Mathematica
Balkanica

New Series Vol. 19, 2005, Fasc. 3-4

Existence of the Minimum and Maximum of Two Self-Adjoint Operators

Afgan Aslanov

Presented by P. Boyvalenkov

The Theory of Positive Operators is very important for understanding of all self-adjoint operators. In this paper some questions of the Theory of Self-adjoint Operators that could be replaced in some visible place of this theory will be discussed. One of the central problems of this article is finding the so-called "natural" minimum or maximum of two linear, self-adjoint operators. The existence of "largest" linear, self-adjoint operator T, so that $A \geq T$ and $B \geq T$ will be established, where A and B are linear, bounded, self-adjoint operators. When A and B are positive definite, bounded operators, the existence of positive definite operator T such that $A > T$ and $B > T$ also will be established. As an elegance application of these results, particularly, is shown that for any commuting, bounded pair of self-adjoint operators A and B there exists the self-adjoint, bounded operators T, A_1 and B_1 such that $A = T + A_1$ and $B = T + B_1$, where operators T, A_1 and B_1 commute with A and B and furthermore, $A_1B_1 = B_1A_1 = 0$.

Key words: Hilbert space, positivity, operator, self-adjoint, closure, extension, completion, resolution of identity.

1. Existence of the "minimum" of two positive operators

A linear operator $A : H \supset D(A) \to H$ is said to be
(i) positive definite, denoted by $A > 0$, if $(Ax, x) > 0$ for all $0 \neq x \in D(A)$,
(ii) strongly positive definite, denoted by $A >> 0$, if $\exists c > 0$ such that $(Ax, x) \geq c(x, x)$ for all $x \in D(A)$,
(iii) positive (or non-negative), denoted by $A \geq 0$, if $(Ax, x) \geq 0$, for all $x \in D(A)$.

It is easily seen that if A and B are strongly positive definite operators on Hilbert Space ($A \geq c_1 I$ and $B \geq c_2 I$, for some positive c_1 and c_2), then there exists a strongly positive operator such that $A \geq F$ and $B \geq F$. For example, we can take $F = cI$, where I is the identity operator and $c = min\{c_1, c_2\}$. In general, F is not maximal operator.

When A commutes with B and both A and B are positive ($A > 0$, $B > 0$), then the operator $F = ||A||^{-1}||B||^{-1}AB$ satisfies the condition $A \geq F$ and $B \geq F$, but it is not maximal operator with this property.

Some results in this paper are not trivial even if A and B are strongly positive definite operators.

Theorem 1. *Let A, B are bounded operators on Hilbert Space H and*

$$A \geq B \geq 0, \qquad A > 0$$

and let H_A is a completion of H by inner product $(x, x)_A = (x, Ax)$. Then

a) $\mathbf{A}^{-1}\mathbf{B}$ *is bounded, and self-adjoint on H_A, where $\mathbf{A}$ and $\mathbf{B}$ are closures of A and B, respectively (on H_A).*

b) *If B is positive definite, so is* $\mathbf{A}^{-1}\mathbf{B}$: $B > 0 \Rightarrow \mathbf{A}^{-1}\mathbf{B} > \mathbf{0}$.

Proof. **a)** For simplicity let $A \leq 1$, otherwise we can replace A and B by $||A||^{-1}A$ and $||A||^{-1}B$, respectively. First note that $\mathbf{A}, \mathbf{B}$ are bounded operators in H_A and A is self-adjoint. Indeed, for $x \in H$ we have

$$(Ax, x)_A = (Ax, Ax) \leq (Ax, x) \quad \text{(and } (Ax, x)_A \geq 0)$$

and

$$(Bx, Bx)_A = (Bx, ABx) \leq (Bx, Bx) \leq (Bx, x) \leq (Ax, x) = (x, x)_A.$$

Then these inequalities are true for all $x \in H_A$. It is obvious that $\mathbf{A}$ is positive on H_A, if $\mathbf{A}x = 0$ for some $x \in H_A$, then $(\mathbf{A}x, y)_A = 0$ for all $y \in H_A$. Hence $(\mathbf{A}x, y)_A = 0$ for all $y \in H$ and so,

$$0 = (\mathbf{A}x, y)_A = (x, \mathbf{A}y)_A = (x, Ay)_A.$$

Since AH is dense in H, it follows that AH is $(,)_A$ -dense (or simply A-dense) in H_A and therefore $x = 0$.

