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An Inverse Problem of Spectral Analysis for Differential Systems on the Half-Line

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An inverse spectral problem is studied for non-selfadjoint systems of ordinary differential equations on the half-line. We give a formulation of the inverse problem, study properties of spectral characteristics, and prove the uniqueness theorem for the solution of the inverse problem.

1. Introduction

Consider the following system of differential equations on the half-line

$$\ell Y(x) := Q_0 Y'(x) + Q(x)Y(x) = \rho Y(x), \quad x > 0. \tag{1.1}$$

Here $Y = [y_k]^T_{k=\overline{1,n}}$ is a column-vector (T is the sign for the transposition), ρ is the spectral parameter, $Q_0 = \mathrm{diag}[q_k]_{k=\overline{1,n}}$, $Q(x) = [q_{kj}(x)]_{k,j=\overline{1,n}}$, where $q_k \neq 0$, $k = \overline{1,n}$ are complex numbers, $q_{kk}(x) \equiv 0$, $q_{kj}(x)$, $k \neq j$ are complex-valued functions, and $q_{kj}(x) \in L(0,\infty)$. The matrix $Q(x)$ is called the potential.

In this paper we study the inverse problem of spectral analysis for non-selfadjoint system (1.1) on the half-line in the general case, i.e. with arbitrary characteristic numbers and with arbitrary behavior of the spectrum. As the main spectral characteristics for (1.1) we introduce and study the so-called Weyl matrix which is an analog of the Weyl function for the Sturm-Liouville operator. We obtain analytical, asymptotical and structural properties of the Weyl matrix, and prove the uniqueness theorem for the solution of the inverse problem from the Weyl matrix. Note that some aspects of the inverse problem theory for differential operators are reflected in the monographs [1]-[4] and other works, but the inverse problem theory for system (1.1) on the half-line and on a finite interval in the general case has not been constructed yet.

Denote $\beta_k = 1/q_k$. Let for definiteness, $\beta_k \neq \beta_j$ for $k \neq j$. It is known (see [5]) that the ρ- plane can be partitioned into sectors $S_j = \{\rho : \arg\rho \in (\theta_j, \theta_{j+1})\}$, $j = \overline{0, 2r-1}$, $0 \le \theta_0 < \theta_1 < \ldots < \theta_{2r-1} < 2\pi$ in which there exist permutations $i_k = i_k(S_j)$ of the numbers $1, \ldots, n$, such that for the numbers $R_k = R_k(S_j)$ of the form $R_k = \beta_{i_k}$ one has

$$\mathrm{Re}(\rho R_1) < \ldots < \mathrm{Re}(\rho R_n), \quad \rho \in S_j. \tag{1.2}$$

Let a matrix $h = [h_{\xi\nu}]_{\xi,\nu=\overline{1,n}}$, $\det h \neq 0$, where $h_{\xi\nu}$ are complex numbers, be given. We introduce linear forms $U(Y) = [U_\xi(Y)]^T_{\xi=\overline{1,n}}$ by the formula $U(Y) = hY(0)$, i.e. $U_\xi(Y) = [h_{\xi 1}, \ldots, h_{\xi n}]Y(0)$. Denote $\Omega^0_{mk}(j_1, \ldots, j_m) = \det[h_{\xi,j_\nu}]$, $\xi = \overline{1, m-1}, k$; $\nu = \overline{1,m}$, $\Omega^0_m(j_1, \ldots, j_m) := \Omega^0_{mm}(j_1, \ldots, j_m)$, $1 \le m \le k \le n$, $\Omega^0_0 := 1$. Let

$$\Omega^0_m(i_1, \ldots, i_m) \neq 0, \quad m = \overline{1, n-1}, \quad j = \overline{0, 2r-1}, \tag{1.3}$$

where $i_k = i_k(S_j)$ is the above-mentioned perturbation for the sector S_j. Condition (1.3) is called the regularity condition for the pair $L = (\ell, U)$. Systems, which do not satisfy the regularity condition, possess qualitatively different properties for investigating inverse problems, and are not considered in this paper. Without loss of generality we assume that the following normalizing conditions are fulfilled: $\det h = 1$, and for a fixed sector (for definiteness, for the sector S_0) one has $\Omega^0_m(i_1, \ldots, i_m) = 1$, $m = \overline{1, n-1}$. Everywhere below we shall assume that the regularity conditions and the normalizing conditions are fulfilled.

