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Averaged Moduli of Smoothness and Runge-Kutta Methods

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We estimate the error in an often used in practice Runge-Kutta method of 3rd order of convergence

$$\begin{aligned}&\tilde{y}_{i+1} = \tilde{y}_i + \tfrac{1}{4}k_1 + \tfrac{3}{4}k_3,\\ &k_1 = hf(x_i, \tilde{y}_i),\ k_2 = hf(x_{i+\frac{1}{3}}, \tilde{y}_i + \tfrac{1}{3}k_1),\ k_3 = hf(x_{i+\frac{2}{3}}, \tilde{y}_i + \tfrac{2}{3}k_2),\\ &\tilde{y}_0 = y_0,\ i = 0, 1, ..., n-1,\end{aligned}$$

for solving the Cauchy problem

$$\begin{cases} y' = f(x,y)\ ,\ 0 \le x \le A\ ,\\ y(0) = y_0\ ,\end{cases}$$

when the function f satisfies only Lipschitz condition

$$|f(x,v) - f(x,z)| \le K|v - z|$$

The so-called averaged moduli of smoothness are used. By the properties of these moduli all rates of convergence follow under the additional assumptions for smoothness (as a rule weaker than well known ones) of the function f, respectively y.

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1. Averaged moduli of smoothness

Let $M_{[a,b]}$ be the set of all bounded by the constant $M > 0$ measurable functions on $[a,b]$. Then, for every function $f \in M_{[a,b]}$ and a real number $p \ge 1$ the averaged modulus of smoothness is defined by [8]

$$\tau_k(f;\delta)_{L_p} = \|\omega_k(f,\cdot;\delta)\|_{L_p} = \left(\frac{1}{b-a}\int_a^b (\omega_k(f,x;\delta))^p\,dx\right)^{\frac{1}{p}},$$

where $k \geq 1$ is an integer number,

$$\omega_k(f,x;\delta) = \sup\{|\Delta_h^k f(t)| : t, t+kh \in [x-\frac{k\delta}{2}, x+\frac{k\delta}{2}] \cap [a,b]\},$$

and

$$\Delta_h^k f(t) = \sum_{m=0}^{k} (-1)^{k+m}\binom{k}{m} f(t+mh).$$

The $\tau_k(f;\delta)_{L_p}$-modulus has the following properties [3],[4],[8]:

1. $\tau_k(f;\delta')_{L_p} \leq \tau_k(f;\delta'')_{L_p}, 0 \leq \delta' \leq \delta''$;
2. $\tau_k(f+g;\delta)_{L_p} \leq \tau_k(f;\delta)_{L_p} + \tau_k(g;\delta)_{L_p}$;
3. $\tau_{k+1}(f;\delta)_{L_p} \leq 2\tau_k(f;\frac{k+1}{k}\delta)_{L_p}$;
4. $\tau_{k+1}(f;\delta)_{L_p} \leq C_k\delta\omega_k(f';\frac{k+1}{k}\delta)_{L_p}$;
5. $\tau_k(f;n\delta)_{L_p} \leq C_k n^{k+1/p}\tau_k(f;\delta)_{L_p}; \tau_1(f;n\delta)_{L_p} \leq n\tau_1(f;\delta)_{L_p}$;
6. $\tau_k(f;\delta)_{L_p} \leq k\delta^k\|f^{(k)}\|_{L_p}$;
7. $\tau(f;\delta)_L \leq \frac{\delta}{b-a}\bigvee_a^b f$, where $\bigvee_a^b f$ is the variation of f on $[a,b]$;
8. $\tau_k(f;\delta)_{L_p} \leq C_{k,m}(\tau_m(f;\delta)_{L_p} + \omega_k(f;\delta)_{L_p}), m > k$;
9. $\omega_k(f;\delta)_{L_p} \leq \tau_k(f;\delta)_{L_p} \leq \omega_k(f;\delta)$.

