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The Inequality of Erdös-Turan-Koksma: Walsh and Haar Functions Over Finite Groups

Vassil S. Grozdanov[1], *Stanislava S. Stoilova*[2]

Presented by V. Kiryakova

Following the main concept of the definition of the Walsh functions over finite abelian groups, the authors introduce a new class of orthonormal functions, the so-called Haar functions over finite abelian groups. The concrete form of the well-known inequality of Erdös-Turan-Koksma for an arbitrary $s-$dimensional net is obtained in the terms of sums of the Walsh and Haar functions over finite abelian groups. The obtained estimations are analogous to the estimations, obtained by Niederreiter and generalize the ones, given by Hellekalek.

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1. Introduction

Let $s \geq 1$ be a fixed integer and $[0,1)^s$ be the $s-$dimensional unit cube. Let λ_s denote the Lebesgue measure on $[0,1)^s$. Let $\mathcal{J}$ and $\mathcal{J}^*$ denote the classes of all subintervals of $[0,1)^s$ of the form $J = \prod_{i=1}^{s}[u_i, v_i)$ and $J = \prod_{i=1}^{s}[0, v_i)$ respectively, where for $1 \leq i \leq s$ $0 \leq u_i < v_i \leq 1$.

For a gived net $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ of $N \geq 1$ points in $[0,1)^s$ and an arbitrary subinterval $J \in \mathcal{J}$ or $J \in \mathcal{J}^*$ we signify $A(\xi_N; J) = \{j : 0 \leq j \leq N-1, \mathbf{x}_j \in J\}$. The extreme discrepancy, or called only discrepancy, $D(\xi_N)$ and the star-discrepancy $D^*(\xi_N)$ of the net ξ_N are defined respectively as

$$D(\xi_N) = \sup_{J \in \mathcal{J}} \left| N^{-1} A(\xi_N; J) - \lambda_s(J) \right|$$

and

$$D^*(\xi_N) = \sup_{J \in \mathcal{J}^*} \left| N^{-1} A(\xi_N; J) - \lambda_s(J) \right|.$$

Let $\mathcal{T} = \{e_{\mathbf{k}}(\mathbf{x}) = \exp\left(2\pi\mathbf{i}\sum_{i=1}^{s} k_i x_i\right) : \mathbf{k} = (k_1, \ldots, k_s) \in \mathbf{Z}^s, \mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s\}$ denote the trigonometric functional system. The well-known inequality of Erdös-Turan-Koksma is the main result of the quantitative theory of uniformly distributed sequences. This inequality gives an estimation of discrepancy of an arbitrary $s-$dimensional net in the terms of the sum of this net with respect to the functions of the trigonometric system.

In Kuipers and Niederreiter [13, theorem 2.5] it is shown that for an arbitrary net $\xi_N = \{x_0, \ldots, x_{N-1}\}$ of $N \geq 1$ points in $[0,1)$ the inequality

$$(1) \qquad D(\xi_N) \leq \frac{6}{m+1} + \frac{4}{\pi}\sum_{k=1}^{m-1}\left(\frac{1}{k} - \frac{1}{m}\right)\left|\frac{1}{N}\sum_{j=0}^{N-1}\exp(2\pi\mathbf{i}kx_j)\right|$$

holds for each integer $m \geq 1$.

Erdös and Turan [2] proved the inequality (1) without concrete values of the constants. Judin [11] gives another proof of the inequality (1).

Multidimensional version of the inequality (1) was given by Koksma [12] and Szüsz [25]: For an arbitrary net $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ of $N \geq 1$ points in $[0,1)^s$ and each positive integer m the inequality

$$(2) \qquad D(\xi_N) \leq C_s\left(\frac{1}{m} + \sum_{0<||\mathbf{k}||\leq m}\frac{1}{r(\mathbf{k})}\left|\frac{1}{N}\sum_{j=0}^{N-1}\exp(2\pi\mathbf{i} < \mathbf{k}, \mathbf{x}_j >)\right|\right)$$

holds, with a constant C_s, depending on the dimension s, for a vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbf{Z}^s$, $||\mathbf{k}|| = \max\limits_{1\leq i\leq s}|k_i|$, $r(\mathbf{k}) = \prod_{i=1}^{s}\max(|k_i|, 1)$ and $< \mathbf{k}, \mathbf{x}_n >$ is the inner product of the vectors $\mathbf{k}$ and $\mathbf{x}_n$.

A generalization of the inequality (2) was given by Niederreiter and Philipp [20] and [21]. Nowadays the inequality (2) is called an inequality of Erdös-Turan-Koksma.

