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Plane Mechanism, and Dual Spatial Motions

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The spherical representations of the roselike curves being generated by hy(t;n,m,r,α) and ep(t;n,m,r,α), the dual developable ruled roselike and hyperbolic surfaces and their graphs are given. It is well-known that the roses are generated by natural mechanism on the plane. Translation operations of curves and surfaces in generally from any Euclidean space E^n to the real sphere are given originally in this paper. The dual ruled or developable ruled surfaces are obtained by the unit spherical representations of the planar or any dimensional Euclidean space curves and surfaces.

Let β(s) be the arclength reparametrization of ξ(t) = (α(t), β(t), γ(t)). Then, we may construct the dual developable ruled surface β(t,ζ) = β(t) $\wedge$ β^*(t) + ζ β(t), ζ $\in$IR.

Let λ(s) be the arclength reparametrized spherical curve obtained from the curve hy (t;n,m,r,α). Then, we may obtain the dual ruled roselike surface λ(t,ζ) = λ(t) $\wedge$ λ^*(t) + ζ λ(t), ζ $\in$IR.

Let x(t) be the spherical curve derived from the hyperbolic curve (acht, bsht, 0), a,b$\in$IR. Then, we may write the equation X(t,ζ) = x(t) $\wedge$ x*(t) + ζ x(t), as the dual developable ruled surface.

1. Introduction

1.1. Key Words: Hypotrochoid, epitrochoid, hypocycloid, epicycloid, roselike curve, spherical representation, reparametrization, dual number, dual sphere.

1.2. Reference Note: There exitsts a vast reference on the subjects, for examle, references from [1] to [13] on roses, references from [14] to [20] on dual spherical motions and references from [21] to [26] on differential geometry. Hall, [13], presents trochoids, roses, and thorns in a clear way. Hacisalihoglu, [17] and [18], explains the dual spherical motions very detaily. Aminov, [25] and [26], presents the old and the contemporary geometric results I have used some of them in this paper.

In section 2 the fundamental definitions and well-known results about hypotrochoids, hypocycloids, epitrochoids, epicycloids, and pedals are presented. In section 3 we give the definitions and the results of spatial motions and developable ruled surfaces. In section 4 some general results about dual developable ruled roselike surfaces related with the arclength reparametrized curve β(t) of the roselike curve ξ (t) being generated by ep (t; 3,1,2,0) are derived. The scope of this manuscript is to obtain some general instantaneous invariants of dual spherical motions and the dual developable ruled roselike surfaces. One figure about the rose being generated by ep (t; 3,1,2,0) and its spherical image is given. Finally we give the graphs of the dual developable ruled roselike and hyperbolic surfaces. I gratefully acknowledge the writer Hall who have written in this paper's reference [13] for his usefull and contemporary suggestions.

2. Trochoids, hypotrchoids, hypocycloids, epitrochoids, epicycloids, roses, pedals, and thorns

2.1. Definition. From the references [1] to [13] we may write the following definitions: "Pedals" will be used instead of "leaves" when describing roselike curves. As for the thorns, we shall see that there are indeed curves with a definite thorny appearance that are closely related to mathematical roses.

If two tangent circles have their centers on the same side of the common tangent line, and one circle remains fixed while the other is rolled around it without slipping, a hypotrochoid is traced by any point on a diameter or extended diameter of the rolling circle. If two tangent circles have their centers on opposite sides of the common tangent line, and one circle remains fixed while the other is rolled around it without slipping, an epitrochoid is traced by any point on a diameter or extended diameter of the rolling circle. A hypocycloid is a hypotrochoid for which the tracing point is on the circumference of the rolling circle, and an epicyloid is an epitrochoid for which the tracing point is on the circumference of the rolling circle. The term trochoid is used to refer to either a hypotrochoid or an epitrochoid. Either radius, but not both, can be infinite, so that cycloids and trochoids obtained by rolling a circle along a straight line, and also certain spirals and involutes are covered by the nomenclature, but in this paper we shall assume both radii are finite.