It is also easily seen that $D(\mathbf{A}^{-1}) \subset H$: Suppose $y \in \mathbf{A}H_A$, that is $y = \mathbf{A}x$ for some $x \in H_A$. Since H is A-dense in H_A, there exists a sequence $\{x_n\} \subset H$ such that $||x_n - x||_A \to 0$ as $n \to \infty$. Let $y_n = Ax_n \in H$. Then

$$||y_n - y_m||^2 = ||A(x_n - x_m)||^2 = (A(x_n - x_m), (x_n - x_m))_A$$
$$\leq ||A||_A(||x_n - x_m||_A)^2 < \varepsilon$$

for any $\varepsilon > 0$ and for n, m sufficiently large. Thus $\{y_n\}$ is Cauchy sequence in H and since H is complete, $y_n \to y \in H$ as $n \to \infty$. Then

$$||y_n - y||_A^2 = (y_n - y, A(y_n - y)) \leq ||A||.||y_n - y||^2$$

and so $y_n \stackrel{A}{\to} y$. However, $y_n = Ax_n \stackrel{A}{\to} \mathbf{A}x = y$, so $y \in H$ and this shows that $D(\mathbf{A}^{-1}) \subset H$.

Let's define bilinear form $q(x, y)$ in H_A by the next formula:

$$q(x, y) = (Bx, y) \; if x, y \in H \text{ and}$$

$$q(x, y) = lim(Bx_n, y_n) \text{ for } x, y \in H_A \text{ and for } x_n \stackrel{A}{\to} x, y_n \stackrel{A}{\to} y.$$

Since

$$|(Bx_n, y_n) - (Bx_m, y_m)| = |(Bx_n - Bx_m, , y_n) + (Bx_m, y_n - y_m)|$$
$$\leq |(B^{\frac{1}{2}}(x_n - x_m), B^{\frac{1}{2}}y_n)| + |(B^{\frac{1}{2}}x_m, B^{\frac{1}{2}}(y_n - y_m)|$$
$$\leq ||B^{\frac{1}{2}}(x_n - x_m)||.||B^{\frac{1}{2}}y_n|| + ||B^{\frac{1}{2}}x_m||.||B^{\frac{1}{2}}(y_n - y_m)||$$

and

$$(B^{\frac{1}{2}}(x_n - x_m), B^{\frac{1}{2}}(x_n - x_m)) \leq ((A^{\frac{1}{2}}(x_n - x_m), A^{\frac{1}{2}}(x_n - x_m)) \to 0,$$

$$(B^{\frac{1}{2}}(y_n - y_m), B^{\frac{1}{2}}(y_n - y_m)) \leq ((A^{\frac{1}{2}}(y_n - y_m), A^{\frac{1}{2}}(y_n - y_m)) \to 0$$

we conclude the existence of the $lim(Bx_n, y_n)$. So $q(x, y)$ is correctly defined for all $x, y \in H_A$. On the other hand, for $x, y \in H$ we obtain

$$|q(x, y)| = |(Bx, y)| \leq \sqrt{(Bx, x)(By, y)} \leq \sqrt{(Ax, x)(Ay, y)} = ||x||_A.||y||_A.$$

