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Properties of Digital n-Surfaces with Boundary

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This paper deals with digital n-dimensional surfaces. On the basis of n-manifolds with boundary in algebraic topology and digital n-surfaces, we introduce digital n-surfaces with boundary and study their properties that are similar to properties of their continuous models. We show that the boundary of a digital n-surface is a digital (n-1)- surface without boundary. We prove that the rim of a boundary point of a digital n- surface is a digital (n-1)-surface with boundary. We define the connected sum of two digital spaces and prove that the connected sum of two digital n-surfaces with boundary is a digital n-surface without boundary.
Key words: digital image, digital topology, digital surface, graph, dimension, boundary.

1. Introduction

The study of geometrical and topological properties of two-, three- and n-dimensional image arrays is an object of digital topology. Concepts and results of digital topology play an important role in image processing operations including pattern analysis, detection of dynamically moving surfaces, algorithms for thinning, boundary extraction, object counting, and contour-filling. The most known now approach to digital topology is based on the use of two graphs for structuring the discrete space Z^n, these graphs correspond to two adjacency relations between the elements of Z^n, for example, the 4- and the 8-adjacency in Z^2 [12,16,17,18]. Nevertheless, serious limitations appear in the discrete space Z^3, in particular, because the borders of a three-dimensional object (i.e. two-dimensional surfaces) do not have a natural cyclic ordering [14]. For the study of scientific data which are often more than three dimensional, a very general concept of a digital space was introduced by Herman [8]; it does not assume that there is an underlying Euclidean space. He studied conditions when a near-Jordan surface was a Jordan surface. A new method of representing a surface in the 3D space as a single digitally continuous sequence of faces (two-dimensional cells) was developed by Kovalevsky [13]. The three-dimensional space was considered as an abstract cell complex with all traditional properties of a topological space. A two-dimensional surface was represented as the boundary surface of an arbitrary connected subset of voxels (three-dimensional cells) in a 3D space. Digital n-dimensional manifolds were introduced and studied within the framework for digital topology in which a digital manifold was represented by means of a combinatorial manifold [1,2]. Using the standard cubical decomposition of three-dimensional Euclidean space, a polyhedral decomposition of a 2-disk was defined, and conditions were found when a digital object was a digital 2-manifold without boundary. In paper [3] an algorithm for calculation of the number of components, the number of holes and the Euler characteristics of a digital picture was considered, and the productivity of the proposed algorithm for the calculation of the Euler characteristics with similar algorithms were evaluated. Paper [9] studied one-simply connected digital spaces that were graphs in which there were no holes larger than a triangle. Boundaries in digital pictures which were defined over finitary one-simply connected digital spaces had some desirable general properties including the partition of the space into a connected interior and a connected exterior. Different definitions for digital surfaces were considered in [4]. In [5,6], a digital n-surface (normal n-dimensional digital space) was considered as a simple graph with some topology defined on it, and some properties of such a space were studied. The important feature of an n-surface was a similarity of its properties with properties of its continuous counterpart in terms of algebraic topology. A digital n-surface was considered as an intersection graph of some cover of its continuous n-dimensional model. The Euler characteristic of digital and continuous n-surfaces was the same and the homology groups coincided. Thus it seems desirable to study more closely properties of digital n-surfaces in a fashion that more closely parallels the classical approach of algebraic topology.

2. Computer experiments as the basis for digital spaces

An important feature of this approach to digital topology is that it is based on computer experiments which results can be applied to computer graphics and animations. The following surprising fact is observed in computer experiments modeling deformation of continuous surfaces and objects in three-dimensional Euclidean space [11]. Suppose that S_1 is a surface in Euclidean space E^3. Divide E^3 into a set of unit cubes with integer vertex coordinates and pick out the family M_1 of unit cubes intersecting S_1. Then construct the digital space D_1 corresponding to M_1 in the following way. D_1 corresponds to M_1 if there exists one-one onto correspondence between unit cubes in M_1 and points in D_1 that retains the adjacency relation between elements of M_1 and D_1. Remark that in graph theory D_1 is called the intersection graph of M_1 [7]. Then slightly move and deform surface S_1 into S_1. Reasoning as in the previous case, we construct D_2. After a series of motions and deformations we obtain a sequence of digital spaces D=D_1,D_2,.. D_n. It is revealed that any space in D can be turned into any other space in D by a sequence of a definite type of transformations called contractible, and the Euler characteristic and homology groups of all spaces in D are the same [10,11]. The same tendency is observed when the scale of cubes is changed. It is reasonable to assume that digital spaces contain topological and geometrical characteristics of continuous surfaces, and the transformations digitally model mapping of continuous spaces.