Here
$$\omega_k(f;\delta) = \sup\{|\Delta_h^k f(x)| : |h| \leq \delta, x, x+kh \in [a,b]\}$$

is the k-th modulus of smoothness and

$$\omega_k(f;\delta)_{L_p} = \sup\left\{\left(\frac{1}{b-a}\int_a^{b-kh} |\Delta_h^k f(x)|^p dx\right)^{\frac{1}{p}} : 0 \leq h \leq \delta\right\}$$

is the integral k-modulus of smoothness in L_p.
We use the following (simple consequence of Whitney's theorem [10]):

Lemma 1. [8] *Let F be a bounded linear functional, defined on $M_{[a,b]}$, such that $F(p) = 0$ for every algebraic polynomial p of degree $\leq k-1$. Then, for every $f \in M_{[a,b]}$,*

$$|F(f)| \leq c(k)\|F\|_{M_{[a,b]}}\omega_k\left(f, \frac{a+b}{2}; \frac{b-a}{k}\right)$$

It is proved in [7], (see also [8]) that $c(k) \leq 6$.

2. Runge-Kutta methods

Our goal is to estimate the error in Runge-Kutta methods for solving the Cauchy problem

$$\left\{\begin{array}{lcl} y' & = & f(x,y),\ 0 \leq x \leq A, \\ y(0) & = & y_0, \end{array}\right. \tag{1}$$

when the function f satisfies only a Lipschitz condition

$$|f(x,v) - f(x,z)| \leq K|v-z| \tag{2}$$

for all v and z, without any additional restrictions on y and f. The idea of Runge-Kutta methods [6] is to construct formulas of the type

$$\begin{array}{l} \tilde{y}_{i+1} = \tilde{y}_i + \sum_{j=1}^{r} p_j k_j(h), \\ \tilde{y}_0 = y_0,\ i = 0,1,2,...,n-1, \end{array} \tag{3}$$

where

$$\begin{array}{l} h = \frac{A}{n},\ x_i = ih,\ x_{i+\alpha} = x_i + \alpha h,\ y_{i+\alpha} = y(x_{i+\alpha}),\ \tilde{y}'_i = f(x_i, \tilde{y}_i), \\ k_j = k_j(h) = hf(x_i + \alpha_j h,\ \tilde{y}_i + \sum_{s=1}^{j-1} \beta_{j,s} k_s),\ j = 1,2,...,r, \end{array}$$

and the coefficients $\alpha_j, \beta_{j,s}, p_j$ are chosen to minimize the error $|y_i - \tilde{y}_i|$.

Let us denote

$$e_i = y_i - \tilde{y}_i, \quad e = \{\max |e_i| : 0 \leq i \leq n\}.$$

The simplest method in (3), $r = 1$, is Euler's method:

$$\left\{\begin{array}{l} \tilde{y}_{i+1} = \tilde{y}_i + hf(x_i, \tilde{y}_i),\ i = 0,1,...,n-1, \\ \tilde{y}_0 = y_0. \end{array}\right. \tag{4}$$

Theorem 1. ([1],[8],p.179) *The numerical solution of* (1),(2) *with method* (4) *satisfies the estimation*

$$e \leq 2Ae^{KA}\tau(y'; h)_L.$$

For $r = 2$ the general Runge-Kutta method is

$$(5) \qquad \begin{cases} \tilde{y}_{i+1} = \tilde{y}_i + phf(x_i, \tilde{y}_i) + qhf(x_{i+\alpha}, \tilde{y}_i + \beta hf(x_i, \tilde{y}_i)), \\ \tilde{y}_0 = y_0, \;\; i = 0, 1, ..., n-1. \end{cases}$$

where the constants p, q, α and β satisfy the system

$$\begin{cases} p + q = 1, \\ \alpha q = \beta q = \frac{1}{2}. \end{cases}$$

Theorem 2. ([1],[8],p.185) *The numerical solution of* (1), (2) *with method* (5) *satisfies the estimation*

$$e \leq c(A, K)\{\tau_2(y'; h)_L + h\tau(y'; h)_L\}.$$

Combining theorems 2 and 3 with the properties 1-9 of $\tau_k(f; \delta)_{L_p}$ a series of estimations can be obtained under the additional assumption for smoothness of the solution of (1), (2) .