Then $q(x, y)$ is bounded for all $x, y \in H_A$:

$$|q(x, y)| = |lim(Bx_n, y_n)| \leq lim||x_n||_A.||y_n||_A = ||x||_A.||y||_A,$$

where $\{x_n\}, \{y_n\} \subset H$, $x_n \to x$, $y_n \to y$. Then there exists a bounded self-adjoint operator $\mathbf{T}$ such that

$$(\mathbf{T}x, y)_A = q(x, y) = (x', y)_A$$

for all $y \in H_A$ and $\mathbf{T}x = x'$ [1, page 71, Theorem 2.24]. Let's prove that $\mathbf{A}x' = Bx$ for all $x \in H$. Indeed, let $\{x_n\} \subset H$ be a sequence such that $x_n \overset{A}{\to} x'$. Then $Ax_n \overset{A}{\to} \mathbf{A}x'$ ($\mathbf{A}$ is A-bounded) and so, for $y \in H$ we have

$$(Bx, y) = (\mathbf{T}x, y)_A = (x', y)_A = \lim_{n\to\infty} (x_n, y)_A = \lim_{n\to\infty} (Ax_n, y).$$

This shows that $Ax_n \overset{w}{\to} Bx$ weakly in H . Then $Ax_n \overset{w,A}{\to} Bx$ weakly in H_A. Indeed, for $y \in H$

$$(Ax_n, y)_A = (Ax_n, Ay) \to (Bx, Ay) = (Bx, y)_A,$$

but H dense in H_A, so $Ax_n \overset{w,A}{\to} Bx$. At the same time we have $Ax_n \overset{A}{\to} \mathbf{A}x'$ strongly. Thus, $Bx = \mathbf{A}x'$.

Now for an arbitrary $x \in H$, there is x' so that $Bx = \mathbf{A}x'$. Then $x' = \mathbf{A}^{-1}Bx$ and this means that the operator $\mathbf{T} = \mathbf{A}^{-1}B$ is defined in all H. It is easily seen that $\mathbf{A}^{-1}B$ is bounded in H_A and its closure is self-adjoint:

$$(\mathbf{A}^{-1}Bx, x)_A = (Bx, x) \le (Ax, x) \quad \text{and} \quad (\mathbf{A}^{-1}Bx, x)_A \ge 0$$

for $x \in H$. Then $\mathbf{A}^{-1}B$ can be defined for all $x \in H_A$ and the closure will be self-adjoint operator on H_A.

b) Now let $B > 0$. Since $KerB = \{0\}$, the set $\{Bx\}$ is dense in H and so $(A+B)$-dense in H_{A+B}. Thus $\{\mathbf{A}^{-1}Bx\}$-is $(A+B)$-dense in H_{A+B} and this means that $\mathbf{A}^{-1}\mathbf{B} > \mathbf{0}$. This completes the proof of Theorem 1. ■

Theorem 2. *Let A, B are positive, bounded operators on Hilbert Space $H : A > 0$, $B > 0$. Then there exists an operator $F > 0$ such that $A > F$ and $B > F$.*

Proof. For simplicity let $A + B \le I$. Let H_{A+B} is a completion of H by the inner product

$$(x, y)_{A+B} = (x, (A + B)y).$$

From Theorem 3 $(\mathbf{A} + \mathbf{B})^{-1}\mathbf{A}$ and $(\mathbf{A} + \mathbf{B})^{-1}\mathbf{B}$ are bounded and self-adjoint operators on H_{A+B}, where $\mathbf{A} + \mathbf{B}$, $\mathbf{A}$ and $\mathbf{B}$ are closures in H_{A+B}. On the other hand

$$(\mathbf{A} + \mathbf{B})^{-1}\mathbf{A} + (\mathbf{A} + \mathbf{B})^{-1}\mathbf{B} = I$$

and so, $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}$ commutes with $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}$:

$$G = (\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B} = (\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}.$$

Then

$$(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A} - G \geq 0 \text{ and } (\mathbf{A}+\mathbf{B})^{-1}\mathbf{B} - G \geq 0 \text{ (on } H_{A+B}).$$