3. Preliminaries

In order to make this paper self-contained we summarize the necessary information from previous papers. A digital space G is a simple undirected graph $G = (V, W)$, where $V = (v_1, v_2, .., v_n)$, is a finite or countable set of points and $W = ((v_p, v_q), ...)$ is a set of edges. Topological properties of G as a digital space in terms of adjacency and connectedness are completely defined by W [5,6]. Since in this paper we use only digital spaces, we say space to abbreviate digital space. We use the notations $v_p \in G$ and $(v_p, v_q) \in G$, if $v_p \in V$ and $(v_p, v_q) \in W$ respectively if no confusion can result. $H = (V_1, W_1)$ is a subspace of $G = (V, W)$ if H is an induced subgraph of the graph $G, H \in G$ [7]. The subspace $O(v)$ containing all neighbors of v (without v) is called the rim of point v in G. The subspace $U(v)$ containing point v and $O(v)$ is called the ball of point v in G [11]. Denote $U(v) - v = O(v)$. If H(G we use the notation O(v)H for the rim of point v in H. Obviously O(v)H=O(v)(H.

The subspace $(v_1, v_2, .., v_p) = O(v_1) \cap (O(v_2) \cap ...(O(v_p)$ is called the joint rim of points $v_1, v_2, ...vp$. The space $v \oplus G$ is called a cone of space G with the center v if point v is adjacent to any point in G [11].

Definition 3.1 A 0-dimensional surface $S^0(a, b)$ is a disconnected graph with just two points a and b. For $n > 0$, an n-dimensional surfaceG_n is a nonempty connected graph such that, for each point v of G_n its rim $O(v)$ is a finite $(n - 1)$-dimensional surface [5,6].

We use notation $dim(G) = n$ for an n-dimensional surface G. In [5] an n-dimensional surface is called a normal n-dimensional space. A point v in G is called n-dimensional in the rim $O(v)$ is an $(n - 1)$-dimensional surface. By definition, the empty space is (-1)-dimensional. Figure 1 depicts zero- and one-dimensional spheres (circles) S^0, S^1_1, S^1_2, S^1_3. Sphere S^1_{min} contains the least number of points. Two-dimensional spheres with different number of points are depicted in figure 2. Figure 3 shows two-dimensional torusT^2 and two-dimensional Klein bottleK^2. The minimal two-dimensional projective plane, three- and four-dimensional spheres are shown in fig. 4. It is easy to check directly that the Euler characteristic and the homology groups of n-surfaces, depicted in figures 1-4 are the same as of their continuous counterparts [10,11].

Definition 3.2. The join $G \oplus H$ of two spaces $G = (X, U)$ and $H = (Y, W)$ is the space that contains G and H and edges joining every point in G with every point in H [5,6]. In graph theory this operation called the join of two graphs [7].

Theorem 3.1. *Let G^n and H^m be n- and m-dimensional surfaces. Then $G_n \oplus H_m$ is an $(n + m + 1)$-dimensional surface, $dim(G^n \oplus H^m) = dim(G^n) + dim(H^m) + 1$ [5,6].*

4. An n-dimensional surface with boundary.

The definition of an n-dimensional surface with boundary should be based on the definition of an n-dimensional surface (without boundary) on one hand, and on the definition of an n-dimensional manifold with boundary in the continuous framework on the other hand. n-dimensional surfaces [5,6] model compact connected n-manifolds in classical approach of algebraic topology. Remind that in classical approach, loosely speaking, G is an n-dimensional manifold with boundary if each point has a neighborhood homeomorphic either to the open n-ball or to the open half-n-ball. However a transition from a continuous space to a digital space raises problems due to different cardinality of these spaces. Therefore, on distinct from continuous case, the result of the operation depends on a number of points. One more distinction of a digital space from a continuous one is that all points of a digital space can be boundary ones.

