Mathematica
Balkanica

New Series Vol. 19, 2005, Fasc. 3-4

On The Zeros of a Certain Class of Polynomials and Related Analytic Functions

W. M. Shah, A. Liman

Presented by Bl. Sendov

The classical Enström-Kakeya theorem states that if a polynomial $P(z) = \sum_{j=0}^{n} a_j z^j$ satisfy $a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0$, then all the zeros of $P(z)$ lie in $|z| \leq 1$. In this paper we prove results concerning the location of the zeros of a more general class of polynomials. From these results, we obtain better bounds and generalizations of Enström-Kakeya type polynomials with less restrictive conditions on their coefficients. We also consider associated analytic functions and obtain zero-free discs for them under various conditions on the coefficients.

1. Introduction and statement of results

The following result due to Enström and Kakeya [8] is well-known in the theory of distribution of the zeros of polynomials.

Theorem A. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \ldots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ *lie in* $|z| \leq 1$.

From this thoerem one can easily deduce

Theorem B. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with real and positive coefficients, then all the zeros of* $P(z)$ *lie in* $|z| \leq t$ *where* $t = Max\left(\frac{a_0}{a_1}, \frac{a_1}{a_2}, \cdots, \frac{a_{n-1}}{a_n}\right)$.

It is clear that Theorem B gives better information regarding the zeros of $P(z)$, if the coefficients are strictly decreasing, that is, if $a_n > a_{n-1} > \ldots > a_1 > a_0 > 0$ because then $t < 1$. In the literature [1-9], there exist some extensions of Enström-Kakeya theorem. As a generalization of Enström-Kakeya theorem, Aziz and Mohammand [1] proved the following.

Theorem C. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real positive coefficients. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \quad r = 1, 2, \ldots, n+1 \, (a_{-1} = a_{n+1} = 0),$$

then all the zeros of $P(z)$ *lie in* $|z| \leq t_1$.

In this paper as a refinement of Theorem C we first prove the following

Theorem 1. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with real and positive coefficients. If* $t_1 > t_2 \geq 0$ *can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0 \quad r = 1, 2, \ldots, n+1 \, (a_1 = a_{n+1} = 0), \tag{1}$$

then all the zeros of $P(z)$ *lie in*

$$\left| z + \frac{a_{n-1}}{a_n} - (t_1 - t_2) \right| \leq t_2 + \frac{a_{n-1}}{a_n} \tag{2}$$

Remark 1. Theorem 1 gives significantly better result than Theorem C. To illustrate this, we consider the polynomial

$$P(z) = 6z^7 + 3z^6 + 3z^5 + z^4 + 2z^2 + z + 2$$

of degree $n = 7$. Here $P(z)$ satisfies the hypothesis of Theorem 1 (and also that of Theorem C) for $t_1 = 2, \ t_2 = 1$. By using Theorem C we see all the zeros of $P(z)$ lie in $|z| \leq 2$, whereas by Theorem 1 it follows that all the zeros of $P(z)$ lie in $\left| z - \frac{1}{2} \right| \leq \frac{3}{2}$.

The following corollary follows from Theorem 1, if we take $t_2 = 0$.

Corollary 1. *Let* $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ *be a polynomial of degree* n *such that*

$$a_n t^n \geq a_{n-1} t^{n-1} \geq \ldots \geq a_1 t \geq a_0 > 0 \tag{3}$$

then all the zeros of $P(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}-t\right|\leq\frac{a_{n-1}}{a_n}$$

The next corollary also follows from Theorem 1, if we take $t_1=t_2=t$.

Corollary 2. *If $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$ is a polynomial of degree n such that for some $t>0$, either $a_nt^n\geq a_{n-2}t^{n-2}\geq\ldots\geq a_3t^3\geq a_1t>0$ and $a_{n-1}t^{n-1}\geq a_{n-3}t^{n-3}\geq\ldots\geq a_2t^2\geq a_0>0$ if n is odd or $a_nt^n\geq a_{n-2}t^{n-2}\geq\ldots a_2t^2\geq a_0>0$ and $a_{n-1}t^{n-1}\geq a_{n-3}t^{n-3}\geq\ldots\geq a_1t>0$, if n is even, then all the zeros of $P(z)$ lie in the circle $\left|z+\frac{a_{n-1}}{a_n}\right|\leq t+\frac{a_{n-1}}{a_n}$.*

For $t=1$, Corollary 2 reduces to Theorem 3 of Aziz and Zargar [3].