Let's consider the operator $F = \mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}B$ on H ! First note that F really can be considered as an operator on H, that is we must show that for an arbitrary $x \in H_{A+B}$, $\mathbf{A}x \in H$. Denote by $y = \mathbf{A}x$, $x \in H_{A+B}$. Since H-is $(A+B)$-dense in H_{A+B}, there exists a sequence $\{x_n\} \subset H$ such that, $||x_n - x|| \to 0$ as $n \to \infty$. Let $y = Ax_n \in H$. Then

$$\begin{aligned} ||y_n - y_m||^2 &= ||A(x_n - x_m)||^2 = (A(x_n - x_m), A(x_n - x_m)) \\ &\leq (x_n - x_m, A(x_n - x_m)) \leq (x_n - x_m, (A+B)(x_n - x_m)) \\ &= ||x_n - x_m||^2_{A+B} < \varepsilon \end{aligned}$$

for m, n sufficiently large. So $\{y_n\}$ is Cauchy sequence in H and H is complete, $y_n \to y \in H$. Thus F is an operator on H. For all $x \in H$,

$$(Fx, x) = (Gx, x)_{A+B} \geq 0 \text{ and } (Fx, x) = (Gx, x)_{A+B} \leq (x, x)_{A+B} \leq (x, x).$$

So F-is bounded and ≥ 0. Let's show that $F > 0$. If $Fx = 0 \Rightarrow Gx = 0 \Rightarrow Ker(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A} \neq \{0\}$ or $Ker(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B} \neq \{0\}$. By Theorem 1 this is impossible. Thus, $F > 0$ and it is clear that $A - F \geq 0$ and $B - F \geq 0$ (for example, $((A-F)x, x) = ((\mathbf{A}+\mathbf{B})^{-1}\mathbf{A} - G)x, x)_{A+B} \geq 0$ for all $x \in H$).

It is easily seen that $A - F > 0$ and $B - F > 0$. For example,

$$B - F = B - \mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}B = B - (\mathbf{A}+\mathbf{B}-\mathbf{B})(\mathbf{A}+\mathbf{B})^{-1}B = \mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}B.$$

This means that the equality $(B - F)x = 0$ results $Bx = 0$ and so $x = 0$. This completes the proof of Theorem 2. ■

Theorem 3. *Let A, B are bounded operators in the Hilbert Space H: $A \geq 0$, $B \geq 0$, $A + B > 0$. Then there exists an operator $F \geq 0$ such that $A \geq F$ and $B \geq F$. The next statement is true for the kernel of F:*

(1) $$KerF = \{Ker\mathbf{A} + Ker\mathbf{B}\ \} \cap H$$

here

$$\{Ker\mathbf{A} + Ker\mathbf{B}\ \} = \{x:\ x = x_1 + x_2, \quad where\ \ x_1 \in Ker\mathbf{A},\ \ x_2 \in Ker\mathbf{B}\},$$

$\mathbf{A}$ *and* $\mathbf{B}$ *are closures of* A *and* B, *respectively, in* H_{A+B} *with the inner product* $(x, y)_{A+B} = (x, (A + B)y)$.

Proof. Let $A + B > 0$. It is obvious that the operator $F = \mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}B$ again satisfies the inequalities $A \geq F$, $B \geq F$. To show that (1) is true from $Fx = 0$ we have

$$(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}Ax = (\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}Bx = 0.$$

Denote by $S = (\mathbf{A}+\mathbf{B})^{-1}\mathbf{B}$ and $T = (\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}$, where S and T are positive (non-negative) operators on Hilbert Space H_{A+B} with inner product $(x, y)_{A+B} = (x, (A + B)y)$. Thus, $S + T = I$. Then the equality $Fx = 0$ follows that $STx = S(I - S)x = 0$ and so $(I - S)x \in KerS$ or $x - Sx = x_1$, where $x_1 \in KerS$. At the same time we have $Sx \in Ker(I - S)$, and so

$$x = Sx + x_1 = x_1 + x_2, \text{ where } x_1 \in KerS,\ \ x_2 \in KerT.$$

Since $Ker\mathbf{B} = KerS$ and $Ker\mathbf{A} = KerT$ we conclude that

$$KerF \subset \{Ker\mathbf{A} + Ker\mathbf{B}\ \} \cap H.$$

The inverse relationship $KerF \supset \{Ker\mathbf{A}+Ker\mathbf{B}\ \}\cap H$ is obvious since

$$x \in \{Ker\mathbf{A} + Ker\mathbf{B}\ \} \cap H \Rightarrow (\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}Bx = 0$$

$$\Rightarrow Fx = \mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}Bx = 0.$$

This completes the proof of Theorem 3. ∎

Note that theorem also holds if $Ker(A+B) \neq \{0\}$. In this case we need to define A and B in the Hilbert Space $\bar{R}(A+B)\oplus Ker(A+B)$, with the inner product $(x, y)_{A+B} = (x, (A+B)y)$ for $x, y \in \bar{R}(A+B)$ and $(x, y)_{A+B} = (x, y)$ if $x \in Ker(A + B)$, $y \in Ker(A + B)$.