Niederreiter [19] considers a class of $s-$dimensional nets, which are defined as follows: For arbitrary integers $M \geq 2$ and $N \geq 1$ the set $\mathcal{P}_N$ is defined as

$$\mathcal{P}_N = \left\{\mathbf{x}_j = \left\{\frac{\mathbf{y}_j}{M}\right\}; \mathbf{y}_j \in \mathbf{Z}^s, 0 \leq j \leq N-1\right\},$$

where $\{x\}$ denotes the fractional part of the real x. A general upper bound of discrepancy $D(\mathcal{P}_N)$ of the concrete set $\mathcal{P}_N$ in terms of the sum of the functions of the trigonometric system is established.

Furthermore, Hellekalek [7] shows the form of the inequality of Erdös-Turan-Koksma for the discrepancy of the net $\mathcal{P}_N$, in the terms of the Walsh functions of base $b \geq 2$ (for the definition of the Walsh functions see **Remark 1**), which is an analogue of the one, given by Niederreiter [19]. Hellekalek [8] shows the form of the inequality of Erdös-Turan-Koksma for the discrepancy of the net $\mathcal{P}_N$ in the terms of the classical Haar functions of base $b = 2$. The form of the inequality of Erdös-Turan-Koksma for an arbitrary $s-$dimensional net in the terms of the classical Haar functions of base $b \geq 2$ (for the definition of the Haar functions see **Remark 2**) was shown by Hellekalek [9].

Let $f = \{\psi_k(x) : k \in \mathbf{Z}, k \geq 0, x \in [0,1)\}$ be an arbitrary complete orthonormal functional system on $[0,1)$ and let us define the system

$$\mathcal{F} = \Big\{ \psi_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} \psi_{k_i}(x_i); \mathbf{k} = (k_1, \ldots, k_s), 1 \leq i \leq s, k_i \in \mathbf{Z}, k_i \geq 0,$$

$$\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s \},$$

which is the set of the multidimensional functions, generated by the functions of the system f.

The authors consider that in general the inequality of Erdös-Turan-Koksma, which corresponds to the system $\mathcal{F}$, has the next form: For an arbitrary net $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ of $N \geq 1$ points in $[0,1)^s$ the discrepancy $D(\xi_N)$ satisfies the inequality

$$D(\mathcal{S}_N) \leq C'_s \left(\delta_{\mathcal{F}}(M) + \sum_{\mathbf{k} \in \Delta_s^*(M)} \rho_{\mathcal{F}}(\mathbf{k}) \left| S(\xi_N; \psi_{\mathbf{k}}) \right| \right), \tag{3}$$

where the parameter $M > 0$ is an arbitrary integer that defines the finite domain $\Delta_s^*(M)$ for the vectors $\mathbf{k}$ with non-negative integer coordinates and $\mathbf{0} \notin \Delta_s^*(M)$, the constant C'_s depends on the dimension s, the function $\delta_{\mathcal{F}}(M)$ depends on the system $\mathcal{F}$ and M, the coefficient $\rho_{\mathcal{F}}(\mathbf{k})$ depends on the system $\mathcal{F}$ and the vector $\mathbf{k} \in \Delta_s^*(M)$ and $S(\xi_N; \psi_{\mathbf{k}}) = \frac{1}{N} \sum_{j=0}^{N-1} \psi_{\mathbf{k}}(\mathbf{x}_j)$ is the sum of the net ξ_N with respect to the function $\psi_{\mathbf{k}} \in \mathcal{F}$.

Following [14] and [15] we remind the definition for the concept of Walsh functions over finite abelian groups. For an arbitrary fixed integer $m \geq 1$ let $\{b_1, \ldots, b_m : 1 \leq l \leq m, b_l \geq 2\}$ be a set of given integers. Let for $1 \leq l \leq m$ $(\mathbf{Z}_{b_l}, \oplus_{b_l})$, where $\mathbf{Z}_{b_l} = \{0, 1, \ldots, b_l - 1\}$ and $\oplus_{b_l}$ be the operation an addition mod b_l, be the discrete cyclic groups. Let $G = \mathbf{Z}_{b_1} \times \ldots \times \mathbf{Z}_{b_m}$, $\oplus = (\oplus_{b_1}, \ldots, \oplus_{b_m})$ and $(G; \oplus)$ be the finite abelian group of order $b = b_1 \ldots b_m$ and $\varphi : \{0, 1, \ldots, b-1\} \to G$ be a bijection with $\varphi(0) = \mathbf{0}$. For $\mathbf{h} = (h_1, \ldots, h_m) \in G$ and $\mathbf{y} = (y_1, \ldots, y_m) \in G$ let $\chi_{\mathbf{h}}(\mathbf{y})$ be defined as $\chi_{\mathbf{h}}(\mathbf{y}) = \prod_{l=1}^{m} \exp\left(2\pi \mathbf{i} \frac{h_l y_l}{b_l}\right)$.