Weyl matrix. Let vector-functions $\Phi_m(x, \rho) = [\Phi_{km}(x, \rho)]^T_{k=\overline{1,n}}$, $m = \overline{1,n}$ be solutions of (1.1) satisfying the conditions $U_\xi(\Phi_m) = \delta_{\xi m}$, $\xi = \overline{1,m}$, and also $\Phi_m(x, \rho) = O(\exp(\rho R_m x))$, $x \to \infty$, $\rho \in S_j$ in each sector S_j with property (1.2). Here and in the sequel, $\delta_{\xi m}$ is the Kronecker symbol. It will be shown in Section 2 that these conditions uniquely determine the solutions $\Phi_m(x, \rho)$. Denote $M_{m\xi}(\rho) = U_\xi(\Phi_m)$, $\xi > m$, $M(\rho) = [M_{m\xi}(\rho)]_{m,\xi=\overline{1,n}}$, $M_{m\xi}(\rho) = \delta_{\xi m}$ for $\xi \le m$, $\Phi(x, \rho) = [\Phi_1(x, \rho), \ldots, \Phi_n(x, \rho)] = [\Phi_{km}(x, \rho)]_{k,m=\overline{1,n}}$. The functions $\Phi_m(x, \rho)$ and $M_{m\xi}(\rho)$ are called the Weyl solutions and the Weyl functions respectively. The matrix $M(\rho)$ is called the Weyl matrix or the spectrum of $L = (\ell, U)$.

Formulation of the inverse problem. Fix Q_0, i.e. the numbers $\beta_k = 1/q_k$, $k = \overline{1,n}$, are known and fixed. The inverse problem is formulated as follows: given the Weyl matrix $M(\rho)$, construct the pair $L = (\ell, U)$.

2. Properties of the Weyl matrix

By definition we have

$$U(\Phi) = h\Phi(0) = \mathcal{N}(\rho), \quad \text{where} \quad \mathcal{N}(\rho) = M^T(\rho). \tag{2.1}$$

Let $C(x, \rho) = [C_{km}(x, \rho)]_{k,m=\overline{1,n}}$ be a matrix-solution of system (1.1) under the initial condition $U(C) = hC(0, \rho) = E$ (E is the identity matrix). In other words, the column-vectors $C_m(x,\rho) = [C_{km}(x,\rho)]^T_{k=\overline{1,n}}$, $m = \overline{1,n}$, are solutions of (1.1) under the initial conditions $U_\xi(C_m) = \delta_{\xi m}$, $\xi, m = \overline{1,n}$. Clearly,

$$(2.2)\quad \Phi(x,\rho) = C(x,\rho)\mathcal{N}(\rho) \text{ or } \Phi_m(x,\rho) = C_m(x,\rho) + \sum_{k=m+1}^{n} M_{mk}(\rho)C_k(x,\rho).$$

It follows from (2.2) and the Gauss-Ostrogradskii theorem that

$$(2.3)\qquad \det C(x,\rho) = \det \Phi(x,\rho) = \exp(\rho(\beta_1 + \cdots + \beta_n)x).$$

Denote $\Gamma_j = \{\rho : \arg\rho = \theta_j\}$, $j = \overline{0, 2r-1}$, $\Gamma_{2r} := \Gamma_0$. We cut the ρ- plane along the rays Γ_j and denote by $\Gamma_j^{\pm} = \{\rho : \arg\rho = \theta_j \pm 0\}$ the sides of the cuts. We put $\overline{S}_j = S_j \cup \Gamma_j^+ \cup \Gamma_{j+1}^-$, and denote by $\Sigma = \bigcup_{j=0}^{2r-1} S_j$ the ρ- plane without the cuts along the rays Γ_j, and denote by $\overline{\Sigma} = \bigcup_{j=0}^{2r-1} \overline{S}_j$ the closure of Σ (we differ the sides of the cuts). Fix $j = \overline{0, 2r-1}$. For $\rho \in \Gamma_j$, strict inequalities from (1.2) in some places become equalities. Let $m_i = m_i(j)$, $p_i = p_i(j)$, $i = \overline{1,s}$, be such that for $\rho \in \Gamma_j$,

$$\mathrm{Re}(\rho R_{m_i-1}) < \mathrm{Re}(\rho R_{m_i}) = \ldots = \mathrm{Re}(\rho R_{m_i+p_i}) < \mathrm{Re}(\rho R_{m_i+p_i+1}), \quad i = \overline{1,s}.$$