Instead of proving Theorem 1, we prove the following more general result.

Theorem 2. *Let $P(z)=\sum\limits_{j=0}^{n}a_jz^j$ be a polynomial of degree n with real coefficients. If $t_1>t_2\geq 0$ can be found such that for certain nonnegative integer k.*

$$a_rt_1t_2+a_{r-1}(t_1-t_2)-a_{r-2}\geq 0,\qquad r=1,2,\ldots,k \tag{4}$$

$$a_rt_1t_2+a_{r-1}(t_1-t_2)-a_{r-2}\leq 0,\qquad r=k+1,k+2,\ldots,n,\quad (a_{-1}=a_{-2}=0) \tag{5}$$

then all the zeros of $P(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}-(t_1-t_2)\right|\leq\frac{2}{|a_n|\,t_1^{n-k}}(t_2a_k+a_{k-1})-\frac{a_{n-1}}{|a_n|}-t_2 \tag{6}$$

For $k=n$, Theorem 2 reduces to Theorem 1.

The following Corollary immediately follows from Theorem 2, if we take $t_2=0$.

Corollary 3. *Let $P(z)=\sum\limits_{j=0}^{n}a_jz^j$ be a polynomial of degree n with real coefficients. If for some real number t*

$$t^na_n\leq t^{n-1}a_{n-1}\leq\ldots\leq t^ka_k\leq t^{k-1}a_{k-1}\geq t^{k-2}a_{k-2}\ldots\geq ta_1\geq a_0>0 \tag{7}$$

then all the zeros of $P(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}-t\right|\leq\frac{2a_{k-1}}{|a_n|t^{n-k}}-\frac{a_{n-1}}{|a_n|}. \tag{8}$$

Corollary 2 of Aziz and Zargar [3] is a special case of Corollary 3 when $k=n$ and $t=1$.

Again, if we take $t_1=t_2$ in Theorem 2, we get the following

Corollary 4. *Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be polynomial of degree n with real coefficients. If for some t*

$$a_nt^n\leq a_{n-2}t^{n-2}\leq\ldots\leq a_kt^k\geq\ldots\geq a_3t^3\geq a_1t>0$$

and

$$a_{n-1}t^{n-1}\leq a_{n-3}t^{n-3}\leq\ldots\leq a_{k-1}t^{k-1}\geq\ldots\geq a_2t^2\geq a_0>0$$

when n is odd, or

$$a_nt^n\leq a_{n-2}t^{n-2}\leq\ldots\leq a_kt^k\geq\ldots\geq a_2t^2\geq a_0>0$$

and

$$a_{n-1}t^{n-1}\leq a_{n-3}t^{n-3}\leq\ldots\leq a_{k-1}t^{k-1}\geq\ldots\geq a_3t^3\geq a_1t>0$$

when n is even, then all the zeros of $P(z)$ lie in

$$\left|z+\frac{a_{n-1}}{a_n}\right|\leq\frac{2(ta_k+a_{k-1})}{|a_n|t^{n-k}}-\frac{(ta_n+a_{n-1})}{|a_n|}.$$

Remark 2. Theorem 3 of Aziz and Zargar [3] is a special case of Corollary 4, when $n=k$ and $t=1$.

Aziz and Mohammand [2] considered a class of analytic functions and proved the following.

Theorem D. *Let $f(z)=\sum_{j=0}^{\infty}a_jz^j\not\equiv 0$ be analytic in $|z|\leq t$. If*

$$a_0\geq ta_1\geq t^2a_2\geq\ldots \tag{9}$$

then $f(z)$ does not vanish in $|z|<t$.

