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Mosaic Labyrinths and Uniform Structures

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We construct an uniform structure which is closed with respect two classes: class of all mosaic labyrinths with a hole and class of all mosaic labyrinths without a hole.

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Some problems of labyrinths recognition with automata were considered in [2], [3], [4], [5], [6] . An infinite class of mosaic labyrinths was investigated in [2], [3], [4] . We proved that automation is useless if elements of this class have got a hole. Some kinds of rectangular labyrinths were considered in [5], [6]. We proved that if corresponding graphs of the labyrinths have got a cycle then automata recognition is not possible. In this paper we will construct an uniform structure which is closed with respect two classes: class of all mosaic labyrinths with a hole and class of all mosaic labyrinths without a hole.

All definitions in this paper are from [1].

Uniform structure (u.s.) is structure $\sigma = \left(Z^k, E_n, V, \varphi\right)$, where $E_n = \{0, 1, .., n-1\}, n \geq 1$, $V = (\alpha_1, \alpha_2, ..., \alpha_{h-1})$ is an ordered collection of different non-zero elements from Z^k, φ is local transition function of u.s. $\varphi : (E_n)^h \longrightarrow E_n, \varphi(0, ..., 0) = 0$. Elements of set Z^k are cells of u.s. σ, elements of set E_n are states of cells u.s. σ, collection V for $\alpha \in Z^k$ define ordered neighborhood of cell α, $\overrightarrow{V}(\alpha) = (\alpha, \alpha + \alpha_1, ..., \alpha + \alpha_{h-1})$. Set $V(\alpha)$ is called neighborhood of cell α. Collection V is pattern of neighborhood for u.s. σ.

State of u.s. σ is a function $f : Z^k \longrightarrow E_n$. If $\alpha \in Z^k$ and f is a state of u.s. σ, then $f(\alpha)$ is state of cell α which is defined by state f of u.s. σ. By Σ denote the set of all states of u.s. σ. Define the basic transition function $\Phi : \Sigma \longrightarrow \Sigma$ of u.s. σ such that $\Phi(f) = g$ where $g(\alpha) = \varphi(f(\alpha), f(\alpha + \alpha_1), ..., f(\alpha + \alpha_{h-1}))$. Behavior of u.s. σ is a sequence of states $f_0, f_1, ...$ such that $f_{i+1} = \Phi(f_i), i = 0, 1, 2, ...$ By Σ' denote subclass of the class Σ such that state f is an element of the class Σ' if the set $\left\{f(\alpha) \neq 0 | \alpha \in Z^k\right\}$ is a finite set. States from $\Sigma^{'}$are called configurations of u.s. σ. In the behavior $f_0, f_1, ..$ of u.s. σ, where f_0 is a configuration, $f_i$is said to be a state of configuration f_0 in the moment $i, i = 0, 1, ...$ Obviously, if $f \in \Sigma^{'}$ then $\Phi(f) \in \Sigma^{'}$. Class of behaviors of u.s. σ will be not changed if we assume that a neighborhood of cell α has got form $\left\{\alpha + \beta | \beta \in V^{'}_{\overline{m}}\right\}$, where $\overline{m} = (m_1, m_2, ..., m_k) \in N^k, V^{'}_{\overline{m}} = \{(\beta_1, ..., \beta_k || \beta_i| \leq m_i, i = 1, ..., k\}$(rectangular with center in the cell). By $V_{\overline{m}}$ denote one ordering of elements of the set $V^{'}_{\overline{m}}$. Let $S(k, n, \overline{m})$ be the set of all u.s. in the form of $(Z^k, E_n, V_{\overline{m}}, \varphi)$.

We shall considered the set of u.s. $S(2, 2, (1, 1))$. Pattern of neighborhood we can represented such as in Figure 1.

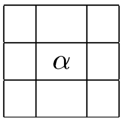

Figure 1.

Here, $V_{\overline{(1,1)}} = ((1, -1), (1, 0), (1, 1), (0, -1), (0, 1), (-1, -1), (-1, 0), (-1, 1)) = (\alpha_1, \alpha_2, ..., \alpha_8)$. The cells $\alpha + (1, -1), \alpha + (1, 0), \alpha + (1, 1), \alpha + (0, -1), \alpha + (0, 1), \alpha + (-1, -1), \alpha + (-1, 0), \alpha + (-1, 1)$ are called *se* (south-east), *e*, *ne*, *s*, *n*, *sw*, *w*, *nw* (north-west) from cell α, successively.