Since the graph of a trochoid depends on four parameters that are fixed and one that is variable, all these quantities will be part of the notation. The hypotrochoid denoted by hy (t;n,m,r,α) is generated by a rolling (moving) circle of radius n, with rm the distance from the center of the rolling circle to the tracing point. Assume the center of the fixed circle is at the origin and denote the initial position of the tracing point (and the center of the rolling circle) by the angle α, measured from the positive x-axis. The corresponding epitrochoid will be denoted by ep (t;n,m,r,α). If t is the counterclockwise angle from the positive x-axis to the line joining the centers of the two circles, using e^{it} = cost + i sint, the parametric equations are easily derived.

2.2. The Parametric Equation of Hypotrochoid.

hy(t;n,m,r,α) = {x[t],y[t]},

where

$$x[t] = (n - m)\cos(t + \alpha) + rm\cos\left(\frac{(n-m)\,t}{m} - \alpha\right)$$

and

$$y[t] = (n - m)\sin(t + \alpha) - rm\sin\left(\frac{(n-m)\,t}{m} - \alpha\right)$$

2.3. The Parametric Equation of Epitrochoid.

ep (t;n,m,r,α) = {x [t],y[t]},

where

$$x[t] = (n + m)\cos(t + \alpha) - rm\cos\left(\frac{(n+m)\,t}{m} + \alpha\right)$$

and

$$y[t] = (n + m)\sin(t + \alpha) - rm\sin\left(\frac{(n+m)\,t}{m} + \alpha\right)$$

2.4. The Parametric Equations of The First Positive Pedal. The first positive pedal of a curve C with respect to P is the locus of points where straight lines through P meet tangents to C at right angles. A point on the pedal is the foot of the perpendicular from P to a tangent line – hence gives the name. The normal pedal of C with respect to P uses the normal line instead of the tangent line. If C is given parametrically by x = f (t) and y = g (t), and P is the origin, then the parametric equations of the first positive pedal are:

$$x(t) = \frac{g'(t)\,(f(t)\,g'(t) - f'(t)\,g(t))}{(f'(t))^2 + (g'(t))^2}, \quad y(t) = -\frac{f'(t)\,(f(t)\,g'(t) - f'(t)\,g(t))}{(f'(t))^2 + (g'(t))^2}$$

These formulas used above may be found in the references [3] and [13].

3. Spatial motions and developable ruled surfaces

The formulas, definitions, and general results we have written in the following may be found in the references from [14] to [21].

3.1. Definition. If, a,a^* $\in$ IR , and $\varepsilon^2 = 0$, then the combination A(ε) = a + εa^* is called a dual number. Hence ε is the dual unit. Dual numbers may be considered as polynomials in ε, $\varepsilon^2 = 0$. Dual numbers form an algebra, not a field. The numbers εa^* are called pure dual numbers. It is clear that (εa^*) (εa^*) = 0. εa^* has not an inverse in the algebra. But the other laws of the algebra of dual numbers are the same as the laws of the algebra of complex numbers. Hence dual numbers form a ring over the real number field. For example, two dual numbers A(ε) and B(ε) are added componentwise. A(ε)+B(ε) = (a + b)

$+ \varepsilon$ (a* + b*), and they are multiplied by $A(\varepsilon).B(\varepsilon)$= ab + ε (a*b + ab*). For the equality of $A(\varepsilon)$ and $B(\varepsilon)$ we write $A(\varepsilon) = B(\varepsilon) \Leftrightarrow$ a = b, and a* = b*. An oriented line in IR ^{3} may be given by two points on it, x and y, if P is any nonzero parameter, the parametrized equation of the line is y = x + Pa, and the moment of the vector a with respect to the origin O is a* = x $\wedge$ a = y $\wedge$ a. This means that the direction vector a of the line, and its moment vector a* are independent of the choice of the points of the line. The two vectors a and a* are not independent of one another, they satisfy the equations, $a^2 = 1$, and a.a* = 0. The six components a_i, a_i^*(i = 1,2,3) of a and a* are Plückerian homogeneous line coordinates. This means that the two vectors a and a* determine the oriented line. A point z is on the line of vectors a, a* if and only if z $\wedge$ a = a*. The set of oriented lines in IR^3 is in one – to – one correspondence with pairs of vectors in IR ^{3}, and so we may expect to represent it as a certain six – dimensional set in IR^6 of sixtuples of real numbers. We may take the space D^3 of triples of dual numbers with coordinates $X_1(\varepsilon) = x_1 + \varepsilon x_1^*$, $X_2(\varepsilon) = x_2 + \varepsilon x_2^*$, $X_3(\varepsilon) = x_3 + \varepsilon x_3^*$. Each line in IR^3 is represented by the dual vector in D^3. Consider $A(\varepsilon) =$ a + εa*. $A(\varepsilon)$ is a dual unit vector, that is, $A(\varepsilon).A(\varepsilon)$ = a.a + 2εa. a* = 1 if we carry over the formal definition of the products of vectors to dual space. We may give the following theorems without proof.