We note the fact that in the paper for an arbitrary real $x \in [0,1)$ we will use the unique $b-$adic representation in the form $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$, where for $j \geq 0$ $x_j \in \{0, 1, \ldots, b-1\}$ and for infinitely many values of j $x_j \neq b-1$.

Definition 1. *For a non-negative integer k and a real $x \in [0,1)$ with the $b-$adic representations $k = \sum_{j=0}^{g} k_j b^j$ and $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$, where for*

$j \geq 0$ $k_j, x_j \in \{0, 1, \ldots, b-1\}$ and $k_g \neq 0$, the function ${}_{G,\varphi}wal_k : [0,1) \to \mathbf{C}$ is defined in the following way ${}_{G,\varphi}wal_k(x) = \prod_{j=0}^{g} \chi_{\varphi(k_j)}\left(\varphi(x_j)\right)$.

The set $\mathcal{W}_{G,\varphi} = \{{}_{G,\varphi}wal_k : k = 0, 1, \ldots\}$ is called the Walsh functional system over the finite group G with respect to the bijection φ, or in brief Walsh functional system over (G, φ). This set is a complete orthonormal functional system on $L_2([0,1); \lambda)$.

Remark 1. *In the case when $G = \mathbf{Z}_b$, $b \geq 2$ and $\varphi = id$ is the identity between $\{0, 1, \ldots, b-1\}$ and $\mathbf{Z}_b$, the obtained functional system $\mathcal{W}_{\mathbf{Z}_b,id}$ from Definition 1 is the functional system of Chrestenson [1] and the system $\mathcal{W}_{\mathbf{Z}_2,id}$ is the classical Walsh-Paley [26] functional system.*

For a vector $\mathbf{k} = (k_1, \ldots, k_s)$ with non-negative integer coordinates we define the $\mathbf{k}-$th Walsh function over (G, φ) as ${}_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} {}_{G,\varphi}wal_{k_i}(x_i)$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s$.

The *purposes* of the presented paper will be as follows:

1. To define the Haar functions over finite group G with respect to the bijection φ (see Definition 2).

2. To show the concrete form of the inequality of Erdös-Turan-Koksma, announced in (3) in the terms of Walsh and Haar functions over (G, φ) for an arbitrary $s-$dimensional net (see Theorems 1 and 2).

The solutions to these problems are given in the statement of the results.

2. Statements of the results

To give the main results of the paper we will need some preliminary notations: For an integer $k \geq 0$ and a real $x \in [0,1)$ with the $b-$adic representations $k = \sum_{j=0}^{\infty} k_j b^j$ and $x = \sum_{j=0}^{\infty} x_j b^{-j-1}$, where for $j \geq 0$ $k_j, x_j \in \{0, 1, \ldots, b-1\}$, let $k(0) = 0$ and $x(0) = 0$, and for an arbitrary integer $g \geq 1$ we put $k(g) = \sum_{j=0}^{g-1} k_j b^j$ and $x(g) = \sum_{j=0}^{g-1} x_j b^{-j-1}$.

We note that each integer k, $b^g \leq k \leq b^{g+1}$, with an integer $g \geq 0$ has the unique $b-$adic representation of the form

$$(4) \qquad k = k_g b^g + k(g), \quad \text{with} \quad k_g \in \{1, \ldots, b-1\} \quad \text{and} \quad 0 \leq k(g) < b^g.$$

Let the abelian group G and the bijection $\varphi : \{0, 1, \ldots, b-1\} \to G$ be as in the Introduction. In the next definition we will introduce the concept of the Haar functions over finite abelian groups:

Definition 2. *For an arbitrary integer $k \geq 0$ the function ${}_{G,\varphi}h_k$: $[0,1) \to \mathbf{C}$ is defined in the following way: If $k = 0$, then ${}_{G,\varphi}h_k(x) = 1$, $\forall x \in [0,1)$. For an arbitrary integer $k \geq 1$ with the $b-$adic representation (4), we define*

$$
{}_{G,\varphi}h_k(x) = \begin{cases} b^{\frac{g}{2}} \chi_{\varphi(k_g)}\left(\varphi(a)\right), & \text{for } \frac{k(g)}{b^g} + \frac{a}{b^{g+1}} \leq x < \frac{k(g)}{b^g} + \frac{a+1}{b^{g+1}}, \\ & a = 0, 1, \ldots, b-1 \\ 0, & \text{otherwise.} \end{cases}
$$

We will call the set $\mathcal{H}_{G,\varphi} = \{{}_{G,\varphi}h_k : k = 0, 1, \ldots\}$ the Haar functional system over finite group G with respect to the bijection φ, or in brief the Haar functional system over (G, φ).