By $\Sigma^{'}_2$ denote the set of all configuration of u.s. $\sigma \in S(2, 2, (1, 1))$. We say that $f \in \Sigma^{'}_2$ is a connected configuration if for all $\alpha, \beta \in Z^2$ such that $f(\alpha) \neq 0$, $f(\beta) \neq 0$, there exists a sequence of cells $\alpha = \alpha_1, \alpha_2, ..., \alpha_p = \beta$, where $f(\alpha_i) \neq 0$ and $\alpha_{i+1} \in \{\alpha_i + (0, 1), \alpha_i + (1, 0), \alpha_i + (0, -1), \alpha_i + (-1, 0)\}, i = 1, ...p$. The sequence $\alpha = \alpha_1, \alpha_2, ..., \alpha_p = \beta$ is called the path which connect cells α and β. We say that $f \in \Sigma^{'}_2$ is a weak connected configuration if for all $\alpha, \beta \in Z^2$ such that $f(\alpha) \neq 0, f(\beta) \neq 0$, there exists a sequence of cells $\alpha = \alpha_1, \alpha_2, ..., \alpha_p = \beta$, where $f(\alpha_i) \neq 0, \alpha_{i+1} \in \left\{\alpha_i + \gamma, \gamma \in V_{(1,1)}\right\}, i = 1, ...p - 1$. Connected components of the configuration f are the maximal connected subsets of the set $1_f = \{\alpha \in \mathcal{Z}^2 | f(\alpha) = 1\}$. We define weak connected components similarly. Arbitrary weak connected component of the set $0_f = \{\alpha \in \mathcal{Z}^2 | f(\alpha) = 0\}$ is called the hole of the configuration f. By $\Sigma^{''}_2$ denote the set of all connected configurations of u.s. $\sigma \in S(2, 2, (1, 1))$. Let $\Sigma^{''}_2 = \mathcal{D}_2 \cup \mathcal{M}_2$ where, $\mathcal{D}_2$ is the set of all configurations from $\Sigma^{''}_2$ with least one finite hole and $\mathcal{M}_2 \subseteq \Sigma^{''}_2$ is determined with configurations f such that $f = 0$ or f has got only one hole (infinity hole). Let $\sigma \in S(2, 2, (1, 1))$ and Φ is a basic transition function of the u.s. σ. The class of states K of u.s. σ is said to be closed with respect to u.s. σ if $\Phi(K) \subseteq K$.

Theorem 1. *There exists u.s.* $\sigma_2 \in S(2, 2, (1, 1))$ *such that*

a) Classes $\mathcal{D}_2$ *and* $\mathcal{M}_2$ *are closed with respect to u.s.* σ_2.

b) For all $f \in \mathcal{D}_2$*, behavior of u.s.* σ_2*,* $f_0 = f$*,* $f_{i+1} = \Phi(f_i), i \geq 0$*, after some time will be in state of stagnation, i.e. for some* $t \in \mathcal{N}$*,* $f_t = f_{t+i}$*, for all* $i > 0$.

c) For all $f \in \mathcal{M}_2$*, behavior of u.s.* σ_2*,* $f_0 = f, f_{i+1} = \Phi(f_i), i \geq 0$*, after some time will be in zero state, i.e. for some* $t \in \mathcal{N}$*,* $f_{t+i} \equiv 0$*, for all* $i > 0$.

Lemma 1. *Let* Φ *is basic transition function of the u.s.* σ_2 *(*σ_2 *is u.s. from Theorem 1.). Then*

a) For all $f \in \mathcal{M}_2$*,* $\min\{t | \Phi^t(f) \equiv 0\} \leq \left\lfloor \frac{|1_f|}{2} \right\rfloor + 1$.

b)For all $f \in \mathcal{D}_2$*,* $\min\{t | \Phi^{t+i}(f) = \Phi^t(f), i \geq 0\} \leq |1_f| - 8$.