Recently Jain [6] considered a more general class of polynomials and obtained the following

Theorem E. *Let* $F(z) = \sum_{j=0}^{\infty} a_j z^j$ *be analytic in* $|z| \leq t$. *If* $a_j = \alpha_j e^{i\phi} + \beta_j e^{i\psi}, j = 0, 1, 2, \ldots,\ 0 < |\phi - \psi| < \pi$ *and for some finite non-negative integer* k

$$0 < \alpha_0 \leq t\alpha_1 \leq \ldots \leq t^k \alpha_k \geq t^{k+1}\alpha_{k+1} \geq \ldots$$

then $f(z)$ *does not vanish in*

$$|z| < \frac{1}{2M_k}\left\{-|a_0 - a_1 t| + \left(|a_0 - a_1 t|^2 + 4M_k|a_0|\right)^{1/2}\right\},$$

where

$$\begin{aligned} M_k &= 2a_k t^k - \alpha_1 t + |\beta_1| t + 2\sum_{j=2}^{\infty} |\beta_j| t^j, \quad k \geq 1 \\ M_0 &= M_1. \end{aligned}$$

In this connection Aziz and Shah [4] proved a more general result and deduced that the result of Jain (Theorem E) also holds true without the restriction $|\phi - \psi| < \pi$.

Very recently Jain [7] considered a new class of analytic functions and obtained zero-free regions for members of this class. Among other things he proved

Theorem F. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0)$ *be analytic in* $|z| \leq t_1$. *If* $a_j = \alpha_j e^{i\phi} + \beta_j e^{i\psi}, j = 0, 1, 2, \ldots, 0 < |\phi - \psi| < \pi$ *and for* $t_2 \geq t_1 > 0$,

$$\begin{aligned} (10)\ b_r &= a_{r-2} + a_{r-1}(t_2 - t_1) - a_r t_1 t_2,\ r = 0, 1, 2, \ldots, (a_{-1} = a_{-2} = 0) \\ (11) C_r &= \alpha_{r-2} + \alpha_{r-1}(t_2 - t_1) - \alpha_r t_1 t_2,\ r = 0, 1, 2, \ldots, (\alpha_{-1} = \alpha_{-2} = 0) \\ (12)\ d_r &= \beta_{r-2} + \beta_{r-1}(t_2 - t_1) - \beta_r t_1 t_2,\ r = 0, 1, 2, \ldots, (\beta_{-1} = \beta_{-2} = 0) \end{aligned}$$

with the characteristics that for certain non-negative integer k,

$$(13) \qquad C_r \leq 0 \qquad r = 0, 1, 2, \ldots, k$$

$$(14) \qquad C_r \geq 0 \qquad r = k+1, \ldots$$

then $f(z)$ *does not vanish in* $|z| < \frac{t_1}{M}$, *where*

$$(15) \qquad M = \frac{1}{|a_0|}\left(-\alpha_0 + 2\alpha_k t_1^k + 2\alpha_{k-1}\frac{t_1^k}{t_2} + |\beta_0| + 2\sum_{j=1}^{\infty} |\beta_j| t_1^j\right)$$

It is again interesting to ask that whether the conclusion of Theorem F remains valid without the restriction $|\phi-\psi|<\pi$. Here we answer this question in affirmative. In fact we prove the following more general result which not only includes Theroem F without the restriction $|\phi-\psi|<\pi$ as a special case but also leads to a standard development of interesting generalizations of some other well-known results.

Theorem 3. *Suppose $f(z)=\sum\limits_{j=0}^{\infty} a_n z^n \not\equiv 0$ is analytic in $|z|\le R$ and for some $t_2 \ge t_1 > 0$*

$$\left|\lim_{n\to\infty} \frac{1}{a_0 t_1 t_2} \sum_{r=1}^{n} b_r z^r\right| \le M^\star \quad \text{for } |z|\le R. \tag{16}$$

where

$$b_r = a_{r-2} + a_{r-1}(t_2-t_1) - a_r t_1 t_2, \quad r=0,1,2,\dots,n \quad (a_{-1}=a_{-2}=0)$$

then $f(z)$ does not vanish in $|z|<\frac{R}{M^\star}$.

We first show that Theorem 1 of Jain [7] (Theorem F) follows from Theorem 3 with a suitable choice of R and the coefficients a_j. For this we take $R=t_1$ and $a_j=\alpha_j e^{i\phi}+\beta_j e^{i\psi}$, $j=0,1,\dots,$ and use the fact that since $f(z)=\sum\limits_{j=0}^{\infty} a_n z^n$ is analytic in $|z|\le t_1$. Therefore $\sum\limits_{j=0}^{\infty}|\alpha_j|t^j$ and $\sum\limits_{j=0}^{\infty}|\beta_j|t^j$ are convergent.