Thus, for the two positive operators A and B we can find some operator F such that $A \geq F$ and $B \geq F$ (and furthermore, this operator has an elegance representation: $F = \mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}B = \mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}A$). Now we are trying to find some maximal operator with this property. In other words we'll find some "natural minimum" of two operators.

Theorem 4. *Let* A, B *are bounded operators in the Hilbert Space* H*:* $A \geq 0$, $B \geq 0$. *Then there exists an operator* $M \geq 0$ *such that* $A \geq M$, $B \geq M$ *and* M *is the maximal, that is if* $A \geq N$, $B \geq N$ *and* $M \geq N$, *then* $M = N$

P r o o f. First we consider the case $A + B = I$. It is easily seen that the minimum of two functions $f(\lambda) = \lambda$ and $g(\lambda) = 1 - \lambda$ is the function

$$m(\lambda) = \frac{1 - |1 - 2\lambda|}{2} = \left\{ \begin{array}{lr} \lambda & \text{if } \lambda < 1/2 \\ 1 - \lambda & \text{if } \lambda \geq 1/2 \end{array} \right\}$$

and so the operator

$$M = \frac{I - |I - 2A|}{2} = \int_{-1}^{1} m(\lambda) dE_\lambda$$

is a minimum of A and B, where E_λ is the resolution of the identity of A. Indeed, since $f(\lambda) \geq m(\lambda)$ and $\mathrm{g}(\lambda) \geq m(\lambda)$ we have $A \geq M$, $B \geq M$. If $A \geq N$, $B \geq N$ and $N \geq M$ then

$$A - M = A - N + N - M \geq N - M \quad \text{and} \quad B - M = B - N + N - M \geq N - M.$$

This shows that $Ker(N - M) \supset Ker(A - M)$ and $Ker(N - M) \supset Ker(B - M)$. If we take into account that

$$Ker(A - M) + Ker(B - M) \supset Ker(E[0, 1/2]) + Ker(E(1/2, 1]) = H,$$

where $E[0, 1/2] = E_{1/2} - E_0$ and $E(1/2, 1] = E_1 - E_{1/2}$, we conclude that $Ker(N - M) = H$ or $N = M$.

Now let A and B are arbitrary positive, bounded operators. Furthermore, let first $A + B > 0$. Instead of A and B let's consider the operators $(\mathbf{A} + \mathbf{B})^{-1}A$ and $(\mathbf{A} + \mathbf{B})^{-1}B$. Since

$$(\mathbf{A} + \mathbf{B})^{-1}A \ + \ (\mathbf{A} + \mathbf{B})^{-1}B = I$$

on H_{A+B} there exists $M' \geq 0$ such that M' is maximal and

$$(\mathbf{A} + \mathbf{B})^{-1}A \ \geq M' \quad \text{and} \quad (\mathbf{A} + \mathbf{B})^{-1}B \geq M'.$$

Then $(A+B)M'$ is maximal positive operator on H so that

$$(A+B)M' \le A \text{ and } (A+B)M' \le B.$$

Indeed for all $x \in H$, we have

$$((A+B)M'x, x) = (M'x, x)_{A+B} \le ((\mathbf{A}+\mathbf{B})^{-1}A\; x, x)_{A+B} = (Ax, x),$$

and similarly $((A+B)M'x, x) \le (Bx, x)$. Assume now that $A \ge N$, $B \ge N$, and $N \ge M$, then

$$((\mathbf{A}+\mathbf{B})^{-1}Ax, x)_{A+B} \ge ((\mathbf{A}+\mathbf{B})^{-1}Nx, x)_{A+B} \;\ge\; ((\mathbf{A}+\mathbf{B})^{-1}Mx, x)_{A+B},$$

similarly

$$((\mathbf{A}+\mathbf{B})^{-1}Bx, x)_{A+B} \ge ((\mathbf{A}+\mathbf{B})^{-1}Nx, x)_{A+B} \ge \; ((\mathbf{A}+\mathbf{B})^{-1}Mx, x)_{A+B}.$$