3.2. Theorem. *The oriented lines in IR^3 are in one–to–one correspondence with the points of the dual unit sphere $A(\varepsilon).A(\varepsilon) = 1$ in D^3, references [14], {17], [18].*

3.3. Theorem. *The Euclidean motions in IR^3 are represented in D^3 by the dual orthogonal matrices $x = (x_{ij})$, $xx^t = I$, where, x_{ij} are dual numbers, references [14], [18], [19].*

3.4. Theorem. *The six-parameter group of motions is the commutative product of the three – parameter group of rotations, and the three – parameter group of translations, references [18], [20], [22].*

According to standard base in IR^3 we may write $u = u_1e_1 + u_2e_2 + u_3e_3$, $u_0^* = u_{01}^*e_1^* + u_{02}^*e_2^* + u_{03}^*e_3^*$.

3.5. Definition. If we write $(u, u_0^*) = (u_1, u_2, u_3, u_{01}^*, u_{02}^*, u_{03}^*)$, then u_i, u_{0i}^*, (i = 1 , 2 , 3), are called normed nonhomogeneous Plückerian vector or line coordinates. For $g > 0$ if we write v = g u instead of u, and $v_0^* = g\ u_0^*$ instead of u_0^* then $< v,\ v_0^* >= 0$, where $v_0^* = x \wedge v$. The six components of the couple of $(v,\ v_0^*) = (gu_1, gu_2, gu_3, gu_{01}^*, gu_{02}^*, gu_{03}^*)$ are called unnormed homogeneous Plückerian vector or line coordinates. The six, u_i, u_{0i}^* (i = 1 , 2 , 3), normed nonhomogeneous Plückerian line coordinates may be considered to be the element $(u_1, u_2, u_3, u_{01}^*, u_{02}^*, u_{03}^*)$ in IR^6, references [18], [19], [20].

3.6. Definition. If x∈IR^3 and x≠0, then the norm $\|x(\varepsilon)\|$ of x(ε) = x + εx* is defined by $\sqrt{x(\varepsilon).x(\varepsilon)}$. Hence we have

$$\|x(\varepsilon)\| = \|x\| + \varepsilon\frac{xx^*}{\|x\|} = \|x(\varepsilon)\| \left(1 + \varepsilon \frac{xx^*}{\|x\|^2}\right)$$

If the relations $x.x = 1$ and $x.x^* = 0$ hold simultaneously, then the dual vector $x(\varepsilon)$ is called a dual unit vector. It is clear that for any dual vector $x(\varepsilon) = x + \varepsilon x^*$, the vector $u(\varepsilon) = \frac{x(\varepsilon)}{\|x(\varepsilon)\|}$, $x \neq 0$, is always a dual unit vector.

3.7. Definition. If $x(\varepsilon) = x + \varepsilon x^*$ is any dual unit vector and $P = x \wedge x^*$, then the vectors x, x^* are called Plückerian vectors, where $\wedge$ denotes the vectorical product.