Definition 4.1. Let G and B be a digital space and its subspace respectively. G is called an n-dimensional surface with boundary B if gluing of a point v to G in such a manner that $O(v) = B$ converts G into an n-dimensional surface. The set A of points belonging to G and not belonging to B is called the interior. Obviously $A = G - B, B = G - A$. Figure 5 depicts one- and two-dimensional surfaces U^1, U_1^2, U_2^2 and U_3^2 with boundary. The interior consists of one point w. Gluing point v to these spaces over the boundary converts them to one- and two-dimensional spheres shown in fig. 1 and 2. We use denotation $B = bd(G)$ and $A = int(G)$ for the boundary and the interior respectively. A simple consequence of this definition is the following. In order to proof the next theorem, we prove two auxiliary propositions first.

Proposition 4.1. *An n-dimensional surface G is not a cone.*

Proof. The proof is by induction. For $n = 0, 1$ it is checked directly. Assume that the theorem is valid whenever $n < k+1$. Suppose the contrary, let $n = k + 1$ and $v \oplus H$, where point v is adjacent to every point of the subspace H. Take a point u in H. Obviously the rim $O(u)$ is a cone, $O(u) = v \oplus O(u)_H$. It yields a contradiction since $O(u)$ is an $(n-1)$-dimensional surface and must not be a cone by the induction hypothesis. Therefore G is not a cone. ■

The next assertion follows directly from this proposition 4.1.

Proposition 4.2. *In an n-dimensional surface G, for any point v there exists a point u such that v and u are not adjacent.*

Proof. Since G is not a cone by proposition 4.1, then for an arbitrary point v in G there exists point u such that u is not adjacent to v. ■

Theorem 4.1. *Let G be an n-dimensional surface with boundary B and interior A. Then:*

(a) BoundaryB is an $(n-1)$-dimensional surface (without boundary).

(b) The rim $O(w)$ of any point w belonging to B is an $(n-1)$-dimensional surface with boundary $O(w) \oplus B$.

(c) Then interior A is a non-empty space, and the rim $O(u)$ of any point u in A is an $(n-1)$-dimensional surface (fig. 5).

Proof. Glue a point v to G in such a manner that $O(v) = B$. Then the obtained space $H = G + v$ is an n-dimensional surface by the definition 4.1.

Proof of (a). Since H is an n-dimensional surface, then $O(v) = B$ is an $(n-1)$-dimensional surface by the definition 3.1.

Proof of (b). Let w belong to B. Denote $O(w)_G$ and $O(w)_H$ the rims of point w in G and H respectively. The operation of gluing of point v to G converts $O(w)_G$ to $O(w)_H$. Since H is an n-dimensional surface then $O(w)_H$ is an $(n-1)$-dimensional surface. Hence $O(w)_G$ is an $(n-1)$-dimensional surface with boundary $B \oplus O(w)_G$.

Proof of (c). By proposition 4.2, for the point v in H there exists a point u, which is not adjacent to v. Therefore, u belongs to interior A. Consider any point u in A. Since H is an n-dimensional surface then $O(u)$ is an $(n-1)$-dimensional surface by definition 3.1 and does not contain point v. ■

Proposition 4.3. *Let G be an n-dimensional surface with boundary B and interior A. Then A is connected if $n > 0$.*

Proof. We have to prove that for any two points x and y in A there exists a path L of points $(v_1, v_2, ...v_n)$ in A, where $v_1 = x, v_n = y$ and v_k is adjacent to v_{k+1}, $k = 1, 2, ...n-1$. The proof is by induction. For dimension $n = 1$, the theorem is verified directly. Assume that the theorem is valid whenever

$n < k+1$. Let $n = k+1$. Glue point v to G in such a manner that $O(v) = B$. We obtain an n-dimensional surface H which is connected by the definition. Take two arbitrary points x and y in A. Then there exists a path L of points $(v_1, v_2, ...v_n)$ in H where $v_1 = x, v_n = y$.