Denote $N_j := \{m : \ m = \overline{m_1, m_1+p_1-1}, \ldots, \overline{m_s, m_s+p_s-1}\}$, $J_m := \{j : m \in N_j\}$, $\gamma_m = \bigcup_{j \in J_m} \Gamma_j$, $\Sigma_m = \mathbf{C} \setminus \gamma_m$ is the ρ- plane without the cuts along the rays from γ_m, $\overline{\Sigma}_m$ is the closure of Σ_m (we differ the sides of the cuts). Clearly, the domain $\Sigma_m = \bigcup_\nu S_{m\nu}$ consists of sectors $S_{m\nu}$, each of which is a union of several sectors S_j with the same collection $\{R_\xi\}_{\xi=\overline{1,m}}$.

Theorem 2.1. *(i) The Weyl functions $M_{mk}(\rho)$, $k > m$ are analytic in Σ_m with the exception of an at most countable bounded set Λ'_m of poles and are continuous in $\overline{\Sigma}_m$ (i.e. they are continuous in each sector $\overline{S}_{m\nu}$) with the exception of a bounded set Λ_m. More precisely, for $j \in J_m$, $\rho \in \Gamma_j \setminus \Lambda_m$ there exist finite limits $M^{\pm}_{mk}(\rho) = \lim M_{mk}(z)$, $z \to \rho$, $z \in \Sigma_m$, $\pm(\arg z - \theta_j) > 0$. On γ_m the functions $M_{mk}(\rho)$ have the cuts.*

(ii) For $|\rho| \to \infty$, $\rho \in \overline{S}_j$, $k > m$,

$$(2.4)\qquad M_{mk}(\rho) = \mu^0_{mk}(S_j) + o(1),$$

$$(2.5)\qquad \mu^0_{mk}(S_j) = \Omega^0_{mk}(i_1, \ldots, i_m)(\Omega^0_m(i_1, \ldots, i_m))^{-1}, \quad i_\nu = i_\nu(S_j).$$

P r o o f. Fix $\alpha \geq 0$. It is known (see [5]) that for $x \geq 0$, $\rho \in \overline{\Sigma}$, $\rho \geq \rho_\alpha$ ($\rho_\alpha = O(\max_{k,j} \|q_{kj}(x)\|_{L(\alpha,\infty)})$) there exists a fundamental system of solutions

(FSS) $\mathcal{D}_\alpha = \{Y_k(x,\rho)\}_{k=\overline{1,n}}$ of (1.1) with the following properties:
(1) for each $j = \overline{0,2r-1}$, the functions $Y_k(x,\rho)$, are continuous for $x \ge 0$, $\rho \in \overline{S}_j$, $|\rho| \ge \rho_\alpha$; (2) for each $j = \overline{0,2r-1}$, $x \ge 0$, the functions $Y_k(x,\rho)$, are analytic in $\rho \in S_j$, $|\rho| \ge \rho_\alpha$; (3) for $|\rho| \to \infty$, $\rho \in \overline{\Sigma}$, uniformly in $x \ge \alpha$,

$$Y_k(x,\rho) = \Big(g_{0k} + o(1)\Big)\exp(\rho\beta_k x), \quad g_{0k} = [\delta_{\nu k}]^T_{\nu=\overline{1,n}}. \tag{2.6}$$

Fix $j = \overline{0,2r-1}$. Let $\rho \in \overline{S}_j$, $\alpha = 0$. Denote

$$\Delta^0_{mk}(\rho) = \det[U_\xi(Y_{i_\nu})]_{\xi=\overline{1,m-1},k;\ \nu=\overline{1,m}}. \tag{2.7}$$

With the help of the FSS $\mathcal{D}_\alpha$ one has the representation

$$\Phi_m(x,\rho) = \sum_{k=1}^{n} d_{mk}(\rho) Y_k(x,\rho), \quad m = \overline{1,n}.$$