Let $\mathcal{D}'_2 \subseteq \Sigma'_2$ such that $f \in \mathcal{D}'_2$ if f has got a finite number of connected components and least one of these components has got a finite hole. Let $\mathcal{M}'_2$ is the set of all configurations $f \in \Sigma'_2$ such that f has got a finite number of connected components and all connected components of f hasn't got a finite hole.

Under these conditions we have:

Corollary 1. *Let u.s.* σ_2 *is from Theorem 1. Then*

a) Classes $\mathcal{D}'_2$ *and* $\mathcal{M}'_2$ *are closed with respect to u.s.* σ_2.

b) For any $f \in \mathcal{D}'_2$ *there exists* $t \in \mathcal{N}$ *such that* $\Phi^t(f) = \Phi^{t+i}(f)$ *for all* $i > 0$.

c) For any $f \in \mathcal{M}_2$ *there exists* $t \in \mathcal{N}$ *such that* $\Phi^{t+i}(f) \equiv 0$ *for all* $i > 0$.

By Lemma 1 and Corollary 1 it follows that

a) for all $f \in \mathcal{D}'_2$, $\min\{t | \Phi^{t+i}(f) \equiv \Phi^t(f), i \geq 0\} \leq \max\limits_{g \in Comp(f)} \{\left\lfloor \frac{|1_g|}{2} \right\rfloor\} + 1$,

b) for all $f \in \mathcal{M}'_2$, $\min\{t | \Phi^t(f) \equiv 0, i \geq 0\} \leq \max\limits_{g \in Comp(f)} \{|1_g|\} - 8$,

where $Comp(f)$ is the set of all components of configuration f.

1)

x	x	x
x	0	x
x	x	x

2)

x	0	x
0	1	x
x	0	x

3)

x	1	x
0	1	0
x	0	x

4)

x	0	x
1	1	0
x	0	x

5)

0	0	x
x	1	0
1	1	x

6)

1	1	x
1	1	0
x	0	x

7)

1	x	x
1	1	0
1	1	x

Figure 2. (x $\in \{0, 1\}$)

Proof of Theorem 1.

Define the local transition function φ_2 of u.s. $\sigma_2 \in S(2,2,(1,1))$.

Set of cell's states, with their neighborhoods which transition function φ_2 move to zero is described in Figure 2.

Remark that:

a) If $f \in \Sigma_2'$ then $|0_f| \leq \left|0_{\Phi(f)}\right|$ (it follows from 1)).

b) By 2), 3), 4), 5) (pattern $\begin{array}{|c|c|c|}\hline 0 & 0 & \mathrm{x} \\ \hline 0 & 1 & 0 \\ \hline 1 & 1 & \mathrm{x} \\ \hline \end{array}$ in 5)) it follows that the state of a cell from which there is only one path (hanging cell) is moved to zero (for example, from centered cell in 3) we can only move to top).

c) Note, that all cells with neighborhoods from 3), 4), 5), 6), 7) have zero at e (east) cell. In 5) there is zero in n cell. In 6) there is zero at s cell. That means that the transition function moves to zero cells which have e, e and ne, e and se zero cells. Thus, situation in which two or more holes would be connected, is impossible. We avoid situations as in Figure 3. For example, in Figure 3 a) west cells from one cell with east cells from another cell will disappear in the same time and two holes are connected. Also, see Figure 3 b) and Figure 3 c).

$$\begin{array}{ccccc} 0&0&0&0&1\\ 0&1&1&1&1\\ 0&1&1&0&1\\ 0&1&1&1&1 \end{array} \quad \begin{array}{ccccc} 0&0&0&0&1\\ 0&1&1&1&1\\ 0&\mathbf{0}&\mathbf{0}&0&1\\ 0&1&1&1&1 \end{array} \qquad \begin{array}{ccccc} 0&0&0&0&1\\ 0&1&1&1&1\\ 0&1&1&0&1\\ 0&1&1&1&1 \end{array} \quad \begin{array}{ccccc} 0&0&0&0&1\\ 0&\mathbf{0}&1&1&1\\ 0&1&\mathbf{0}&0&1\\ 0&1&1&1&1 \end{array}$$

a) b)

$$\begin{array}{ccccc} 1&1&1&1&0\\ 1&1&1&0&1\\ 1&1&1&1&1\\ 0&0&1&1&1 \end{array} \quad \begin{array}{ccccc} 1&1&1&1&0\\ 1&1&\mathbf{0}&0&1\\ 1&\mathbf{0}&1&1&1\\ 0&0&1&1&1 \end{array}$$

c)

