}^i_\alpha} p_{\mu_{n-1}}(x_\alpha) \int_{\mathbb{R}^2} D_{y_{\alpha,i}} D^{n-1}_{z^\mu} f(t) (M(t|X^\mu,x) - M(t|X^\mu,x_\alpha))dt\Bigg) +$$

$$+\sum_{j=1}^{n} \sum_{\mu\in\mathcal{C}_{j-1}} \sum_{\beta\in I'_j} p^{\perp}_{\beta}(x) \Pi_\mu(X^\mu,\mathbf{y}^\mu) \int_{\mathbb{R}^2} D_{d_{\mu_{j-1},\beta}} D^{j-1}_{z^\mu} f(t) M(t|X^\mu,x)dt$$

Using the equality $M(t|X^\mu,x) - M(t|X^\mu,x_\alpha) = -D_{x-x_\alpha} M(t|X^\mu,x_\alpha,x)$ and the integration by parts we obtain:

$$(12)\qquad b_{n+1}[X^{(0)},\dots,X^{(n)},x;f] = \sum_{\mu\in\mathcal{C}_{n-1}} \Pi_\mu(X^\mu,\mathbf{y}^\mu) \sum_{\alpha\in J_n} p_\alpha(x)\cdot$$

$$\cdot\Bigg(D^{\sigma_\alpha}_{\mathbf{y}_\alpha} p_{\mu_{n-1}}(x_\alpha) \int_{\mathbb{R}^2} D_{x-x_\alpha} D_{x_\alpha - x_{\mu_{n-1}}} D^{n-1}_{z^\mu} f(t) M(t|X^\mu,x_\alpha,x)dt +$$

$$+\sum_{i=1}^{\sigma_\alpha} D_{\hat{\mathbf{y}}_\alpha^i}^{\sigma_\alpha-1} p_{\mu_{n-1}}(x_\alpha) \int\limits_{\mathbb{R}^2} D_{x-x_\alpha} D_{y_{\alpha,i}} D_{z^\mu}^{n-1} f(t) M(t|X^\mu, x_\alpha, x)dt \Bigg) +$$

$$+\sum_{j=1}^{n} \sum_{\mu\in\mathcal{C}_{j-1}} \sum_{\beta\in I'_j} p_\beta^\perp(x) \Pi_\mu(X^\mu, \mathbf{y}^\mu) \int\limits_{\mathbb{R}^2} D_{d_{\mu_{j-1},\beta}} D_{z^\mu}^{j-1} f(t) M(t|X^\mu, x)dt$$

The first sum from the first term in (12) vanishes for $x_\alpha = x_{\mu_{n-1}}$. For $\alpha = \mu_n$ and $x_\alpha \neq x_{\mu_{n-1}}$ we have $D_{x_\alpha - x_{\mu_{n-1}}} = D_{x_{\mu_n} - x_{\mu_{n-1}}} = D_{z_{\mu_{n-1},\mu_n}}$. If $x_\alpha = x_{\mu_n} = x_{\mu_{n-1}}$, taking into account the regularity property, for any $i = 1, \ldots, \sigma_\alpha$, there exists a multiindex γ^i, with $|\gamma^i| < |\alpha|$ such that $\mathbf{y}_{\gamma^i} = \hat{\mathbf{y}}_\alpha^i$ and $x_{\gamma^i} = x_\alpha$. The definition of Newton polynomials, implies, for any $|\beta| = |\alpha| - 1$ the following equality:

$$D_{\mathbf{y}_{\gamma^i}}^{\sigma_\alpha-1} p_\beta(x_\alpha) = \begin{cases} 0 & \text{if} \quad x_\beta \neq x_\alpha \\ 1 & \text{if} \quad \beta = \gamma^i, \ i \in \{1, \ldots \sigma_\alpha\} \end{cases}$$

Consequently, the operator $D_{y_{\alpha,i}}$ from the second sum will become for $\alpha = \mu_n$ the operator $D_{z_{\mu_{n-1},\mu_n}}$. Taking into account the definition of $\Pi_\mu(X^\mu, \mathbf{y}^\mu)$, we can observe that in the first sum we have the operator $\mathcal{D}_{\mu_{n-1},\mu_n} p_{\mu_{n-1}}(x_{\mu_n})$ and $\sum_{i=1}^{\sigma_\alpha} D_{\hat{\mathbf{y}}_\alpha^i}^{\sigma_\alpha-1} p_{\mu_{n-1}}(x_{\mu_n})$ will supply $\mathcal{D}_{\mu_{n-1},\mu_n} p_{\mu_{n-1}}(x_{\mu_n})$. Therefore formula (12) turns into (6). ■