Using the boundary conditions on $\Phi_m(x,\rho)$, we obtain

$$\Phi_m(x,\rho) = \sum_{k=1}^{m} d_{m,i_k}(\rho) Y_{i_k}(x,\rho), \tag{2.8}$$

$$d_{m,i_k}(\rho) = (-1)^{m+k}(\Delta^0_{mm}(\rho))^{-1}\det[U_\xi(Y_{i_\nu})]_{\xi=\overline{1,m-1};\ \nu=\overline{1,m}\setminus k}. \tag{2.9}$$

In particular, it follows from (2.6), (2.8) and (2.9) that for $|\rho| \to \infty$, $\rho \in \overline{S}_j$, uniformly in $x \ge 0$,

$$\Phi_m(x,\rho) = \sum_{k=1}^{m} \exp(\rho R_k x)\Big([\delta_{m,i_k} d^0_{m,i_k}]^T_{k=\overline{1,n}} + o(1)\Big), \tag{2.10}$$

where $d^0_{m,i_k} = (-1)^{m+k}(\Omega^0_m(i_1,\dots,i_m))^{-1}\det[h_{\xi,i_\nu}]_{\xi=\overline{1,m-1};\ \nu=\overline{1,m}\setminus k}$. In particular, this yields

$$d^0_{m,i_m} = \Omega^0_{m-1}(i_1,\dots,i_{m-1})(\Omega^0_m(i_1,\dots,i_m))^{-1} \ne 0. \tag{2.11}$$

It follows from (2.8) and (2.9) that $M_{mk}(\rho) = \Delta^0_{mk}(\rho)/\Delta^0_{mm}(\rho)$. From this, taking into account (2.6) and (2.7), we arrive at (2.4).

Fix $\alpha \ge 0$, $m = \overline{1,n-1}$. By a well-known method (see, for example, [4]) one can prove that there exists a FSS $\mathcal{D}_{\alpha m} = \{Y^0_{km}(x,\rho)\}_{k=\overline{1,n}}$, $x \ge 0$, $\rho \in \overline{\Sigma}$, $|\rho| \ge \rho_\alpha$ of system (1.1) with the following properties:
(1) $Y^0_{km}(x,\rho) = Y_{i_k}(x,\rho)$, $k = \overline{m+1,n}$, $i_k = i_k(S_j)$; (2) for $k = \overline{1,m}$, the functions $Y^0_{km}(x,\rho)$ are analytic for $\rho \in \Sigma_m$, $|\rho| \ge \rho_\alpha$ and continuous for $x \ge 0$, $\rho \in$

$\overline{\Sigma}_m$, $|\rho| \geq \rho_\alpha$; (3) $Y^0_{km}(x,\rho) = O(\exp(\rho R_m x))$, $|\rho| \to \infty$, $\rho \in \overline{\Sigma}_m$, uniformly in $x \geq \alpha$. Thus, the solutions $Y^0_{km}(x,\rho)$, $k = \overline{1,m}$ are analytic in sectors $S_{m\nu}$ which are wider than S_j. Repeating the preceding arguments for the FSS $\mathcal{D}_{\alpha m}$ we get

$$M_{mk}(\rho) = \frac{\Delta^1_{mk}(\rho)}{\Delta^1_{mm}(\rho)}, \ \Delta^1_{mk}(\rho) := \det[U_\xi(Y^0_{\nu m})]_{\xi=\overline{1,m-1},k;\ \nu=\overline{1,m}}. \tag{2.12}$$

The functions $\Delta^1_{mk}(\rho)$ are analytic for $\rho \in \Sigma_m$, $|\rho| \geq \rho_\alpha$ and continuous for $\rho \in \overline{\Sigma}_m$, $|\rho| \geq \rho_\alpha$. From this, using (2.12), (2.4) and the arbitrariness of α, we obtain the assertion (i) of Theorem 2.1. □

It follows from (2.4) and (2.10) that

$$|M_{mk}(\rho)| \leq C, \ |\Phi_{km}(x,\rho)| \leq C|\exp(\rho R_m x)|, \quad \rho \in \overline{S}_j, \ |\rho| \geq \rho_0. \tag{2.13}$$