(i) If point $v = v_k$ belongs to L, then points v_{k-1} and v_{k+1} belong to B which is also a connected space. Then part $L_1 = (v_{k-1}, v_k, v_{k+1})$ of L can be replaced by part $L_2 = (v_{k-1}, ...v_{k+1})$ belonging to B. The obtained path does not contain point v.

(ii) Suppose that point v_k is the first point in path L belonging to B. Then point v_{k-1} belongs to A and $O(v_k)$. Since $O(v_k)$ is an $(n-1)$-dimensional surface by theorem 4.1, then by the inductive assumption, there is a path $(v_{k-1}, ...v_{k+1})$ belonging to $O(v_k)$. Therefore, part $L1 = (v_{k-1}, v_k, v_{k+1})$ of L can be replaced by part $L_2 = (v_{k-1}, ...v_{k+1})$ without point v_k. This operation is repeated until all points belonging to B are excluded from path L. Hence, all points of path L belong to A. ■

Theorem 4.2. *Let G be an n-dimensional surface. Then deleting of any point v converts G into an n-dimensional surface $H = G - v$ with boundary $bd(H) = O(v)$ (fig. 6).*

Proof. Delete a point v from G and denote $H = G - v$ the obtained space. Obviously, H is an n-dimensional surface with boundary $O(v)$ since converse gluing of v to H turns H into G. ■

In fig. 6, deleting of point v from two-dimensional sphere S^2 and projective plane P^2 converts these spaces into two-surfaces with boundary.

Theorem 4.3. *Let G be an n-dimensional surface, $n > 0$. Then deleting of any edge (v_1v_2) converts G into an n-dimensional surface $H = G - (v_1v_2)$ with boundary $bd(H) = S^0(v_1, v_2) \oplus O(v_1, v_2)$.*

Proof. The proof is by induction. For small dimensions n, it is checked directly. Assume that the theorem is valid whenever $n < k+1$. Let $n = k+1$. Take an edge (v_1v_2) in G. By the definition of an n-dimensional surface, the joint rim $O(v_1, v_2)$ is an $(n-2)$-dimensional surface. Delete an edge (v_1v_2) from G and denote $H = G - (v_1v_2)$ the obtained space. Glue a point v to H in such a manner that $O(v) = S^0(v_1, v_2) \oplus O(v_1, v_2)$ meaning that v is adjacent to points v_1, v_2 and any point in $O(v_1, v_2)$. Denote $X = H + v$ the obtained space. Denote $O(u)_H$, $O(u)_H$ and $O(u)_X$ the rims of a point u in G, H and X respectively.

(i) Since $O(v_1, v_2)_G$ in G is an $(n-2)$-dimensional surface, then $O(v)_X = S^0(v_1, v_2) \oplus O(v_1, v_2)$ is an $(n-1)$-dimensional surface in X by theorem 3.2.

(ii) It is easy to see that $O(v_1)_G$ and $O(v_1)_X$ are the same, except for that point v_2 in G is replaced with point v in X. Since $O(v_1)_G$ is an (n-1)-

dimensional surface, then $O(v_1)_X$ is an $(n-1)$-dimensional surface. Similarly, $O(v_2)_X$ is an $(n-1)$-dimensional surface.

(iii) Let a point u belong to $O(v_1, v_2)$ in G. By the definition of an n-dimensional surface, the rim $O(u)_G$ is an $(n-1)$-dimensional surface. Since points v_1 and v_2 belong to $O(u)_G$, then the operations of deleting of edge $(v_1 v_2)$ and gluing of point v take place in $O(u)_G$. Hence, by the induction hypothesis, $O(u)_X$ is an $(n-1)$-dimensional surface.