Theorem 3.2. *Let $f \in C^{n+1}(\mathbb{R}^2)$, $\Omega \in \mathbb{R}^2$ a convex domain containing the points x_α, $\alpha \in I_n$. If $H_n(f)$ is the projection of f on a minimal block Hermite interpolation space, of n order, then for any $x \in \Omega$ the following inequality holds:*

$$|(f - H_n(f))(x)| \leq \frac{\|f\|_{n+1,\Omega}}{(n+1)!} \sum_{\alpha\in J_n} \sum_{i=1}^{2} |p_\alpha(x)(\xi_i - (\xi_\alpha)_i)| \cdot c_\alpha \tag{13}$$

$$+\sum_{j=1}^{n} \frac{\|f\|_{j,\Omega}}{j!} \sum_{\beta\in I'_j} \|p_\beta^\perp(x)\| \cdot b_{j,\beta}, \qquad x = (\xi_1, \xi_2) \in \Omega,$$

with c_α, $b_{j,\beta} \in \mathbb{R}^2$, given by

$$c_\alpha = \sum_{\mu\in\mathcal{C}_n(\alpha)} |\Pi_\mu(X^\mu, \mathbf{y}^\mu)| \sum_{(\beta_1,\ldots,\beta_n)\in\{1,2\}^n} |(z_{\mu_{n-1},\mu_n})_{\beta_n} \cdots (z_{\mu_0,\mu_1})_{\beta_1}|,$$

$$b_{j,\beta} = \sum_{\mu\in\mathcal{C}_{j-1}} |\Pi_\mu(X^\mu, \mathbf{y}^\mu)| \sum_{(\gamma_1,\ldots,\gamma_n)\in\{1,2\}^n} |(d_{\mu_{j-1},\beta})_{\gamma_j} (z_{\mu_{j-1},\mu_{j-2}})_{\gamma_{j-1}} \cdots (z_{\mu_0,\mu_1})_{\gamma_1}|$$

and $\|f\|_{j,\Omega} = \sup_{x\in\Omega} \max_{|\beta|=j} \dfrac{\partial^j}{\partial x^\beta} f(x)$.

P r o o f. The right term in (6) can be rewritten in the form:

$$(R_n(f))(x)=\sum_{\alpha\in J_n}\sum_{i=1}^{2}p_\alpha(x)(\xi_i-(\xi_\alpha)_i)\sum_{\mu\in\mathcal{C}_n(\alpha)}\Pi_\mu(X^\mu,\mathbf{y}^\mu)\int_{\mathbb{R}^2}D_{e^i}D^n_{z_\mu}f(t)M(t|X^\mu,x)dt$$

$$+\sum_{\beta\in I'_n}p^\perp_\beta(x)\sum_{j=|\beta|}^{n}\sum_{\mu\in\mathcal{C}_{j-1}}\Pi_\mu(X^\mu,\mathbf{y}^\mu)\cdot\int_{\mathbb{R}^2}D_{d_{\mu_{j-1},\beta}}D^{j-1}_{z^\mu}f(t)M(t|X^\mu,x)dt$$

Taking into account that, for any $x=(\xi_1,\xi_2)\in\mathbb{R}^2$, $D^n_{z^\mu}f = \sum_{(\beta_1,\ldots,\beta_n)\in\{1,2\}^n}(z_{\mu_{n-1},\mu_n})_{\beta_n}\cdots(z_{\mu_0,\mu_1})_{\beta_1}\cdot\frac{\partial^n f}{\partial\mu_{\beta_1}\ldots\partial\mu_{\beta_n}}$, and a similar relation holds for $D^{j-1}_{z^\mu}$ and using the equality $\int_{\mathbb{R}^2}M(t|X^\mu,x)dt = \frac{1}{(n+1)!}$, $\forall\mu\in\mathcal{C}_n$, we easily obtain relation (13). ∎

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"Lucian Blaga" University of Sibiu
Faculty of Sciences
Department of Computer Science
Sibiu, ROMANIA
e-mail: d_simian@yahoo.com

Received 30.09.2003