In the case $r = 3$ two methods are widely used in practice. Until now we manage to handle just one of them:

$$(6) \qquad \begin{array}{l} \tilde{y}_{i+1} = \tilde{y}_i + \frac{1}{4}k_1 + \frac{3}{4}k_3, \\ k_1 = hf(x_i, \tilde{y}_i), \;\; k_2 = hf(x_{i+\frac{1}{3}}, \tilde{y}_i + \frac{1}{3}k_1), \;\; k_3 = hf(x_{i+\frac{2}{3}}, \tilde{y}_i + \frac{2}{3}k_2), \\ \tilde{y}_0 = y_0, \;\; i = 0, 1, ..., n-1; \end{array}$$

Another one

$$(7) \qquad \begin{array}{l} \tilde{y}_{i+1} = \tilde{y}_i + \frac{1}{6}(k_1 + 4k_2 + k_3), \\ k_1 = hf(x_i, \tilde{y}_i), k_2 = hf(x_{i+\frac{1}{2}}, \tilde{y}_i + \frac{1}{2}k_1), k_3 = hf(x_{i+1}, \tilde{y}_i - k_1 + 2k_2), \\ \tilde{y}_0 = y_0, \;\; i = 0, 1, ..., n-1. \end{array}$$

causes some difficulties and it is unclear for us whether the error for method (7) can be estimated by averaged modulus.

Theorem 3. *The numerical solution of (1),(2) with method (6) satisfies the estimation*

$$e \leq c(A, K)\{\tau_3(y'; h)_L + h\tau_2(y'; h)_L + h^2\tau(y'; h)_L\}.$$

P r o o f. From (1) and (6) we get

$$\begin{aligned} e_{i+1} &= y_{i+1} - \tilde{y}_{i+1} = y_{i+1} - \tilde{y}_i - \frac{1}{4}k_1 - \frac{3}{4}k_3 \\ &= y_{i+1} - \tilde{y}_i \pm y_i \pm \frac{h}{4}f(x_i, y_i) - \frac{h}{4}f(x_i, \tilde{y}_i) \\ &- \frac{3h}{4}f(x_{i+\frac{2}{3}}, \tilde{y}_i + \frac{2h}{3}f(x_{i+\frac{1}{3}}, \tilde{y}_i + \frac{h}{3}f(x_i, \tilde{y}_i))) \\ &\pm \frac{3h}{4}f(x_{i+\frac{2}{3}}, y_i + \frac{2h}{3}f(x_{i+\frac{1}{3}}, y_i + \frac{h}{3}f(x_i, y_i))) \end{aligned}$$

and, since (2),

$$\begin{aligned} |e_{i+1}| &\le |y_i - \tilde{y}_i| + \tfrac{Kh}{4}|y_i - \tilde{y}_i| + \tfrac{3Kh}{4}|y_i + \tfrac{2h}{3}f(x_i + \tfrac{h}{3}, y_i \\ &+ \tfrac{h}{3}f(x_i, y_i)) - \tilde{y}_i - \tfrac{2h}{3}f(x_i + \tfrac{h}{3}, \tilde{y}_i + \tfrac{h}{3}f(x_i, \tilde{y}_i))| + |P| \\ &\le (1 + \tfrac{Kh}{4})|y_i - \tilde{y}_i| + \tfrac{3Kh}{4}(|y_i - \tilde{y}_i| \\ &+ \tfrac{2Kh}{3}(|y_i - \tilde{y}_i| + \tfrac{h}{3}|f(x_i, y_i) - f(x_i, \tilde{y}_i)|)) + |P| \\ &\le (1 + Kh + \tfrac{K^2h^2}{2} + \tfrac{K^3h^3}{6})|y_i - \tilde{y}_i| + |P|, \end{aligned} \tag{8}$$

where

$$P = y_{i+1} - y_i - \frac{h}{4}f(x_i, y_i) - \frac{3h}{4}f(x_{i+\frac{2}{3}}, y_i + \frac{2h}{3}f(x_{i+\frac{1}{3}}, y_i + \frac{h}{3}f(x_i, y_i))).$$