(iiii) All other points in G (except for v_1, v_2 and points in $O(v_1, v_2)$) retain their rims. Therefore, the rim of any point in X is an $(n-1)$-dimensional surface, and X is n-dimensional. Therefore, by definition 4.1, H is an n-dimensional surface with boundary $bd(H) = S^0(v_1, v_2) \oplus O(v_1, v_2)$. ■

Proposition 4.4. *Let G be an n-dimensional surface. Then the cone $v \oplus G$ is an $(n+1)$-dimensional surface with boundary G and interior v, $bd(v \oplus G) = G$, $int(v \oplus G) = v$ (compare fig. 1 and 5).*

Proof. Let G is an n-dimensional surface in cone $H = v \oplus G$. Glue a point u to H in such a way that $O(u) = G$. Obviously, the obtained space X can be considered as the join $S^0(v, u) \oplus G$ and, therefore, is an $(n+1)$-dimensional surface by theorem 3.1. Hence by definition 4.1, the cone $v \oplus G$ is an $(n+1)$-dimensional surface with boundary G and interior v, $bd(v \oplus G) = G$, $int(v \oplus G) = v$. ■

Theorem 4.4. *If G is an n-dimensional surface with boundary B and interior A, H is an m-dimensional surface, then the join $G \oplus H$ is an $(n+m+1)$-dimensional surface with boundary $B \oplus H$ and interior A.*

Proof. Glue point v to G in such a manner that $O(v) = B$. Then the obtained space $X = G + v$ is an n-dimensional surface by definition 4.1. By theorem 3.1, the join $Y = X \oplus H$ is an $(n+m+1)$-dimensional surface. Delete point v from Y. It is obvious that the obtained space $Z = Y - v$ is the join $G \oplus H$. On the other hand by theorem 4.2, the obtained space Z is an $(n+m+1)$-dimensional surface with boundary $O(v)Y = O(v)_X \oplus H = B \oplus H$ and interior A. ■

On the basis of an n-dimensional surface G, it is possible to construct a space H homotopic to G in which every point has the topological structure of a boundary point. It means the rim $O(v)$ of any point v is an n-dimensional surface with boundary. Using continuous terminology, we can say that a neighborhood of any point in H is a half-ball. This kind of digital spaces has no direct continuous counterparts since for any continuous n-dimensional manifold with boundary, the interior is always non-empty. Figure 7 shows such a space H where all points are boundary points; the rim of any point v is an one-dimensional surface $O(v)$ with boundary.

5. The connected sum of two n-dimensional surfaces with boundary.

Gluing of point v to an n-dimensional surface G with boundary B in definition 4.1 is equal to two operations: constructing the n-dimensional surface $v \oplus B$ with boundary B and the only interior point v and identifying points in B in space $v \oplus B$ with corresponding points in B in space G. In algebraic topology [15], a similar operation of gluing is used to obtain a set of compact connected 2-manifolds and classify them. Arbitrary examples are constructed by gluing (fig. 8). We generalize these operations to arbitrary n-surfaces with boundaries. Using topological terminology, call this operation the connected sum of two digital n-surfaces with boundary. Remind the isomorphism of digital spaces. Note that an isomorphism of digital spaces is an isomorphism of graphs [7] if we see a digital space as a graph.

Definition 5.1. A digital space G with a set of points $V = (v_1, v_2, ...v_n)$ and a set of edges $W = ((v_p v_q), ...)$ is said to be isomorphic to a digital space H with a set of points $X = (x_1, x_2, ...x_n)$ and a set of edges $Y = ((x_p x_q), ...)$ if there exists one-one onto correspondence $f : V \longrightarrow X$ such that $(v_i v_k)$ is an edge in G iff $(f(v_i) f(v_k))$ is an edge in H. Map f is called an isomorphism of G to H. We write $G \approx H$ to denote the fact that there is an isomorphism of G to H.

Definition 5.2. Let G be an n-dimensional surface with boundary B and interior A, H be an n-dimensional surface with boundary D and interior C. If B and D are isomorphic, then the space $G \triangle H$ obtained by identifying points in B with corresponding points in D is said to be the connected sum of G and H by boundary B (fig. 9).