The Plückerian vectors x, x^* determine the line L containing them considered as orientable or a spear. We also call L the carrier of the spear, and x the spear vector. With respect to 0, the vector x^* is usually called the moment of the spear. Let L be the set of all oriented lines in IR^3, then it is evident that L ≅ x(ε) = x + εx*

4. Dual spherical motions and dual developable ruled surfaces of hypotrochoids, epitrochoids and hyperbole

4.1. Definition. The sphere determined by $x_1^2 + x_2^2 + x_3^2 = 1$ and $x_1x_1^* + x_2x_2^* + x_3x_3^* = 0$ is called the dual unit sphere with 0 as its center. If we denote the real spheres by S^2, S^3, ..., S^n without pole points, and the dual sphere by S$^2(\varepsilon)$ without pole points in IR^2, IR^3, ..., IR^n and in IR$^2(\varepsilon)$, respectively, it is clear that there are one–to–one correspondence between them, that is, as topologically and as topologic continually we may write $S^2 \cong S^2(\varepsilon)$.

4.2. Definition. If X(t,ε) be a differentiable curve on the dual unit sphere, depending on a t time parameter, it defines a differentiable family of straight lines, then it is called a ruled surface of 3-dimensional space.

4.3. Definition. For a dual point X(ε) on the dual unit sphere we may write $\sum_{i=1}^{3} x_i(\varepsilon)\, e_i(\varepsilon) = \sum_{i=1}^{3} x_i(\varepsilon)\, r_i(\varepsilon)$. Let

$$X(\varepsilon) = \begin{bmatrix} X_1(\varepsilon) \\ X_2(\varepsilon) \\ X_3(\varepsilon) \end{bmatrix} \text{ and } x(\varepsilon) = \begin{bmatrix} x_1(\varepsilon) \\ x_2(\varepsilon) \\ x_3(\varepsilon) \end{bmatrix}$$

be the position vectors of X(ε) and x(ε) with respect to E(ε) = {e$_1(\varepsilon)$,e$_2(\varepsilon)$,e$_3(\varepsilon)$} and R(ε) = {r$_1(\varepsilon)$,r$_2(\varepsilon)$,r$_3(\varepsilon)$}. Then we may write:

$$x_i(\varepsilon)=(e_i(\varepsilon).r_1(\varepsilon))\, r_1(\varepsilon)+(e_i(\varepsilon).r_2(\varepsilon))\, r_2(\varepsilon)+(e_i(\varepsilon).r_3(\varepsilon))\, r_3(\varepsilon)\,,\ i=1,2,3.$$

Taking $e_i(\varepsilon).r_k(\varepsilon) = \alpha_{ik}$, and using the dual matrix

$$A(\varepsilon) = \alpha_{ik}(\varepsilon) = (\alpha_{ik}) + \varepsilon\,(\alpha^*_{ik}) = A + \varepsilon A^*$$

we may write

$$X(\varepsilon) = A(\varepsilon)x(\varepsilon) = (A+\varepsilon A^*)\,(x+\varepsilon x^*) = Ax + \varepsilon\,(Ax^* + A^*x + \varepsilon^2 A^* x^*)$$

or

$$X(\varepsilon) = X + \varepsilon X^*.$$

If the matrix $A(t,\varepsilon) = A(t) + \varepsilon A^*(t)$ is a periodic function of the time parameter t, that is, $A\ (t + 2\pi,\varepsilon) = A\ (t,\varepsilon)$ the motion K/K′ is called a closed motion, otherwise it is called an open motion. During the closed or open motion K/K′, the orbits of the fixed points of K are closed or open curves on K′, then the corresponding ruled surfaces are closed surfaces or open surfaces in K′, where K′ is the fixed space in the motion.

