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Some Aspects in Discrete Asymptotic Analysis

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In this paper we show the principal three methods used to find asymptotic formulae of first order for functions of natural variable. Some nontrivial examples are given.

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0. Introduction. The discrete asymptotic analysis

Usually, the asymptotic analysis is considered in relation with the functions of real or complex variable, to help the solution of some problems of differential equations, mechanics, astronomy and other domains. But some important theoretical examples were of the domain of the natural variables; there are some old and classical today, e.g.

(α) The asymptotic formula of the harmonic sum $H_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, $H_n = \ln n + \gamma + \varepsilon_n$, where $\gamma = \lim\limits_{n\to\infty}(H_n - \ln n)$ is the constant of Euler and $\varepsilon \to 0$.

(β) The formula of Stirling $n! \approx n^n \mathrm{e}^{-n}\sqrt{2\pi n}$, which means that the limit of the ratio of the two parts is equal to 1.

(γ) The formula of the prime numbers not exceeding n, $\pi(n)$, namely $\pi(n) \approx \frac{n}{\ln n}$, which has a rich and beautiful history, including Legendre, Gauss, Tchebycheff, Hadamard, De la Vallée Poussin, and also, Selberg and Erdös.

The first modern form of the Asymptotic Analysis was given by Th. Stieltjes and H.Poincaré, in 1886. Later E.Landau introduced the symbols O and o.

Some of the most important texts in this domain are the works [17], [6]-[11], [3], [5], [13], [12], [18], [16],[20], [22].

Generally speaking about the asymptotic analysis, if D is a domain in $\mathbb{R}$ or in $\mathbb{C}$, $f, g : D \to \mathbb{C}$, and $x_0 \in D'$, we denote that (in a neighborhood U of x_0):

a) $f = O(g)$ if there are two constants $M > 0$ and $c > 0$ so that $f(x) < cg(x)$ for any $x \in U$, with $|x| < M$.

b) $f = o(g)$ if $\lim\limits_{\substack{x \to x_0 \\ (x \in U)}} \dfrac{f(x)}{g(x)} = 0$.

Also,

c) $O(1)$ is a notation for an expression which is bounded for $x \to \infty$;

d) $o(1)$ is a notation for an expression which tends to zero, where $x \to \infty$.

e) The functions f and g are called asymptotic equivalent (in U) and we write $f \sim g$ if $\lim\limits_{\substack{x \to x_0 \\ (x \in U)}} \dfrac{f(x)}{g(x)} = 1$.

Now let $(u_k)_k$ be a sequence of functions defined on D so that $u_{k+1} = o(u_k)$ in U, for every $k = 0, 1, 2, \ldots$. If there is a sequence of constants $(a_k)_k$ so that $f(x) \sim a_0 u_0(x) + a_1 u_1(x) + \ldots + a_k u_k(x)$, for all $k = 0, 1, 2, \ldots, n$, and for all $n \in \mathbb{N}$, we say that the series $\sum\limits_{k=0}^{\infty} a_k u_k(x)$ is an asymptotic expansion of the function f in the point $x \in U$, respecting the sequence of functions $(u_k)_k$; the coefficients $a_0, a_1, a_2, \ldots, a_n$ are called the coefficients of the expansion, or, still, the iterated limits of the function f, because they are given by the formulas: $a_0 = \lim\limits_{x \to x_0} \dfrac{f(x)}{u_0(x)}$, $a_1 = \lim\limits_{x \to x_0} \dfrac{f(x) - a_0 u_0(x)}{u_1(x)}$, $a_2 = \lim\limits_{x \to x_0} \dfrac{f(x) - a_0 u_0(x) - a_1 u_1(x)}{u_2(x)}$ etc. So $f(x)$ is structured respecting the order of the successive functions u_k; the term $a_0 u_0(x)$ "extracts" the "principal" part of $f(x)$, the term $a_1 u_1(x)$ "extracts" the "principal" part of $f(x_1) - a_0 u_0(x)$ etc.

The precedent considerations also are valid if $\infty \in D'$ and $x_0 \to \infty$.

For the functions of discrete variable, more precisely, "equidistant" discrete variable, i.e. situated in an infinite arithmetical progression (which is equivalent with the natural variable), all the precedent facts are valid, but for the unique accumulation point of $\mathbb{N}$, namely $x_0 = \infty$. One of the purposes of the discrete asymptotic analysis is to find asymptotic evaluations of first order or asymptotic expansion of the sequences ([2],[16], [18], [19], [25]-[38]).