Since M' is maximal we have that

$$(\mathbf{A}+\mathbf{B})^{-1}N = M' = (\mathbf{A}+\mathbf{B})^{-1}M.$$

This means that $(\mathbf{A}+\mathbf{B})^{-1}(N-M) = 0$ or $M = N$.

In case $Ker(A+B) \neq \{0\}$, we just need to take into account the note at the end of the proof of Theorem 3.

2. Existence of the "minimum" and "maximum" of two self-adjoint operators

Theorem 5. *Let A and B are bounded, self-adjoint operators in the Hilbert Space H. Then there exists a self-adjoint operator T such that $A-T \ge 0$, $B-T \ge 0$ and T is the maximal. That is if $A-S \ge 0$ and $B-S \ge 0$ for some operator S and $S \ge T$ then $T = S$.*

Proof. We just need to note that the operators $A+cI$ and $B+cI$ satisfy the conditions of Theorem 4, where

$$c = \inf_{||x||=1} \{(Ax, x), (Bx, x)\}.$$

Indeed, from Theorem 4 there exists $S \ge 0$ such that $A+cI \ge S$ and $B+cI \ge S$. Then operator $S-cI$ satisfies the conditions of Theorem 5. Therefore Theorem 5 is proved. ∎

Note that by applying the same method we can define the "maximum of two operators". Instead of A and B we consider the operators $-A$ and $-B$. Then we define

$$Max(A, B) = -Min(-A, -B).$$

Now consider the differences $A - T \geq 0$ and $B - T \geq 0$. They are orthogonal by some inner product, namely $(\mathbf{A}+\mathbf{B})^{-1}(\mathbf{A} - \mathbf{T})(\mathbf{A}+\mathbf{B})^{-1}(\mathbf{B} - \mathbf{T}) = 0$, where $\mathbf{A}, \mathbf{B}, \mathbf{T}$ are closures on Hilbert Space H_{A+B}, with inner product $(x, y)_{A+B} = ((A + B)x, y)$ (otherwise, we can find an operator $\mathbf{T}_1 \geq 0$ and $\neq 0$, such that $\mathbf{A} - \mathbf{T} - \mathbf{T}_1 \geq 0$ and $\mathbf{B} - \mathbf{T} - \mathbf{T}_1 \geq 0$, which contradicts the maximality of T).

Now using this note we can prove the next elegance theorem.

Theorem 6. *Let A and B are bounded, self-adjoint operators in the Hilbert Space H and let A commutes with B. Then there exists a self-adjoint operator T such that $A - T \geq 0$ and $B - T \geq 0$ and T is the maximal, furthermore, A and B can be represented in the form*

$$A = T + A_1,\ B = T + B_1 \tag{2}$$

where $A_1B_1 = B_1A_1 = 0$ and A and B commute with T, A_1 and B_1.

Proof. For simplicity, let $A \geq 0, B \geq 0$. (Otherwise, we take $A+cI$ and $B+cI$ instead of A and B). Furthermore, let $A+B > 0$: If $Ker(A+B) \neq \{0\}$, we consider the space $\overline{Re(A + B)}$-the completion of the range of the operator $A + B$ instead of H.