Figure 3.

d) It follows from definition of transition function φ_2 that "thin connected neighborhood" cells function φ_2 doesn't move to zero (for example, patterns $\begin{array}{|c|c|c|}\hline 1 & 1 & \mathrm{x} \\ \hline 0 & 1 & 0 \\ \hline 1 & 1 & \mathrm{x} \\ \hline \end{array}$, $\begin{array}{|c|c|c|}\hline 0 & 1 & 0 \\ \hline 0 & 1 & 1 \\ \hline 0 & 0 & 1 \\ \hline \end{array}$, $\begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline 1 & 1 & 0 \\ \hline \end{array}$, $\begin{array}{|c|c|c|}\hline 0 & 0 & \mathrm{x} \\ \hline 1 & 1 & 0 \\ \hline 0 & 1 & \mathrm{x} \\ \hline \end{array}$, ...)

By a), b), c), d) it follows that if $f \in \Sigma_2''$ then configurations $f_{i+1} = \Phi(f_i), i \geq 0$ keep number of holes and connectivity as in the configuration $f_0 = f$. Therefore, the classes $\mathcal{D}_2$ and $\mathcal{M}_2$ are closed with respect to u.s. σ_2.

Combining a) and finiteness of set 1_f, where $f \in \Sigma_2''$, we obtain that there exists $t \in \mathcal{N}$ such that $f_{t+i} = f_t$, $i \geq 0$.

If $0 \neq f \in \mathcal{M}_2$ then basic transition function Φ of u.s. σ_2 will move least one cell of configuration f to zero (see the proof of Lemma 1.). Hence, by finiteness of set 1_f, it follows that there exists $t \in \mathcal{N}$ such that $\Phi^t(f) = 0$. ■

Proof of Lemma 1.

a) The Lemma 1 is a consequence of statement: if $f \in \mathcal{M}_2, |1_f| \geq 2$, then basic transition function Φ of u.s. σ_2 will move least two cells of configuration f to zero. The proof of the statement is by induction of cells number in a configuration from $\mathcal{M}_2$. Let $f \in \mathcal{M}_2, |1_f| \geq 2$. If $|1_f| \in \{2, 3, 4\}$ the proof is trivial. Let $f \in \mathcal{M}_2$ such that $|1_f| = n$ for some $n \in N$. Suppose that the statement is true for $f \in \mathcal{M}_2$ such that $|1_f| < n$. Look at the most higher most right cell α of configuration f. If neighborhood of the cell α is as in Figure 4 a) then, if we change (for a moment) the state of the cell α we will increase the number of connected components of the configuration f. By induction hypothesis it follows that the statement is true for these components. So, there are least two cells in configuration f (two "remote" cells in different connected components) which basic transition function Φ of u.s. σ_2 will move to zero. If the neighborhood of the cell α is as in Figure 4 b) then, local transition function φ_2 of u.s. σ_2 will change the state of the cell α (by 5) in Figure 2). Let g is configuration that is different from configuration f only at the state of cell α. By induction hypothesis for configuration g the statement is true. So, except cell α, there exists least one more cell in configuration f which state the basic transition function Φ of u.s. σ_2 will move to zero.

0	0	0
1	1	0
0	1	x

a)

0	0	0
1	1	0
1	1	x

b)

Figure 4. (x $\in \{0, 1\}$)

b) Let $f \in \mathcal{D}_2$. The configurations $f_{i+1} = \Phi(f_i)$, $i \geq 0$ have the same number of holes as in the configuration $f = f_0$. The smallest connected component with one hole has eight cells (see Figure 5). It follows that sequence of states $f_{i+1} = \Phi(f_i), i \geq |1_f| - 8$ is a stationary.

0	0	0	0	0
0	1	1	1	0
0	1	0	1	0
0	1	1	1	0
0	0	0	0	0

Figure 5.

This completes the proof. ■

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