4.4. Spherical images of the curves hypotrochoids, epitrochoids, hypocycloids, epicycloids, pedals and hyperbole

Consider the curve ep (t; n,m,r,α) expressed in 2.3. Then, it is easy to see that ep (t; 3,1,2,0) = (4 cost – 2 cos4t, 4 sint – 2 sin4t). Let S^2 be the unit sphere $x^2 + y^2 + (z-1)^2 = 1$ with center (0,0,1), radius = 1, and tangent to xy plane at the origin (0,0,0), without the pole point (0,0,2). Let (a,b,0) be any point of the roselike curve being generated by ep (t; 3,1,2,0). Hence, we may write the straight line passing through the points (0,0,2) and (a,b,0) by the equation $\frac{x}{a} = \frac{y}{b} = \frac{2-z}{2}$. Let ($\alpha$,$\beta$,$\gamma$) be the intersection of the line and the unit sphere without the pole point (0,0,2). If we use the point (α,β,γ) in the straight line equation, then we may write $\frac{\alpha}{a} = \frac{\beta}{b} = \frac{2-\gamma}{2}$, $a = \frac{2\alpha}{2-\gamma}$, $b = \frac{2\beta}{2-\gamma}$, *and* $a^2 + b^2 = \frac{4\,(\alpha^2+\beta^2)}{(2-\gamma)^2}$. The equation of the unit sphere S^2 satisfies the point (α,β,γ). Then, we have $\alpha^2 + \beta^2 + (\gamma-1)^2 = 1$, *and* $\alpha^2 + \beta^2 = \gamma\,(2-\gamma)$. From these equations we may write $a^2 + b^2 = \frac{4\gamma}{2-\gamma}$, or adding 4 to each sides, $a^2 + b^2 + 4 = \frac{8}{2-\gamma}$. From the previous results we may have

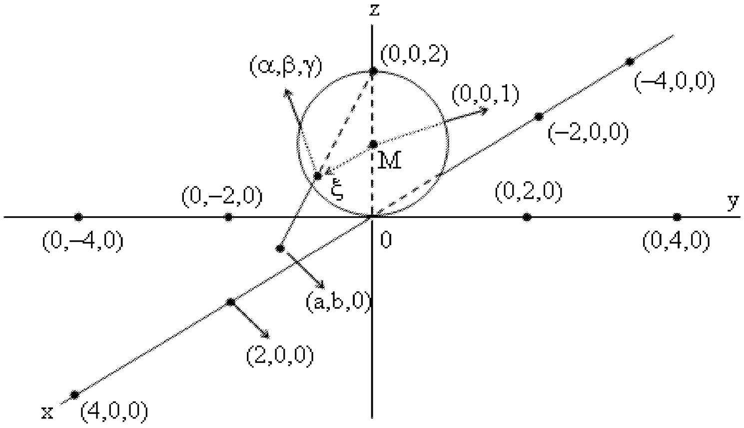
$$\alpha = \frac{a\,(2-\gamma)}{2} = \frac{4a}{a^2+b^2+4}\,, \quad \beta = \frac{b\,(2-\gamma)}{2} = \frac{4\,b}{a^2+b^2+4}\,,$$

$$\gamma = \frac{2\,(a^2 + b^2)}{a^2 + b^2 + 4}\,.$$

Using the components of the roselike curve being generated by ep (t; 3,1,2,0) in the above equations, then it is easy to verify that we may write the spherical image of the curve,

$$\xi = (\alpha,\beta,\gamma) = \left(\frac{2\cos t - \cos 4t}{3 - 2\cos 3t}, \frac{2\sin t - \sin 4t}{3 - 2\cos 3t}, \frac{5 - 4\cos 3t}{3 - 2\cos 3t}\right).$$

The graph of the transformation is given in the following figure.



The rose being generated by ep (t; 3,1,2,0) and its spherical image.

For the spherical curve $\xi = (\alpha(\mathrm{t}),\ \beta(\mathrm{t}),\ \gamma(\mathrm{t}))$, it is possible to obtain a curve $\beta = (\beta_1(\mathrm{s}),\beta_2(\mathrm{s}),\beta_3(\mathrm{s}))$ parametrized by arclength which has the same trace as ξ. In fact, let

$\mathrm{s} = \mathrm{s(t)} = \int_0^t \|\xi'\,(t)\|\,dt$, t,to$\in$ the domain of the curve ξ. Since $\frac{ds}{dt} = \|\xi'\| \neq 0$, the function s = s(t) has a differentiable inverse function s^{-1} of s. Now set $\beta = \xi$o t. Clearly, $\|\beta'\,(s)\| = \left\|\xi'\,(t)\,.\,\frac{dt}{ds}\right\| = 1$. This shows that β has the same trace as ξ and is parametrized by arclength. It is usual to say that β is a reparametrization of ξ by arc length. This fact allows us to extend all local concepts defined to the curve ξ. Thus, we say that the curvature k(t) of ξ at t is the curvature of a reparametrization β of ξ by arclength at the corresponding point s = s(t). This is clearly independent of the choice of β. It is often convenient to use t as a parameter istead of s in the reparametrized curve β.