In this work we present some methods for finding the asymptotic expressions of first order in the discrete asymptotic analysis.

1. First method: The use of the convergence of certain adequate sequences

This method is able to give us asymptotic formulas of first order for many sequences.

In the classical example of Euler, using the well-known inequality of Neper

$$\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}$$

and its consequence called as double inequality of Schlomilch-Lemonnier (see[1])

$$\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+1} < \ln(n+1) < 1 + \frac{1}{2} + \ldots + \frac{1}{n},$$

a classical and easy procedure shows us that the sequence $(\gamma_n)_{n\geq 1}$, of general term $\gamma_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n$ is convergent. Let $\gamma = \lim_{n\to\infty} \gamma_n$ be. So we obtain that $\lim_{n\to\infty}(H_n - \ln n - \gamma) = 0$; putting $\varepsilon_n = H_n - \ln n - \gamma$, we obtain the asymptotic formula (mentioned is the point (α) of the introduction), which is of the form

$$H_n = \ln n + \gamma + o(1). \tag{1.1}$$

We have $\frac{1}{2n+1}, \gamma_n - \gamma < \frac{1}{2n}$. The best lower and upper bounds were established in [4], $\frac{1}{2n+\alpha} < \gamma_n - \gamma < \frac{1}{2n+1/3}$, where $\alpha = \frac{1}{1-\gamma} - 2 = 0,365272113\ldots$.

Another classical example consists in the evaluation of the divergent sum $T_n = \sum_{k=2}^{n} \frac{1}{k\ln k}$. Using the known inequality (analogue of the inequality of Neper)

$$\frac{1}{(n+1)\ln(n+1)} < \ln(\ln(n+1)) - \ln(\ln n) < \frac{1}{n\ln n}$$

and its consequence

$$\sum_{k=2}^{n} \frac{1}{(k+1)\ln(k+1)} < \ln(\ln(n+1)) - \ln(\ln 2) < \sum_{k=2}^{n} \frac{1}{k\ln k}$$

it is easy to show that the sequence $(\beta_n)_{n\geq 2}$ of general term

$$\beta_n = \left(\sum_{k=2}^{n} \frac{1}{k\ln k}\right) - \ln(\ln n)$$

is convergent to a limit β ($\beta = 0,794684074$). So we obtain the asymptotic formula

$$T_n = \ln(\ln n) + \beta + o(1). \tag{1.2}$$

Let now $S_n = \log_2 3 + \log_3 4 + \ldots + \log_n(n+1)$ $(n \geq 2)$ be (from [23]). In our work [34], taking into account that $\log_k(k+1) = 1 + \frac{\ln(1+1/k)}{\ln k}$, $k \geq 2$ and so $S_n = (n-1) + \sum_{k=1}^{n} \frac{1}{\ln k} \ln\left(1 + \frac{1}{k}\right)$, we have proved that the sequence of general term $\alpha_n = S_n - (n-1) - \sum_{k=2}^{n} \frac{1}{k \ln k}$ is convergent to a limit α; $\alpha = -0,242840285\dots$. So, in a first step, we have obtained the asymptotic formula of first order

$$S_n = (n-1) + \sum_{k=2}^{n} \frac{1}{k \ln k} + \alpha + o(1). \tag{1.3}$$

Because (1.2), finally we have obtained the asymptotic formula of first order

$$S_n = (n-1) + \ln(\ln n) + A + o(1), \tag{1.4}$$

where $A = \alpha + \beta = 0,551843789\dots$.

2. Second method. The use of theorem of Cesàro-Stolz if we search the first iterated limit and we know the limit or the principal part of a sequence

We note first that the classical theorem of Cesàro-Stolz also is valid for the monotone sequences converging to zero ([15], pag. 56).

So, as a first example, if we consider the sequence $e_n = \left(1 + \frac{1}{n}\right)^n$, we find the limit $l = \lim_{n\to\infty} \frac{\mathrm{e} - \left(1 + \frac{1}{n}\right)^n}{\frac{1}{n}}$ passing to the real variable $t \to 0$ or using first the theorem of Cesàro-Stolz and after this the real variable and we obtain that $l = \frac{\mathrm{e}}{2}$. So, we have

$$\left(1 + \frac{1}{n}\right)^n = \mathrm{e} - \frac{\mathrm{e}}{2n} + o(1). \tag{2.1}$$

The inequality from [24], page 38 and [21], pp. 104, 105, 175, 291, 293.

$$\frac{\mathrm{e}}{2n+2} < \mathrm{e} - \left(1 + \frac{1}{n}\right)^n < \frac{\mathrm{e}}{2n+1}$$

also characterizes the first order of convergence of the sequence $(e_n)_{n\geq 1}$ to e.