Theorem 5.1. *If G is an n-dimensional surface with boundary B and interior A, H is an n-dimensional surface with boundary D and interior C, and $B \approx D$, then the connected sum $G \triangle H$ by B is an n-dimensional surface (without boundary).*

Proof. The proof is by induction. For dimension $n = 1$, it is checked directly. Assume that the theorem is valid whenever $n < k + 1$. Let $n = k + 1$. Suppose that G is an n-dimensional surface with boundary B and interior A, H is an n-dimensional surface with boundary D and interior C, and f is an isomorphism of B to D. Obviously, in the connected sum $G \triangle H$ point v in B merges point $f(v)$ in D and its rim in $G \triangle H$ is the connected sum $O(v) \triangle O(f(v))$. $O(v)$ and $O(f(v))$ are $(n-1)$-dimensional surfaces with boundaries $O(v) \cap B$ and $O(f(v)) \cap D$. Therefore, $O(v) \triangle O(f(v))$ is an $(n-1)$-dimensional surface (without boundary) by the inductive assumption. The rims of all points belonging to interiors A and C are $(n-1)$-dimensional surfaces and do not change. Therefore, the rim of any point in $G\triangle H$ in an $(n-1)$-dimensional surface. Hence, $G\triangle H$ is an n-dimensional surface. ∎

Figure 9 shows the connected sum of two-dimensional disks with interior points v and u respectively and boundary points 1, 2, 3 and 4. The connected sum $U_1^2\triangle U_1^2$ is the minimal two-dimensional sphere. This theorem can be used to construct two-dimensional surfaces as it is in the classical approach of algebraic topology.

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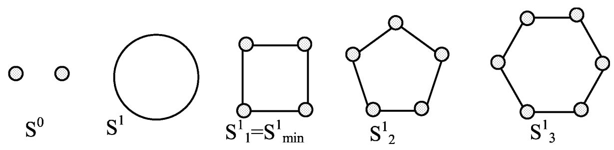

Figure 1. Zero- and one-dimensional spheres S^0, $S^1{}_1$, $S^1{}_2$, $S^1{}_3$.

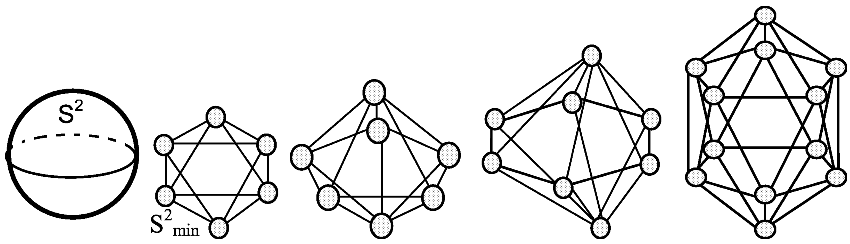

Figure. 2. Two-dimensional spheres with a different number of points. All of spheres can be converted into the minimal sphere $S^2{}_{min}$ by contractible transformations.

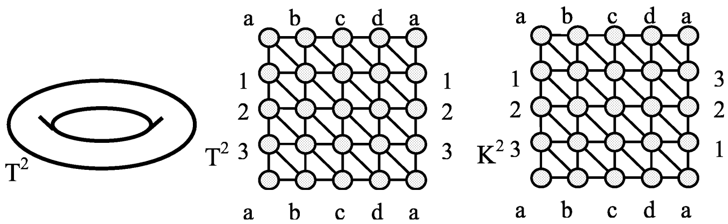

Fig. 3. Two-dimensional torus $T^2 = T^2{}_{min}$ and Klein bottle $K^2 = K^2{}_{min}$ with sixteen points.

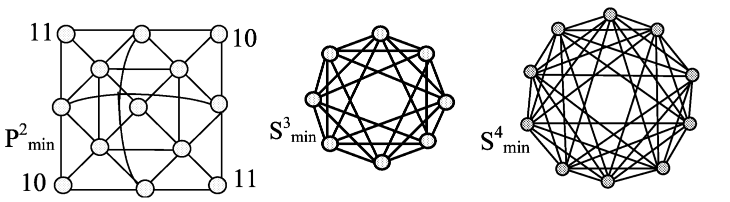

Fig. 4. $P^2{}_{min}$, $S^3{}_{min}$ and $S^4{}_{min}$ are the minimal two-dimensional projective plane, three- and four-dimensional spheres respectively.