Let us estimate P:

$$\begin{aligned} |P| &= |P \pm \frac{3h}{4}f(x_{i+\frac{2}{3}}, y_i + \frac{2h}{3}f(x_{i+\frac{1}{3}}, y_i + \frac{h}{3}))| \\ &\le |y_{i+1} - y_i - \frac{h}{4}f(x_i, y_i) - \frac{3h}{4}f(x_{i+\frac{2}{3}}, y_i + \frac{2h}{3}f(x_{i+\frac{1}{3}}, y_{i+\frac{1}{3}}))| \\ &+ \frac{3h}{4} \cdot \frac{2h}{3}K|f(x_{i+\frac{1}{3}}, y_i + \frac{h}{3}f(x_i, y_i)) - f(x_{i+\frac{1}{3}}, y_{i+\frac{1}{3}})| \\ &\le |Q| + \frac{K^2h^2}{2}|y_i + \frac{h}{3}y'_i - y_{i+\frac{1}{3}}|, \end{aligned}$$

where

$$Q = y_{i+1} - y_i - \frac{h}{4}y'_i - \frac{3h}{4}f(x_{i+\frac{2}{3}}, y_i + \frac{2h}{3}y'_{i+\frac{1}{3}}).$$

From

$$|y_i + \frac{h}{3}y'_i - y_{i+\frac{1}{3}}| = \frac{h}{3}\left|\frac{3}{h}\int_{x_i}^{x_{i+\frac{1}{3}}}(y'(t) - y'_i)dt\right| \le \frac{h}{3}\omega(y', x_{i+\frac{1}{2}}; h),$$

it follows

$$|P| \le |Q| + \frac{K^2h^3}{6}\omega(y', x_{i+\frac{1}{2}}; h). \tag{9}$$

On the other hand

$$Q = y_{i+1} - y_i - \frac{h}{4}y'_i \pm \frac{3h}{4}f(x_{i+\frac{2}{3}}, y_{i+\frac{2}{3}}) - \frac{3h}{4}f(x_{i+\frac{2}{3}}, y_i + \frac{2h}{3}y'_{i+\frac{1}{3}})$$

and

$$\begin{aligned} |Q| &\le |y_{i+1} - y_i - \tfrac{h}{4}y'_i - \tfrac{3h}{4}y'_{i+\frac{2}{3}}| \\ &+ \tfrac{3Kh}{4}|y_{i+\frac{2}{3}} - y_i - \tfrac{2h}{3}y'_{i+\frac{1}{3}}| = |L(y')| + \tfrac{3Kh}{4}|N(y')|. \end{aligned} \tag{10}$$

The linear functional

$$L(y') = y_{i+1} - y_i - \frac{h}{4}y'_i - \frac{3h}{4}y'_{i+\frac{2}{3}} = \int_{x_i}^{x_{i+1}} (y'(t) - \frac{1}{4}y'_i - \frac{3}{4}y'_{i+\frac{2}{3}})dt$$

satisfies $L(q) = 0$, if q is a quadratic algebraic polynomial and

$$|L(y')| \le 2h \max\{|y'(x)| : x_i \le x \le x_{i+1}\}.$$

Therefore $\|L\|_{M[x_i, x_{i+1}]} \le 2h$. Similarly, the linear functional

$$N(y') = y_{i+\frac{2}{3}} - y_i - \frac{2h}{3}y'_{i+\frac{1}{3}} = \int_{x_i}^{x_{i+\frac{2}{3}}} (y'(t) - y'_{i+\frac{1}{3}})dt$$

vanishes for all linear functions and $\|N\|_{[x_i, x_{i+1}]} \le 2h$. By Lemma 1

$$|L(y')| \le 12h\omega_3(y', x_{i+\frac{1}{2}}; \frac{h}{3}), \;\; |N(y')| \le 12h\omega_2(y', x_{i+\frac{1}{2}}; \frac{h}{2}). \tag{11}$$