A second example is given by the sequence $(\gamma_n)_{n\geq 1}$ which converges to the constant of Euler γ. Using analog methods, we obtain $\lim_{n\to\infty} \frac{\gamma_n - \gamma}{\frac{1}{n}} = \frac{1}{2}$ and so

$$\gamma_n = \gamma - \frac{1}{2n} + o(1). \tag{2.2}$$

An inequality which characterizes the order of convergence of $(\gamma_n)_{n\geq 1}$ to γ is

$$\frac{1}{2n+1} < \gamma_n - \gamma < \frac{1}{2n}$$

(see [28]). It also permits to obtain the precedent limit, equal to $\frac{1}{2}$.

In a short paper of 1998*), starting from a result of Andrica and Tóth, we have obtained for the convergent sequence $(a_n(\alpha))_n$ of general term

$$a_n(\alpha) = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots + \frac{1}{n^\alpha} - \frac{n^{1-\alpha}}{1-\alpha}, \quad \alpha \in (0,1),$$

the inequality

$$\frac{1}{(2+1/n)\, n^\alpha} < a_n(\alpha) - a(\alpha) < \frac{1}{2n^\alpha} \quad n \geq 1$$

$(\lim\limits_{n\to\infty} a_n(\alpha) = a(\alpha))$. This gives us

$$\lim_{n\to\infty} n^\alpha (a_n(\alpha) - a(\alpha)) = \frac{1}{2}$$

and so

$$a_n(\alpha) = a(\alpha) + \frac{1}{2n^\alpha} + o(1). \tag{2.3}$$

In [19] A. Lupaş and I have obtained for the sequence of T. Lalescu $(L_n)_{n\geq 1}$, of general term $L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}$ $(n \geq 2)$ the inequality

$$\frac{1}{2en}\left(1 - \alpha\frac{\ln n}{n}\right) < L_n - \frac{1}{e} < \frac{1}{2en} \qquad \left(\alpha > \frac{1}{4},\ n \geq n_0\right)$$

and so $\lim\limits_{n\to\infty} n\left(L_n - \frac{1}{e}\right) = \frac{1}{2e}$. We have the asymptotic formula of first order

$$L_n = \frac{1}{e} + \frac{1}{2en} + o(1). \tag{2.4}$$

In all these examples $o(1)$ is also $o(\varepsilon_n)$, where $(\varepsilon_n)_n$ is the sequence tending to zero which appears in the formulas (2.1) - (2.4) as difference between the general term and its limit and anticipates certain asymptotic expansions.

*) D. Andrica, V. Berinde, L. Tóth, A. Vernescu: *The order of convergence of certain sequences*, Gazeta Matematică 103(1998) no 7-8, 282-286 (in Romanian).

3. Third method. The use of the generating functions if we search an asymptotic expression for the regular lacunary sums of a given sum for which we know an asymptotic expression

As a first example let H_n be the harmonic sum, $H_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, and $r, \alpha \in \mathbb{N}^*$, $r>1$, $1 \le \alpha \le r$, fixed. We consider the regular lacunary sums

$$H_{n,\alpha}^{(r)} = \frac{1}{\alpha} + \frac{1}{\alpha+r} + \frac{1}{\alpha+2r} + \ldots + \frac{1}{\alpha+(n-1)r} \tag{3.1}$$

for which we search an asymptotic expression.

In our previous paper [32], using the generating function $G_n^{[r|}: \mathbb{C} \to \mathbb{C}$, $G_n^{[r]}(u) = \frac{u}{1}+\frac{u^2}{2}+\frac{u^3}{3}+\ldots+\frac{u^{rn}}{rn}$, its values in the points $1, \omega_1, \omega_2, \ldots, \omega_{r-1}$, where $\omega^r = 1$, and its decomposition $G_n^{[r]}(u) = -\log(1-u) - I_n^{(r)}(u)$, where $I_n^{(r)}(u) = \int\limits_0^u \frac{z^{rn}}{1-z}\mathrm{d}z$ (for $u \in \mathbb{C}\backslash[1,\infty)$), and $I_n^{(r)}(u) \xrightarrow[(n\to\infty)]{} 0$ if $u \in \mathbb{C}\backslash\{1\}$, $|u| = 1$, we have found the asymptotic expressions of first order of the sums $H_{n,r}^{(\alpha)}$. We denote by log the complex logarithm and by ln the real logarithm. So, we have obtained, in particular, the following equalities:

a) For the case $r = 2$:

$$\begin{cases} H_{n,1}^{(2)} = 1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1} = \frac{1}{2}\ln n + \frac{1}{2}(\gamma + 2\ln 2) + o(1); \\ H_{n,2}^{(2)} = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n} = \frac{1}{2}\ln n + \frac{1}{2}\gamma + o(1). \end{cases} \tag{3.2}$$

b) For the case $r = 3$:

(3.3)
$$\begin{cases} H_{n,1}^{(3)} = 1+\frac{1}{4}+\frac{1}{7}+\ldots+\frac{1}{3n-2} = \frac{1}{3}\ln n + \frac{1}{3}\left(\gamma + \frac{3}{2}\ln 3 + \frac{\pi\sqrt{3}}{6}\right) + o(1); \\ H_{n,2}^{(3)} = \frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\ldots+\frac{1}{3n-1} = \frac{1}{3}\ln n + \frac{1}{3}\left(\gamma + \frac{3}{2}\ln 3 - \frac{\pi\sqrt{3}}{6}\right) + o(1); \\ H_{n,3}^{(3)} = \frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\ldots+\frac{1}{3n} = \frac{1}{3}\ln n + \frac{1}{3}\gamma + o(1). \end{cases}$$

c) For the case $r = 4$:

(3.4)
$$\begin{cases} H_{n,1}^{(4)} = 1 + \frac{1}{5} + \frac{1}{9} + \ldots + \frac{1}{4n-3} = \frac{1}{4}\ln n + \frac{1}{4}\left(\gamma + 3\ln 2 + \frac{\pi}{2}\right) + 0(1) \\ H_{n,2}^{(4)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{1}{4n-2} = \frac{1}{4}\ln n + \frac{1}{4}(\gamma + 2\ln 2) + o(1) \\ H_{n,3}^{(4)} = \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots + \frac{1}{4n-1} = \frac{1}{4}\ln n + \frac{1}{4}\left(\gamma + 3\ln 2 - \frac{\pi}{2}\right) + o(1) \\ H_{n,4}^{(4)} = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \ldots + \frac{1}{4n} = \frac{1}{4}\ln n + \frac{1}{4}\gamma + o(1) \end{cases}.$$

In these formulas the constants involve π because the complex logarithm had appeared in the decomposition of the generating function $G_n^{[r]}(u)$.

In our work [38], using the same method for the sums
$$E_{n,\alpha}^{(r)} = \frac{1}{\alpha!} + \frac{1}{(\alpha+r)!} + \frac{1}{(\alpha+2r)!} + \ldots + \frac{1}{(\alpha+(n-1)r)!},$$
we have obtained, with $G_n^{[r]}(u) = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \ldots + \frac{u^{rn}}{(rn)!}$, the formulas

$$E_{n,0}^{(2)} = 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots + \frac{1}{(2n)!} = \frac{1}{2}\left(\mathrm{e} + \frac{1}{\mathrm{e}}\right) + o(1) = \mathrm{ch}1 + o(1);$$
$$E_{n,1}^{(2)} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots + \frac{1}{(2n+1)!} = \frac{1}{2}\left(\mathrm{e} - \frac{1}{\mathrm{e}}\right) + o(1) = \mathrm{sh}1 + o(1).$$
$$E_{n,0}^{(3)} = 1 + \frac{1}{3!} + \frac{1}{6!} + \ldots + \frac{1}{(3n)!} = \frac{1}{3}\left(E_{3n} + G^{[3]}(\omega) + G^{[3]}(\omega^2)\right)$$
$$= \frac{1}{3}\left(\mathrm{e} + \mathrm{e}^{\omega} + \mathrm{e}^{\omega^2}\right) + o(1) = \left(\mathrm{e} + \frac{2}{\sqrt{\mathrm{e}}}\cos\frac{\sqrt{3}}{2}\right) + o(1)$$
$$E_{n,1}^{(3)} = \frac{1}{1!} + \frac{1}{4!} + \frac{1}{7!} + \ldots + \frac{1}{(3n+1)!} = \frac{1}{3}\left(E_{3n} + \omega^2 G^{[3]}(\omega) + \omega G^{[3]}(\omega^2)\right)$$
$$= \frac{1}{3}\left(\mathrm{e} + \omega^2\mathrm{e}^{\omega} + \omega\mathrm{e}^{\omega^2}\right) + o(1) = \frac{1}{3}\left(\mathrm{e} + \frac{2}{\sqrt{\mathrm{e}}}\cos\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right) + o(1)$$
$$E_{n,2}^{(3)} = \frac{1}{2!} + \frac{1}{5!} + \frac{1}{8!} + \ldots + \frac{1}{(3n+2)!} = \frac{1}{3}\left(E_{3n} + \omega G(\omega) + \omega^2 G(\omega^2)\right)$$
$$= \frac{1}{3}\left(\mathrm{e} + \omega^2\mathrm{e}^{\omega} + \omega\mathrm{e}^{\omega^2}\right) + o(1) = \frac{1}{3}\left(\mathrm{e} + \frac{2}{\sqrt{\mathrm{e}}}\cos\left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right)\right) + o(1).$$
$$E_{n,0}^{(4)} = 1 + \frac{1}{4!} + \frac{1}{8!} + \ldots + \frac{1}{(4n)!} = \frac{1}{4}\left(E_{4n} + G^{[4]}(\mathrm{i}) + G^{[4]}(-1) + G^{[4]}(-\mathrm{i})\right)$$
$$= \frac{1}{4}\left(\mathrm{e} + \frac{1}{\mathrm{e}} + 2\cos 1\right) + o(1) = \frac{1}{2}\left(\mathrm{ch}1 + \cos 1\right) + o(1)$$