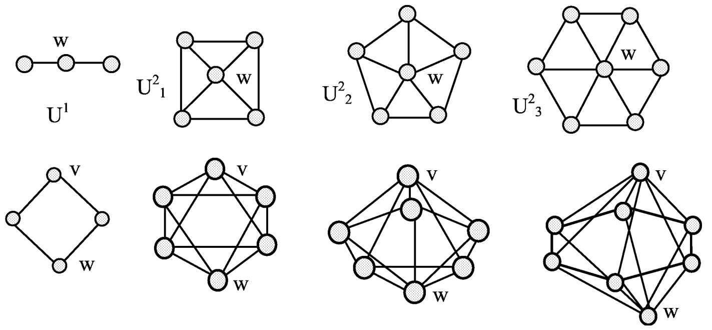

Fig. 5. One- and two-dimensional surfaces U^1, U^2_1, U^2_2 and U^2_3 with boundary. The interior consists of one point w. Gluing point v to these spaces over the boundary converts them to one- and two-dimensional spheres.

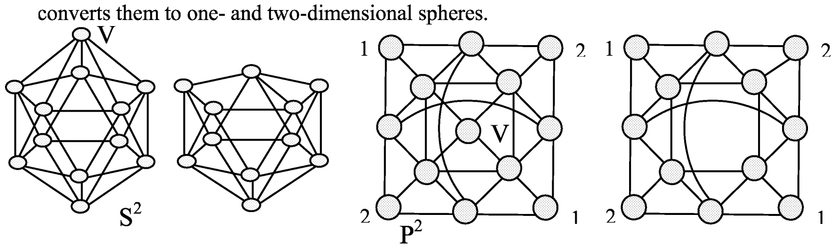

Figure 6. Deleting of point v from normal two-dimensional sphere S^2 and projective plane P^2 converts these spaces into normal spaces with boundary.

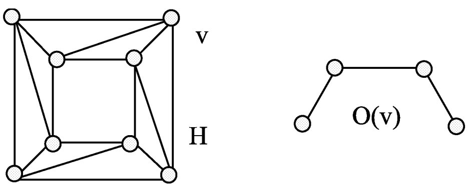

Fig. 7. The space H in which every point is a boundary point of a normal two-dimensional space. The rim O(v) of point v is a normal one-dimensional space with boundary.

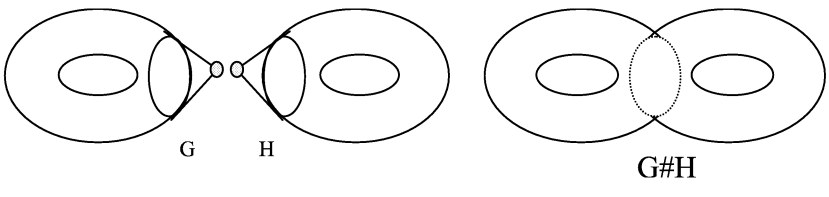

Fig. 8. The connected sum of two tori G and H is a sphere with two handles .G#H.

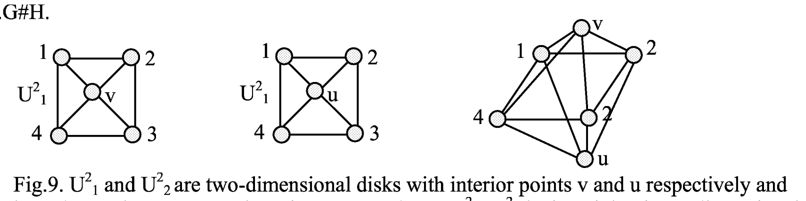

Fig.9. U^2_1 and U^2_2 are two-dimensional disks with interior points v and u respectively and boundary points 1, 2, 3 and 4. The connected sum $U^2_1\#U^2_1$ is the minimal two-dimensional sphere.