The inequalities (8), (9), (10), (11) give

$$\begin{aligned} |e_{i+1}| &= |y_{i+1} - \tilde{y}_{i+1}| \le e^{Kh}|y_i - \tilde{y}_i| + \frac{K^2h^3}{6}\omega(y', x_{i+\frac{1}{2}}; h) \\ &+ 12h\omega_3(y', x_{i+\frac{1}{2}}; \frac{h}{3}) + \frac{3Kh}{4}12h\omega_2(y', x_{i+\frac{1}{2}}; \frac{h}{2}) \\ \le e^{Kh}|e_i| &+ c(K)(h\omega_3(y', x_{i+\frac{1}{2}}; \frac{h}{3}) + h^2\omega_2(y', x_{i+\frac{1}{2}}; \frac{h}{2}) + h^3\omega(y', x_{i+\frac{1}{2}}; h)), \end{aligned}$$

where $c(K) = \max(12, \frac{K^2}{6}, 9K)$. Using recursively this inequality we find

$$\begin{aligned} |e_{i+1}| &\le c(K)\sum_{k=0}^{i} e^{Kh(i-k)}(h\omega_3(y', x_{k+\frac{1}{2}}; \frac{h}{3}) \\ &+ h^2\omega_2(y', x_{k+\frac{1}{2}}; \frac{h}{2}) + h^3\omega(y', x_{k+\frac{1}{2}}; h)). \end{aligned}$$

Since $e^{Kh(i-k)} \leq e^{Khn} = e^{KA}$ for $0 \leq k \leq i \leq n$, $nh = A$, setting $c = c(K)e^{KA}$, we get finally

$$\begin{aligned}
e &= \max\{|e_i| : 0 \leq i \leq n\} \\
&\leq c\sum_{k=0}^{n-1}(h\omega_3(y', x_{k+\frac{1}{2}}; \frac{h}{3}) + h^2\omega_2(y', x_{k+\frac{1}{2}}; \frac{h}{2}) + h^3\omega_(y', x_{k+\frac{1}{2}}; h)) \\
&\leq c\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}(\omega_3(y', x_{k+\frac{1}{2}}; \frac{h}{3}) + h\omega_2(y', x_{k+\frac{1}{2}}; \frac{h}{2}) + h^2\omega(y', x_{k+\frac{1}{2}}; h))dx \\
&\leq c\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}}(\omega_3(y', x; \frac{2}{3}h) + h\omega_2(y', x; h) + h^2\omega(y', x; 2h))dx \\
&= c\int_0^A(\omega_3(y', x; \frac{2}{3}h) + h\omega_2(y', x; h) + h^2\omega(y', x; 2h))dx \\
&= Ac(\tau_3(y'; \frac{2}{3}h)_L + h\tau_2(y'; h)_L + h^2\tau(y'; 2h)_L) \\
&\leq c_1((\tau_3(y'; h)_L + h\tau_2(y'; h)_L + h^2\tau(y'; h)_L), \ c_1 = 2Ac.
\end{aligned}$$

Remark . Since

$$|y_i' - \tilde{y}_i'| \leq K|y_i - \tilde{y}_i| \text{ for } 0 \leq i \leq n,$$

it follows that Theorem 3 provides also error estimation for the derivative of the solution.

3. Numerical experiments and conclusions

Let us consider three Cauchy problems:

Problem 1. Let

$$y' = |(x - \frac{1}{2})y|, \qquad y(0) = A, \qquad 0 \leq x \leq 3. \tag{12}$$

For the function in the right side of the above equation Lipschitz condition is fulfilled, e.g. using the inequality

$$||a| - |b|| \leq |a - b|$$

we get

$$||(x - \frac{1}{2})y| - |(x - \frac{1}{2})z|| \leq |(x - \frac{1}{2})(y - z)| \leq \frac{5}{2}|y - z|.$$

Simple calculations show that for $A > 0$ the solution y of (12) satisfies:

	$0 \le x < \frac{1}{2}$	$\frac{1}{2} \le x \le 3$
$y(x)$	$Ae^{\frac{1}{8}-\frac{(x-\frac{1}{2})^2}{2}}$	$Ae^{\frac{1}{8}+\frac{(x-\frac{1}{2})^2}{2}}$
$y'(x)$	$A(\frac{1}{2}-x)e^{\frac{1}{8}-\frac{(x-\frac{1}{2})^2}{2}}$	$A(x-\frac{1}{2})e^{\frac{1}{8}+\frac{(x-\frac{1}{2})^2}{2}}$
$y''(x)$	$Ae^{\frac{1}{8}-\frac{(x-\frac{1}{2})^2}{2}}((x-\frac{1}{2})^2-1)$	$Ae^{\frac{1}{8}+\frac{(x-\frac{1}{2})^2}{2}}((x-\frac{1}{2})^2+1)$