We consider the differential system and the linear forms $L^* = (\ell^*, U^*)$:

$$\ell^* Z(x) := -Z'(x)Q_0 + Z(x)Q(x) = \rho Z(x), \tag{2.14}$$

$U^*(Z) = Z(0)h^*$, where $Z = [z_k]_{k=\overline{1,n}}$ is a row-vector, $h^* = [h^*_{k\xi}]_{k,\xi=\overline{1,n}} := Q_0 h^{-1}$. Then $U^*(Z) = [U^*_n(Z), \ldots, U^*_1(Z)]$, where $U^*_{n-\xi+1}(Z) = Z(0)[h^*_{k\xi}]^T_{k=\overline{1,n}}$. Clearly,

$$Z(0)Q_0 Y(0) = U^*(Z)U(Y) = \sum_{\xi=1}^{n} U^*_{n-\xi+1}(Z)U_\xi(Y). \tag{2.15}$$

Denote $R^*_m := -R_{n-m+1}$. Let vector-functions $\Phi^*_m(x,\rho) = [\Phi^*_{km}(x,\rho)]_{k=\overline{1,n}}$, $m = \overline{1,n}$ be solutions of (2.14) satisfying the conditions $U^*_\xi(\Phi^*_m) = \delta_{\xi m}$, $\xi = \overline{1,m}$, and also $\Phi^*_m(x,\rho) = O(\exp(\rho R^*_m x))$, $x \to \infty$, $\rho \in S_j$ in each sector S_j with property (1.2). We put $\Phi^*(x,\rho) = [\Phi^*_{n-m+1}(x,\rho)]^T_{m=\overline{1,n}} = [\Phi^*_{n-m+1,k}(x,\rho)]_{m,k=\overline{1,n}}$, $M^*_{mk}(\rho) = U_k(\Phi^*_m)$, $M^*(\rho) = [M^*_{n-\xi+1,n-k+1}(\rho)]_{k,\xi=\overline{1,n}}$, $\mathcal{N}^*(\rho) = (M^*)^T(\rho)$. Then

$$\Phi^*(0,\rho)h^* = U^*(\Phi^*) = \mathcal{N}^*(\rho). \tag{2.16}$$

Denote $\gamma^*_m = \gamma_{n-m}$, $\Sigma^*_m = \Sigma_{n-m}$, $\Lambda^*_m = \Lambda_{n-m}$, ${\Lambda'_m}^* = \Lambda'_{n-m}$. Properties of the matrix $M^*(\rho)$ are completely analogous to those of the matrix $M(\rho)$. In particular, the functions $M^*_{mk}(\rho)$, $k > m$ are analytic in Σ^*_m with the exception of the set ${\Lambda'_m}^*$ of poles and are continuous in $\overline{\Sigma}^*_m$ with the exception of the set Λ^*_m. On γ^*_m the functions $M^*_{mk}(\rho)$ have the cuts.

Lemma 2.1. *The following relations hold*

$$\Phi^*(x,\rho) = (Q_0\Phi(x,\rho))^{-1}, \tag{2.17}$$

(2.18) $$M^*(\rho) = M^{-1}(\rho), \quad \mathcal{N}^*(\rho) = \mathcal{N}^{-1}(\rho),$$

P r o o f. Denote $Z = (Q_0\Phi)^{-1}$. Since $ZQ_0\Phi = E$, it follows that $Z'Q_0\Phi + ZQ_0\Phi' = 0$ or $Q_0\Phi' = -Z^{-1}Z'Q_0\Phi$. Since Φ satisfies (1.1), we get that $\rho\Phi - Q\Phi = -Z^{-1}Z'Q_0\Phi$ or $-Z^{-1}Z'Q_0 + Q = \rho E$. Thus,

(2.19) $$\ell^* Z(x,\rho) = \rho Z(x,\rho).$$

Let $Z(x,\rho) = [Z_k(x,\rho)]^T_{k=\overline{1,n}}$, where $Z_k(x,\rho)$ are rows of the matrix $Z(x,\rho)$. Using (2.15) we calculate $E = Z(0,\rho)Q_0\Phi(0,\rho) = \sum_{\xi=1}^n U^*_{n-\xi+1}(Z)U_\xi(\Phi)$ or