Now, we have, for $|z|\le t_1$

$$\begin{aligned}
\left|\lim_{n\to\infty} \frac{1}{a_0 t_1 t_2} \sum_{r=1}^{n} b_r z^r\right| &\le \lim_{n\to\infty} \frac{1}{|a_0| t_1 t_2} \sum_{r=1}^{n} |C_r e^{i\phi} + d_r e^{i\psi}| t_1^n \\
\le \quad & \lim_{n\to\infty} \frac{1}{|a_0| t_1 t_2} \left\{ \sum_{r=1}^{n} |C_r| t_1^n + \sum_{r=1}^{n} |d_r| t_1^n \right\} \\
= \quad & \lim_{n\to\infty} \frac{1}{|a_0| t_1 t_2} \left\{ \sum_{r=1}^{n} |\alpha_{r-2} + \alpha_{r-1}(t_2-t_1) - \alpha_r t_1 t_2|\, t_1^n \right. \\
& \left. + \sum_{r=1}^{n} |\beta_{r-2} + \beta_{r-1}(t_2-t_1) - \beta_r t_1 t_2|\, t_1^n \right\} \\
\le \quad & \lim_{n\to\infty} \frac{1}{|a_0| t_1 t_2} \left\{ \sum_{r=1}^{k} (\alpha_r t_1 t_2 - \alpha_{r-1}(t_2-t_1) - \alpha_{r-2})\, t_1^n \right.
\end{aligned}$$

$$+\sum_{r=k+1}^{n}\left(\alpha_{r-2}+\alpha_{r-1}(t_2-t_1)-\alpha_r t_1 t_2\right)t_1^n$$
$$\left.+\sum_{r=1}^{n}\left(|\beta_{r-2}|+|\beta_{r-1}|\,(t_2-t_1)+|\beta_r|\,t_1t_2\right)t_1^n\right\}$$
$$=\lim_{n\to\infty}\frac{1}{|a_0|t_1t_2}\Big\{-\alpha_0t_1t_2+\alpha_kt_1^{k+1}t_2+\alpha_{k-1}t_1^{k+1}+\alpha_kt_1^{k+1}t_2$$
$$--\alpha_{n-1}t_1^{n+1}t_1+|\beta_0|t_1t_2+2t_1t_2+2t_1t_2\sum_{j=1}^{n-1}|\beta_j|t_1^j$$
$$+|\beta_n|t_1^{n+1}t_1-|\beta_{n-1}|t_1^{n+1}\Big\}. \tag{17}$$

Using the fact that $\sum\limits_{j=0}^{\infty}|\alpha_j|t^j$ and $\sum\limits_{j=0}^{\infty}|\beta_j|t^j$ are convergent, we get from (16) by using (15)

$$\left|\lim_{n\to\infty}\frac{1}{|a_0|t_1t_2}\sum_{r=1}^{n}b_rz^r\right|$$
$$\le\frac{1}{|a_0|t_1t_2}\left\{-\alpha_0t_1t_2+2\alpha_kt_1^{k+1}t_2+2\alpha_{k-1}t_1^{k+1}+|\beta_0|t_1t_2+2t_1t_2\sum_{j=1}^{n-1}|\beta_j|t_1^j\right\}$$
$$=\frac{1}{|a_0|}\left\{-\alpha_0+2\alpha_kt_1^k+2\alpha_{k-1}\frac{t_1^k}{t_2}+|\beta_0|+2\sum_{j=1}^{n-1}|\beta_j|t_1^j\right\}=M$$

Therefore for $R=t_1$ and $M=M^\star$ gives precisely the conclusion of Theorem F.

Similarly, it can be easily shown that Theorem 2 of Jain [7] can be deduced from Theorem 3 without the restriction $|\phi-\psi|<\pi$.

Finally, if we take $t_1=t_2$ in Theorem 3, we get the following Corollary, wherein we relax the hypothesis of Theorem D by assuming that the alternate coefficients of $f(z)=\sum\limits_{j=0}^{\infty}a_jz^j$ satisfy (9).