From

$$y''(\frac{1}{2}-) = -Ae^{\frac{1}{8}}, \qquad y''(\frac{1}{2}+) = Ae^{\frac{1}{8}}.$$

it follows that y'' is a discontinues function with bounded variation. Theorem 3 and the properties of the moduli show that method (6) provides second order of convergence (not third order) for the problem (12). At the same time using the convergence theorems like **Theorem 7.2.2.3** in [9],page 416, we can prove only linear rate of convergence.

In the next we compare numerical results for the problem (12) with initial condition $A = 500$ and step $h = 0.1$ using the exact solution y, Euler's method (4)-RK1, method (5)-RK2 (with $p = q = \frac{1}{2}$, $\alpha = \beta = 1$) and method (6)-RK3. Theorems 2 and 3 state that both methods RK2 and RK3 are comparable (they have second order of convergence) and must provide better results than RK1.

x	**y**	**RK1**	**RK2**	**RK3**
0.00	500.00000	500.00000	500.00000	500.00000
0.10	523.01393	525.00000	523.00000	523.01398
0.20	541.64353	546.00000	541.61880	541.64335
0.30	555.35531	562.38000	555.32176	555.35463
0.40	563.74843	573.62760	563.70711	563.74707
0.50	566.57423	579.36388	566.52565	566.57209
0.60	569.41419	579.36388	569.35828	569.41124
0.70	578.01979	585.15751	577.95559	578.01589
0.80	592.65243	596.86067	592.57786	592.64732
0.90	613.76253	614.76649	613.67364	613.75577
1.00	642.01271	639.35714	641.90262	642.00363
1.50	934.12298	896.33988	933.56705	934.07568
2.00	1745.17148	1579.02752	1741.36277	1744.88466
2.50	4186.44874	3460.67136	4160.46875	4184.54805
3.00	12895.16996	9350.04289	12710.96899	12881.19189

	Problem 2 :	Problem 3 :
(13)	$y' = (x+\frac{1}{2})\sin\lvert(x-\frac{1}{2})y+1\rvert$	$y' = (x+\frac{1}{2})\sin\lvert(x-\frac{1}{2})y+1\rvert$
	$y(0) = 1, \qquad 0 \le x \le 1$	$y(0) = 5, \qquad 0 \le x \le 1$

For the equations (13) $f(x,y) = (x+\frac{1}{2})\sin|(x-\frac{1}{2})y+1|$ and by simple calculations it is clear that this function satisfies Lipschitz condition, e.g.

$$|f(x,y)-f(x,z)| \;=\; |x+\tfrac{1}{2}|.|\sin|(x-\tfrac{1}{2})y+1| - \sin|(x-\tfrac{1}{2})z+1||$$

$$\le\; \tfrac{3}{2}.2\left|\cos\tfrac{|(x-\frac{1}{2})y+1|+|(x-\frac{1}{2})z+1|}{2}\right|\left|\sin\tfrac{|(x-\frac{1}{2})y+1|-|(x-\frac{1}{2})z+1|}{2}\right|$$

$$\le\; 3\left|\tfrac{|(x-\frac{1}{2})y+1|-|(x-\frac{1}{2})z+1|}{2}\right| \le \tfrac{3}{4}|y-z|$$