Let $A - T \geq 0$ and $B - T \geq 0$ and T is the maximal in terms of inner product $(x, y)_{A+B} = ((A + B)x, y)$. That is let

$$(\mathbf{A} + \mathbf{B})^{-1}(\mathbf{A} - \mathbf{T})(\mathbf{A} + \mathbf{B})^{-1}(\mathbf{B} - \mathbf{T}) = 0. \tag{3}$$

Let's show that T commutes with A and B. Since A commutes with B, we have that $\mathbf{A}$ and $\mathbf{B}$ are $(A + B)$-self-adjoint operators in the Hilbert Space H_{A+B}. Let E^+_λ is the resolution of the identity for $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}$: that is $(\mathbf{A} + \mathbf{B})^{-1}\mathbf{A} = \int_{-1}^{1} \lambda dE^+_\lambda$. Then

$$(\mathbf{A} + \mathbf{B})^{-1}\mathbf{T} = \int_{-1}^{1} m(\lambda)dE^+_\lambda = (\mathbf{A} + \mathbf{B})^{-1}\int_{-1}^{1} m(\lambda)dE_\lambda,$$

where

$$m(\lambda) = \frac{1-|1-2\lambda|}{2} = \left\{ \begin{array}{ll} \lambda & \text{if } \ \lambda < 1/2 \\ 1-\lambda & \text{if } \ \lambda \geq 1/2 \end{array} \right\}.$$

Let's show that $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}$ is the $(A+B)$-closure of $(A+B)^{-1}A$. Since A commutes with B, operator $(A+B)^{-1}A$ is densely defined on H: $(A+B)^{-1}A \supset A(A+B)^{-1}$ [2, page 171, (5.22)], which means that for $\forall x \in D(A+B)^{-1}$, $Ax \in D(A+B)^{-1}$, and similarly, $Bx \in D(A+B)^{-1}$. This gives

$$\begin{aligned} ((A+B)^{-1}Ax, x) &= ((A+B)^{-1}(A+B-B)x, x) \\ &= (x,x) - ((A+B)^{-1}Bx, x) \leq (x,x) \end{aligned}$$

for $\forall x \in D(A+B)^{-1}$ and so, $(A+B)^{-1}A$ is bounded, positive operator on H. Then

$$(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}x = (A+B)^{-1}Ax \quad \text{for} \quad \text{all} \quad x \in D(A+B)^{-1},$$

and therefore, $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A} = \mathbf{A}(\mathbf{A}+\mathbf{B})^{-1} = \overline{(A+B)^{-1}A}$. We conclude that $\mathbf{A}$ commutes with $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A} = \int \lambda dE_\lambda^+$, which means that $\mathbf{A}$ commutes with E_λ^+ and with $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{T} = \int m(\lambda) dE_\lambda^+$ [1, page 270, Theorem 6.80]. Since $\mathbf{A}$ and $(\mathbf{A}+\mathbf{B})^{-1}\mathbf{T}$ are bounded and $(A+B)$-bounded operators, we have

$$\begin{aligned} (\mathbf{A}+\mathbf{B})^{-1}\mathbf{AT}x &= \mathbf{A}\ (\mathbf{A}+\mathbf{B})^{-1}\mathbf{T}x = A\ (A+B)^{-1}Tx \\ &= (\mathbf{A}+\mathbf{B})^{-1}\mathbf{TA}x = (A+B)^{-1}TAx. \end{aligned}$$

This shows that $ATx = TAx$. Similarly, we can show that $BTx = TBx$. It follows from Eq. (3) that

$$(A+B)^{-1}(A-T)(A+B)^{-1}(B-T) = (A+B)^{-2}(A-T)(B-T) = 0,$$

and $(A-T)(B-T) = 0$, which means that the representation (2) is true, where $A_1 = A - T$ and $B_1 = B - T$ and the result is proved. ■

References

[1] N.I. Akhiezer and I.M. Glazman, "Theory of Linear Operators in Hilbert Space", Vol. 1, Pitman Press, London, 1981

[2] Tosio Kato, "Perturbation Theory for Linear Operators", 2-nd Edition, Springer-Verlag, New York, 1980

Department of Mathematics
Faculty of Art and Sciences
Fatih University,
34900. Buyukcekmece,
Istanbul, Turkey
E-mail: aaslanov@fatih.edu.tr
Phone: 0 212 889 08 10
Fax: 0 212 889 09 12

Received 30.09.2003