$$E^{(4)}_{n,1} = \frac{1}{1!} + \frac{1}{5!} + \frac{1}{9!} + \ldots + \frac{1}{(4n+1)!} = \frac{1}{4}\Big(E_{4n} - \mathrm{i}G^{[4]}(\mathrm{i}) - G^{[4]}(-1) + \mathrm{i}G^{[4]}(-\mathrm{i})\Big)$$
$$= \frac{1}{4}\left(\mathrm{e} - \frac{1}{\mathrm{e}} + 2\sin 1\right) + o(1) = \frac{1}{2}\,(\mathrm{sh}1 + \sin 1) + o(1)$$
$$E^{(4)}_{n,2} = \frac{1}{2!} + \frac{1}{6!} + \frac{1}{10!} + \ldots + \frac{1}{(4n+2)!} = \frac{1}{4}\Big(E_{4n} - G^{[4]}(\mathrm{i}) + G^{[4]}(-1) - G^{[4]}(-\mathrm{i})\Big)$$
$$= \frac{1}{4}\left(\mathrm{e} + \frac{1}{\mathrm{e}} - 2\cos 1\right) + o(1) = \frac{1}{2}(\mathrm{ch}1 - \cos 1) + o(1)$$
$$E^{(4)}_{n,3} = \frac{1}{3!} + \frac{1}{7!} + \frac{1}{11!} + \ldots + \frac{1}{(4n+3)!} = \frac{1}{4}\Big(E_{4n} + \mathrm{i}G^{[4]}(\mathrm{i}) - G^{[4]}(-1) - \mathrm{i}G^{[4]}(-\mathrm{i})\Big)$$
$$= \frac{1}{4}\left(\mathrm{e} - \frac{1}{\mathrm{e}} - 2\sin 1\right) + o(1) = \frac{1}{2}\,(\mathrm{sh}1 - \sin 1) + o(1).$$

Now, we raise the following problem (which we consider open): let $\zeta_n(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots + \frac{1}{n^s}$, $s \in \mathbb{R}$, $s > 1$ (or $s \in \mathbb{C}$, $|s| > 1$) be the n-th partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$, which defines the famous function ζ of Riemann: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, (if $s > 1$, respectively $\mathrm{Re}(s) > 1$). We consider for $\alpha, r \in \mathbb{N}^*$, $r > 1, 1 \leq \alpha \leq r$ the sums:

$$\zeta^{(r)}_{n,\alpha}(s) = \frac{1}{a^s} + \frac{1}{(\alpha + r)^s} + \frac{1}{(\alpha + 2r)^s} + \ldots + \frac{1}{(\alpha + (n-1)r)^s}.$$

The problem which we raise is: to find the asymptotic expressions (of first order) for $\zeta^{(r)}_{n,\alpha}(s)$. Of course, the asymptotic expression of first order of $\zeta_n(s)$, as

$$\zeta_n(s) = \zeta(s) - \frac{1}{sn^{s-1}} + o(1/n^{s-2}),$$

are known ([14], pg. 262, [30], but to express the generating function in a convenient manner seems unknown to us, now.

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