At the same time the partial derivative $\frac{\partial f(x,y)}{\partial y}$ does not exist. Therefore, it is not sure that the well-known error estimates for Runge-Kutta methods work properly for the equations (13). Theorem 3 indicates that method (6) is applicable regardless of the fact that we lose the advantage of the higher order

of convergence. Numerical experiments for the above two problems, carried out with step $h = 0.1$, show that we have respectively:
Problem 2:

x_i	y_i	$y1_i$	$(x_i - \frac{1}{2})y_i + 1$	$f(x_i, y_i)$	$f(x_i, y1_i)$	y'_i	$y1'_i$
0.0	1.00000	1.00000	0.50000	0.23971	0.23971	0.23518	0.19256
0.1	1.02853	1.02397	0.58859	0.33311	0.33402	0.33538	0.28687
0.2	1.06708	1.05737	0.67988	0.44009	0.44167	0.44236	0.38785
0.3	1.11700	1.10154	0.77660	0.56069	0.56245	0.56295	0.50206
0.4	1.17967	1.15779	0.88203	0.69483	0.69608	0.69694	0.62926
0.5	1.25639	1.22739	1.00000	0.84147	0.84147	0.84302	0.76878
0.6	1.34827	1.31154	1.13483	0.99711	0.99539	0.99731	0.91843
0.7	1.45585	1.41108	1.29117	1.15339	1.15038	1.15092	1.07289
0.8	1.57845	1.52612	1.47354	1.29386	1.29171	1.28683	1.22105
0.9	1.71322	1.65529	1.68529	1.39083	1.39417	1.37737	1.34294
1.0	1.85393	1.79471	1.92696	1.40586	1.42073	1.43688	1.44539

$s = 0.03524, s1 = 0.20222$

Problem 3:

x_i	y_i	y_i1	$(x_i - \frac{1}{2})y_i + 1$	$f(x_i, y_i)$	$f(x_i, y1_i)$	y'_i	$y1'_i$
0.0	5.00000	5.00000	−1.50000	0.49875	0.49875	0.49321	0.49250
0.1	5.05179	5.04987	−1.02072	0.51149	0.51125	0.51277	0.50500
0.2	5.09658	5.10100	−0.52897	0.35325	0.35405	0.28040	0.43265
0.3	5.11646	5.13641	−0.02329	0.01863	0.02182	0.13234	0.18794
0.4	5.13717	5.13859	0.48628	0.42061	0.42050	0.43628	0.21116
0.5	5.20106	5.18064	1.00000	0.84147	0.84147	0.85491	0.63098
0.6	5.30017	5.26478	1.53002	1.09909	1.09892	1.10693	0.97020
0.7	5.41057	5.37468	2.08211	1.04652	1.05071	1.03957	1.07481
0.8	5.49661	5.47975	2.64898	0.61480	0.62059	0.59374	0.83565
0.9	5.52389	5.54181	3.20956	−0.09508	−0.10508	−0.07524	0.25775
1.0	5.47614	5.53130	3.73807	−0.84260	−0.87650	−0.90683	−0.46792

$s = 0.22041, s1 = 0.69273,$

where:

- y_i is the solution obtained by the method (6);
- $y1_i$ is the solution obtained by Euler's method (4);
- y'_i and $y1'_i$ are the derivatives obtained by making use of the values y_i and $y1_i$ and formulas for numerical differentiation of order $O(h^2)$;

- the quantities s and $s1$ defined by

$$s = \sqrt{\sum_{i=0}^{10}(f(x_i, y_i) - y_i')^2}, \quad s1 = \sqrt{\sum_{i=0}^{10}(f(x_i, y1_i) - y1_i')^2}$$

  can be considered as a measure of accuracy.

The practical conclusions from these numerical results are the following:

1. The Problem 2 is in fact

$$y' = (x + \frac{1}{2}) \sin((x - \frac{1}{2})y + 1), \quad y(0) = 1, \quad 0 \leq x \leq 1$$

   e.g. the partial derivatives of the function $f(x, y)$ do exist. This means that we are able to make use of the well known estimations that involved the derivatives of the function $f(x, y)$, see [5], p.98, [2], p. 367. In our case we have third order of convergence for the method (6).

2. The problem 3 is more complicated. The values of $(x_i - \frac{1}{2})y_i + 1$ are different by sign, and $\frac{\partial f(x,y)}{\partial y}$, $\frac{\partial f(x,y)}{\partial x}$ do not exist. Theorems 1 and 3 show that both methods (4) and (6) provide first order of convergence. In spite of these theoretical considerations numerical experiments give better results when the method (6) is used.

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