(2.20) $$\sum_{\xi=1}^n U^*_{n-\xi+1}(Z_k)U_\xi(\Phi_m) = \delta_{mk}, \quad k,m = \overline{1,n}.$$

Taking in (2.20) $m = n, n-1, \dots, 1$ successively and using the relations $U_\xi(\Phi_m) = \delta_{\xi m}$, $\xi = \overline{1,m}$, we obtain

(2.21) $$U^*_{n-\xi+1}(Z_k) = \delta_{\xi k}, \quad \xi = \overline{k,n}.$$

Moreover, by virtue of (2.13) and (2.3) we have

(2.22) $$|Z_k(x,\rho)| \le C|\exp(\rho R^*_{n-k+1}x)|.$$

It follows from (2.19), (2.21) and (2.22) that $Z_k(x,\rho) = \Phi^*_{n-k+1}(x,\rho)$, i.e. $Z(x,\rho) = \Phi^*(x,\rho)$, and (2.17) is proved.

Furthermore, using (2.15)-(2.17) and (2.1) we get $E = \Phi^*(0,\rho)Q_0\Phi(0,\rho) = U^*(\Phi^*)U(\Phi) = \mathcal{N}^*(\rho)\mathcal{N}(\rho)$, i.e. (2.18) holds. □

Let us now establish the so-called structural properties of the Weyl matrix, which play an important role for the solution of the inverse problem. For $\xi = \overline{0, n-2}$ we construct functions $B^\xi_{mk}(\rho)$, $m = \overline{1, n-\xi-1}$, $k = \overline{m+\xi+1, n}$ by the following recurrent formulae $B^0_{mk}(\rho) = M_{mk}(\rho)$, $B^\xi_{mk}(\rho) = B^{\xi-1}_{mk}(\rho) - B^{\xi-1}_{m,m+\xi}(\rho)B^0_{m+\xi,k}(\rho)$.

Lemma 2.2. *The functions $B^\xi_{mk}(\rho)$ are analytic in $\Sigma_{m+\xi} \setminus \Lambda'_{m+\xi}$, are continuous in $\overline{\Sigma}_{m+\xi} \setminus \Lambda_{m+\xi}$, and have cuts on $\gamma_{m+\xi}$.*

P r o o f. By virtue of (2.18), for $k = m+\xi+1$ one has $B^\xi_{m,m+\xi+1}(\rho) = -M^*_{n-m-\xi,n-m+1}(\rho)$. This implies the assertion of the lemma for $k = m+\xi+1$. For $k > m+\xi+1$ we interchange places of the linear forms U_k and $U_{m+\xi+1}$ and repeat the preceding arguments. □

Theorem 2.2. *The functions $B^{m-\nu}_{\nu k}(\rho)$ are analytic on $\Gamma_j \setminus \Lambda'_m$ for $j \notin J_m$, $1 \le \nu \le m \le n-1$, $m+1 \le k \le n$.*

Indeed, since $j \notin J_m$, one has $\Gamma_j \subset \Sigma_m$, hence the theorem follows from Lemma 2.2. □

3. Uniqueness of the solution of the inverse problem

We agree that together with $L = (\ell, U)$ we consider a pair $\tilde{L} = (\tilde{\ell}, \tilde{U})$ of the same form but with different matrices $\tilde{Q}, \tilde{h}$ (we remind that the matrix Q_0 is known a priori and fixed). Everywhere below if a symbol α denotes an object related to L, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{L}$.

We define a matrix $\mathcal{P}(x,\rho) = [\mathcal{P}_{\xi k}(x,\rho)]_{\xi,k=\overline{1,n}}$ by the formula

$$\mathcal{P}(x,\rho) = \Phi(x,\rho)\tilde{\Phi}^{-1}(x,\rho). \tag{3.1}$$

In other words,

$$\mathcal{P}_{\xi k}(x,\rho) = (\det \tilde{\Phi}(x,\rho))^{-1} \det[\tilde{\Phi}_{1\nu}(x,\rho), \ldots, \tilde{\Phi}_{k-1,\nu}(x,\rho),$$

$$\Phi_{\xi\nu}(x,\rho), \tilde{\Phi}_{k+1,\nu}(x,\rho), \ldots, \tilde{\Phi}_{n\nu}(x,\rho)]_{\nu=\overline{1,n}}\,. \tag{3.2}$$