Corollary 5. *Suppose $f(z)=\sum\limits_{j=0}^{\infty}a_jz^j(\not\equiv 0)$ is analytic in $|z|\le t$. If $a_0\ge a_2t^2\ge a_4t^4\ge\dots$ and $a_1t\ge a_3t^3\ge a_5t^5\ge\dots$ then $f(z)$ does not vanish in $|z|<t$.*

2. Proofs of the theorems

Proof. **of Theorem 2.** Consider the polynomial

$$f(z) = (t_2 + z)(t_1 - z)P(z) =$$
$$(t_1t_2 + (t_1 - t_2)z - z^2)(a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0) =$$
$$-a_nz^{n+2} + (a_n(t_1 - t_2) - a_{n-1})z^{n+1} + \sum_{r=0}^{n} \{a_rt_1t_2 + a_r(t_1 - t_2) - a_{r-2}\} z^r$$

Let $|z| > t_1$, then we have by using (4) and (5) $|F(z)| \le$

$$|F(z)| \le \left|a_nz^{n+2} - (a_n(t_1 - t_2) - a_{n-1})z^{n+1}\right| - \left|\sum_{r=0}^{n} \{a_rt_1t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}\} z^r\right|$$
$$\ge |z|^{n+1} \left\{|a_nz - a_n(t_1 - t_2) + a_{n-1}| - \sum_{r=0}^{n} |a_rt_1t_2 + a_{r-1}(t_1 - t_2) - a_{r-2}| \frac{1}{|z|^{n-r+1}}\right\}$$
$$\ge |z|^{n+1} \left\{|a_nz - a_n(t_1 - t_2) + a_{n+1}| - \sum_{r=0}^{k}(a_rt_1t_2 + a_{r-1}(t_1 - t_2) - a_{r-2})\frac{1}{t_1^{n-r+1}}\right.$$
$$\left. - \sum_{r=k+1}^{n} (a_{r-2} - a_{r-1}(t_1 - t_2) - a_rt_1t_2)\frac{1}{t_1^{n-r+1}}\right\}$$
$$= |z|^{n+1} \left\{|a_nz - a_n(t_1 - t_2) + a_{n-1}| - \frac{2a_kt_2}{t_1^{n-k}} - \frac{2a_{k-1}}{t_1^{n-k}} + a_nt_2 + a_{n-1}\right\} > 0,$$

if $\left|z + \frac{a_{n-1}}{a_n} - (t_1 - t_2)\right| > \frac{2}{|a_n|t_1^{n-k}}(a_kt_2 + a_{k-1}) - \frac{1}{|a_n|}(a_nt_2 + a_{n-1})$. Therefore all the zeros of $F(z)$ and hence $P(z)$ lie in $\left|z + \frac{a_{n-1}}{a_n} - (t_1 - t_2)\right| \le \frac{2}{|a_n|t_1^{n-k}}(a_kt_2 + a_{k-1}) - \frac{1}{|a_n|}(a_nt_2 + a_{n-1})$ This completes proof of Theorem 2. ■

Proof. **of Theorem 3.** We have for $|z| \le R$

$$(z - t_1)(z + t_2)\sum_{j=0}^{n} a_jz^j = -t_1t_2a_0 + b_1z + b_2z^2 + \ldots$$
$$+ b_nz^n + a_{n-1}z^{n+1} + a_n(t_2 - t_1)z^{n+1} + a_nz^{n+2}$$

As $n \to \infty$, we get

$$F(z) = (z - t_1)(z + t_2)f(z) = -t_1t_2a_0 + G(z) \tag{18}$$

where $G(z) = \lim_{n\to\infty} \sum_{r=1}^{n} b_r z^r$.

Now $G(z)$ is analytic in $|z| \le R$, $G(0) = 0$ and by (16) $|G(z)| \le t_1 t_2 |a_0| M^\star$ for $|z| = R$. Hence by Schwarz's Lemma

$$|G(z)| \le \frac{t_1 t_2 |a_0| M^\star |z|}{R}.$$

Now from (18), we have for $|z| = R$

$$\begin{aligned} |F(z)| &\ge t_1 t_2 |a_0| - |G(z)| \\ &\ge t_1 t_2 |a_0| - \frac{t_1 t_2 |a_0| M^\star |z|}{R} > 0, \end{aligned}$$

if $|z| < \frac{R}{M}$

This shows that $F(z)$ and hence $f(z)$ does not vanish in $|z| < \frac{R}{M}$ and Theorem 3 is completely proved. ∎

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Department of Mathematics
Bemina College, Srinagar
Kashmir - 190 012, India
E-MAIL:wmshah@rediffmail.com

Received 21.06.2003

Department of Mathematics
National Institute of Technology
Srinagar, Kashmir - 190 006, India
E-MAIL: abliman@rediffmail.com