4.5. Theorem. *Let $\beta(t)$ be the arclength reparametrization of $\xi(t) = (\alpha(t),\beta(t),\ \gamma(t))$. Then, we may construct the dual developable ruled surface $\beta(t,\zeta) = \beta(t) \wedge\ \beta^*(t) + \zeta\ \ \beta(t)$, where $\wedge$ denotes the vectorical product.*

P r o o f. Firstly, we must compute the dual curve

$\beta^*(\mathrm{t}) = \mathrm{OM} \wedge \xi(\mathrm{t}) = (0,0,1) \wedge \left(\frac{2\cos t - \cos 4t}{3-2\cos 3t}, \frac{2\,sin\, t - \sin 4t}{3-2\cos 3t}, \frac{2-2\cos 3t}{3-2\cos 3t}\right)$

$$\beta^*(\mathrm{t}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 0 & 1 \\ \frac{2\cos t - \cos 4t}{3-2\cos 3t} & \frac{2\sin t - \sin 4t}{3-2\cos 3t} & \frac{2-2\cos 3t}{3-2\cos 3t} \end{vmatrix}$$

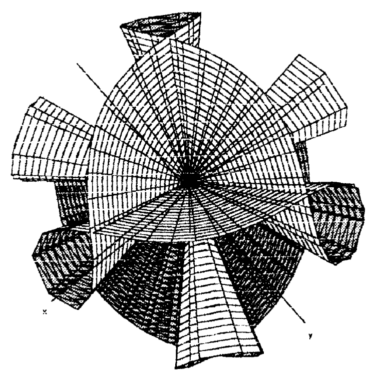
$$\beta^*(\mathrm{t}) = \left(-\frac{2\sin t - \sin 4t}{3-2\cos 3t}, \frac{2\cos t - \cos 4t}{3-2\cos 3t}, 0\right)$$

where $\wedge$ denotes the vectorical product. Then, the dual unit curve is written by the equation.

$$\beta(\mathrm{t},\varepsilon) = \left(\frac{2\cos t - \cos 4t}{3-2\cos 3t}, \frac{2\sin t - \sin 4t}{3-2\cos 3t}, \frac{2-2\cos 3t}{3-2\cos 3t}\right)$$

$$+\varepsilon\left(-\frac{2\sin t - \sin 4t}{3-2\cos 3t}, \frac{2\cos t - \cos 4t}{3-2\cos 4t}, 0\right)$$

where $\varepsilon^2 = 0$, that is, ε is the dual unit. The figure of the dual developable ruled roselike surface $\beta(\mathrm{t},\varepsilon) = \beta(\mathrm{t}) \wedge \beta^*(\mathrm{t}) + \zeta\ \beta(\mathrm{t})$ is the following:



The dual developable ruled roselike surface related with ep (t; 3, 1, 2, 0),
$\beta(\mathrm{t},\zeta) = \beta(\mathrm{t}) \wedge \beta^*(\mathrm{t}) + \zeta\ \beta(\mathrm{t})$, $\zeta \in$IR.

4.6. Theorem. *Let $\lambda(t)$ be the arclength reparametrized spherical curve obtained from the curve hy (t;n,m,r,α). Then, we may also obtain the dual developable ruled roselike surface $\lambda(t,\zeta) = \lambda(t) \wedge \lambda^*(t) + \zeta\ \lambda(t)$, where $\wedge$ denotes the vectorical product.*