Lemma 3.1. *(i) For $|\rho| \ge \rho_0$, uniformly in $x \ge 0$,*

$$|\mathcal{P}_{\xi k}(x,\rho)| \le C, \quad \xi, k = \overline{1,n}. \tag{3.3}$$

(ii) For $|\rho| \to \infty$, $\rho \in S_j$, $\arg\rho = const$, uniformly in $x \ge 0$,

$$\mathcal{P}_{\xi k}(x,\rho) = o(1),\ \xi \ne k, \quad \mathcal{P}_{kk}(x,\rho) = \mathcal{P}_k^j + o(1), \quad \xi, k = \overline{1,n}, \tag{3.4}$$

$$\mathcal{P}_k^j := \frac{\Omega_{k-1}^0(i_1,\ldots,i_{k-1})}{\Omega_k^0(i_1,\ldots,i_k)} \cdot \frac{\tilde{\Omega}_k^0(i_1,\ldots,i_k)}{\tilde{\Omega}_{k-1}^0(i_1,\ldots,i_{k-1})}, \quad i_k = i_k(S_j). \tag{3.5}$$

Moreover, if $h = \tilde{h}$, then (3.4) holds for $\rho \in \overline{S_j}$, and $\mathcal{P}_k^j = 1$ for all $k = \overline{1,n}$, $j = \overline{0, 2r-1}$.

The assertions of the lemma follow easily from (3.2) in view of (2.10), (2.11) and (2.13). Let us now prove the uniqueness theorem for the solution of the inverse problem of recovering L from the Weyl matrix.

Theorem 3.1. *If $M(\rho) = \tilde{M}(\rho)$, then $L = \tilde{L}$. Thus, the specification of the Weyl matrix $M(\rho)$ uniquely determines $Q(x)$ and h.*

P r o o f. We transform the matrix $\mathcal{P}(x,\rho)$. For this we use (3.1) and (2.2). Under the conditions of the theorem,

$$\mathcal{P}(x,\rho) = \Phi(x,\rho)\tilde{\Phi}^{-1}(x,\rho) = C(x,\rho)\mathcal{N}(\rho)\tilde{\mathcal{N}}^{-1}(\rho)\tilde{C}^{-1}(x,\rho) = C(x,\rho)\tilde{C}^{-1}(x,\rho).$$

Therefore, it follows from (2.3) that for each fixed $x \geq 0$, the matrix-function $\mathcal{P}(x,\rho)$ is entire in ρ. Using (3.3) and Liouville's theorem, we get that $\mathcal{P}_{\xi k}(x,\rho) \equiv \mathcal{P}_{\xi k}(x)$, $\xi, k = \overline{1,n}$, i.e. the functions $\mathcal{P}_{\xi k}$ do not depend on ρ. Together with (3.5) this yields $\mathcal{P}_{\xi k}(x,\rho) \equiv 0$ for $\xi \neq k$, and $\mathcal{P}_{kk}(x,\rho) \equiv \mathcal{P}_k$. On the other hand, $\mathcal{P}_k = \mathcal{P}_k^j$, where $\mathcal{P}_k^j$ is defined by (3.5). Since h and $\tilde{h}$ satisfy the normalizing conditions, we get for the sector S_0 that $\Omega_k^0(i_1,\ldots,i_k) = \tilde{\Omega}_k^0(i_1,\ldots,i_k)$, $k = \overline{0,n}$, and consequently, $\mathcal{P}_k = 1$, $k = \overline{1,n}$. Thus, $\mathcal{P}(x,\rho) \equiv E$, where E is the identity matrix. Together with (3.1) this yields $\Phi(x,\rho) \equiv \tilde{\Phi}(x,\rho)$, hence $Q(x) = \tilde{Q}(x)$. In view of (2.1), $h\Phi(0,\rho) = \mathcal{N}(\rho)$, and consequently, $h = \tilde{h}$. □

R e m a r k 3.1. Using the method of spectral mappings [4] and the properties of the Weyl matrix, obtained above in Theorems 1.1 and 1.2, one can obtain a constructive procedure for the solution of the inverse problem along with necessary and sufficient conditions for its solvability.

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