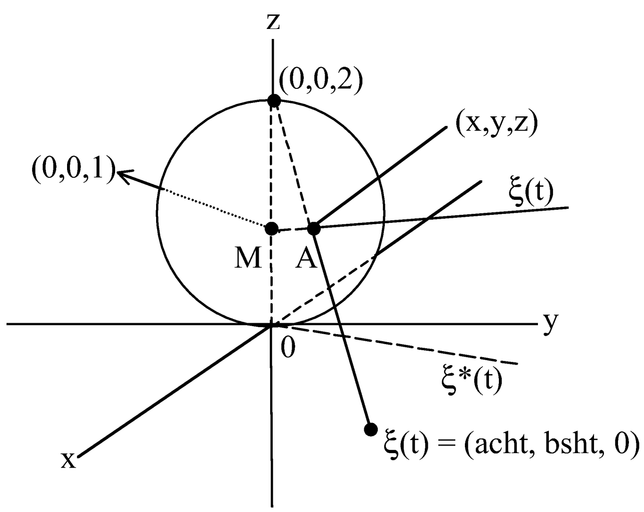
P r o o f. If we use the previous explanations, the proof is evidence.

4.7. Theorem. *Let $x(t)$ be the spherical curve derived from the hyperbolic curve (acht, bsht, 0), a,b$\in$IR. Then, we may write the equation $X(t,\varepsilon)$*

$= x(t) \wedge x^*(t) + \zeta x(t)$, $\zeta \in IR$ as the dual ruled surface, where $\wedge$ denotes the vectorical product.

P r o o f. The proof is given in the explanations of 4.8 and 4.9:

4.8. Spherical image of the hyperbolic curve (acht, bsht, 0)



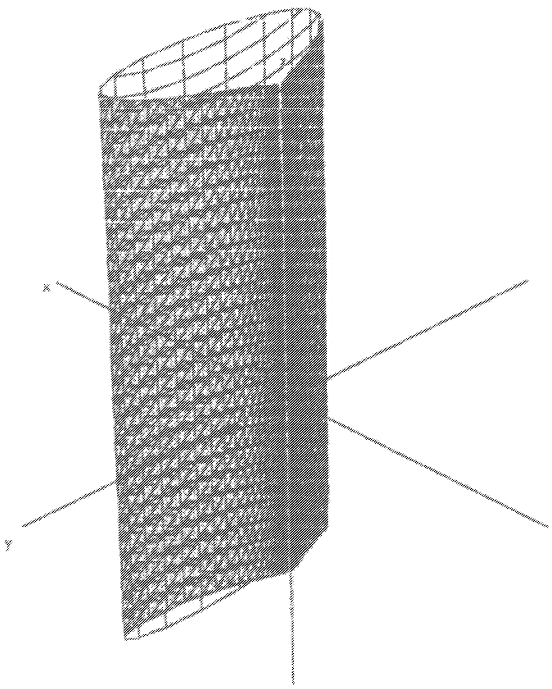
Consider the hyperbolic curve (asect, btgt, 0), or (acht, bsht, 0). Let S^2 be the unit sphere $x^2+y^2+(z-1)^2 = 1$, with center (0,0,1), radius = 1, and tangent to xy plane at the origin (0,0,0), without the pole point (0,0,2). Let (acht, bsht, 0) be any point of the hyperbolic curve. Then we may write the equation of the straight line passing through the points (0,0,2) and (acht, bsht, 0) by $\frac{x}{acht} = \frac{y}{bsht} = \frac{2-z}{2}$. From the last equation we may write $acht = \frac{2x}{2-z}, bsht = \frac{2y}{2-z}$, and $a^2ch^2t + b^2sh^2t = \frac{4(x^2+y^2)}{(2-z)^2}$. If we put the values x,y in the equation of S^2 we have $\frac{a^2ch^2t}{4}(2-z)^2 + \frac{b^2sh^2t}{4}(2-z)^2 = z(2-z)$, $x = \frac{4acht}{a^2ch^2t + b^2sh^2t+4}$, $y = \frac{4bsht}{a^2ch^2t + b^2sh^2t+4}$ and $z = \frac{2(a^2ch^2t+b^2sh^2t)}{a^2ch^2t+b^2sh^2t+4}$, that is, the spherical image of the hyperbole is

$$\xi = \left(\frac{4acht}{a^2ch^2t + b^2sh^2t + 4}, \frac{4bsht}{a^2ch^2t + b^2sh^2t + 4}, \frac{2(a^2ch^2t + b^2sh^2t)}{a^2ch^2t + b^2sh^2t + 4}\right).$$

If we write a = 1 and b = 2, we have

$$\xi(t) = \left(\frac{4cht}{5+5sh^2t}, \frac{8\ sht}{5+5sh^2t}, \frac{2(1+5sh^2t)}{5+5sh^2t}\right).$$

We may write the curve ξ(t) as an arclength reparametrized curve. The line moving parallel to Oz axis and support to the arclength reparametrized spherical ξ(t) curve derives the developable ruled surface which appears in the following figure. In reality, the developable surface is equivalent to the dual developable ruled surface X(t,ζ) = ξ(t) $\wedge$ ξ*(t) + ζ(0,0,1) where ξ*(t) is the dual curve of the spherical representation ξ(t), ζ $\in$IR , and $\wedge$ denotes the vectorical product.



The dual developable ruled surface X(t,ζ) = ξ(t) $\wedge$ ξ*(t) + ζ(0,0,1), ζ $\in$IR.

4.9. The dual curve ξ*(t) of the curve ξ(t).

To find the dual curve ξ*(t) used in the above expressions we must compute the vectorial product $OM \wedge \xi$, such that

$$\xi^*(t) = \left(\frac{4cht}{5+5sh^2t}, \frac{8sht}{5+5sh^2t}, \frac{-3+5sh^2t}{5+5sh^2t}\right) \wedge (0,0,1)$$

$$\xi^*(t) = \begin{vmatrix} e_1 & e_2 & e_3 \\ 0 & 0 & 1 \\ \frac{4cht}{5+5sh^2t} & \frac{8sht}{5+5sh^2t} & \frac{-3+5sh^2t}{5+5sh^2t} \end{vmatrix}$$

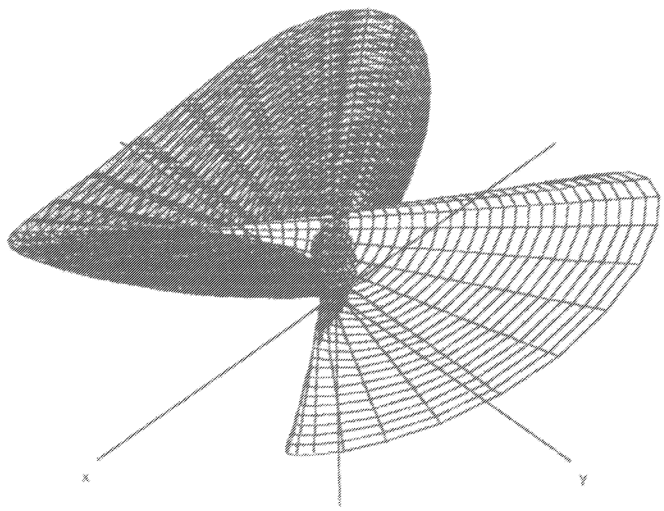
$$\xi^*(t) = \left(-\frac{8sht}{5+5sh^2t}, \frac{4cht}{5+5sh^2t}, 0\right).$$

Hence, the equation of the dual spherical unit curve is

$$X(t,\varepsilon) = \left(\frac{4cht}{5+5sh^2t}, \frac{8sht}{5+5sh^2t}, \frac{-3+5sh^2t}{5+5sh^2t}\right)$$

$$+\,\varepsilon\left(-\frac{8sht}{5+5sh^2t}, \frac{4cht}{5+5sh^2t}, 0\right), \ \varepsilon^2 = 0$$

where ε is the dual unit. The figure of the dual spherical ruled surface is the following.



The dual ruled hyperbolic surface $X(t,\zeta) = \xi(\mathrm{t}) \wedge \xi^*(\mathrm{t}) + \zeta \quad \xi(\mathrm{t}), \zeta \in$IR.

The dual unit spherical ruled surfaces of the pedal curves may be written as easyly as in the previous methodes and ways. For example, the real unit spherical representation of the pedal curve (x(t),y(t),0) in the definition 2.4 is $\left(\frac{4x^2(t)}{x^2(t)+y^2(t)+4}, \frac{4y^2(t)}{x^2(t)+y^2(t)+4}, \frac{2(x^2(t)+y^2(t))}{x^2(t)+y^2(t)+4}\right)$.

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