Remark 2. *In the case when $G = \mathbf{Z}_b$, $b \geq 2$ and $\varphi = id$ is the identity between $\{0, 1, \ldots, b-1\}$ and $\mathbf{Z}_b$, the obtained functional system $\mathcal{H}_{\mathbf{Z}_b, id}$ from Definition 2 is the Haar system of order b, see Schipp, Wade, Simon, Pal [22] and the system $\mathcal{H}_{\mathbf{Z}_2, id}$ is the classical Haar [6] functional system.*

We define the fundamental domain D_k of the $k-$th Haar function ${}_{G,\varphi}h_k$ over (G, φ) as follows: If $k = 0$, then $D_0 = [0, 1)$. If $k \geq 1$, $b^g \leq k < b^{g+1}$ with $g \geq 0$, then $D_k = \left[\frac{k(g)}{b^g}, \frac{k(g)+1}{b^g}\right)$.

For a vector $\mathbf{k} = (k_1, \ldots, k_s)$ with non-negative integer coordinates we define the $\mathbf{k}-$th Haar function over (G, φ) as ${}_{G,\varphi}h_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{s} {}_{G,\varphi}h_{k_i}(x_i)$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0, 1)^s$ and the fundamental domain $D_{\mathbf{k}}$ of this function as $D_{\mathbf{k}} = \prod_{i=1}^{s} D_{k_i}$.

We will note some *properties of the Haar functions over* (G, φ).

1. The Haar system $\mathcal{H}_{G,\varphi}$ is a complete orthonormal functional system on $L_2([0, 1); \lambda)$.

2. The functions ${}_{G,\varphi}h_k$ are concentrated on D_k. They vanish outside of D_k.

3. For each integer $g \geq 0$ there are exactly $b - 1$ Haar functions over (G, φ) ${}_{G,\varphi}h_k$ with $b^g \leq k < b^{g+1}$, that have the same fundamental domain.

4. If $0 \leq k < b$, then ${}_{G,\varphi}h_k(x) = {}_{G,\varphi}wal_k(x)$, $\forall x \in [0, 1)$.

For a vector $\mathbf{k} = (k_1, \ldots, k_s)$ with non-negative integer coordinates we define

$$
\omega_{G,\varphi walsh}(\mathbf{k}) = \prod_{i=1}^{s} \omega_{G,\varphi walsh}(k_i),
$$

with

$$
\omega_{G,\varphi walsh}(k) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{2(C+1)}{b^{g+1}} & \text{if } b^g \leq k < b^{g+1}, g \geq 0, g \in \mathbf{Z}; \end{cases}
$$

$$
\omega'_{G,\varphi walsh}(\mathbf{k}) = \prod_{i=1}^{s} \omega'_{G,\varphi walsh}(k_i),
$$

with

$$
\omega'_{G,\varphi walsh}(k) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{C+1}{b^{g+1}} & \text{if } b^g \leq k < b^{g+1}, g \geq 0, g \in \mathbf{Z}, \end{cases}
$$

where $C = \max_{1\le\mu\le b-1}\max_{1\le\nu\le b-1}\left|\sum_{a=0}^{\nu-1}\chi_{\varphi(\mu)}(\varphi(a))\right|$.

Let $n \geq 1$ be an arbitrary fixed integer. We introduce the sets

$$\Delta_s(n) = \{\mathbf{k} = (k_1,\ldots,k_s) \in \mathbf{Z}^s : 0 \leq k_i < b^n, 1 \leq i \leq s\}, \quad \Delta_s^*(n) = \Delta_s(n) \setminus \{\mathbf{0}\}.$$

Theorem 1. (The inequality of Erdös-Turan-Koksma in the terms of the Walsh functions over finite groups) *Let $n \geq 1$ be an arbitrary fixed integer. For an arbitrary net $\xi_N = \{\mathbf{x}_0,\ldots,\mathbf{x}_{N-1}\}$ of $N \geq 1$ points in $[0,1)^s$ we have the following:*

(i) The extreme discrepancy $D(\xi_N)$ of the net ξ_N satisfies the inequality

$$D(\xi_N) \leq 1 - \left(1 - \frac{2}{b^n}\right)^s + 2\sum_{\mathbf{k}\in\Delta_s^*(n)} \omega_{G,\varphi walsh}(\mathbf{k})|S(\xi_N; {}_{G,\varphi}wal_{\mathbf{k}})|;$$

(ii) The star-discrepancy $D^*(\xi_N)$ of the net ξ_N satisfies the inequality

$$D^*(\xi_N) \leq 1 - \left(1 - \frac{1}{b^n}\right)^s + 2\sum_{\mathbf{k}\in\Delta_s^*(n)} \omega'_{G,\varphi walsh}(\mathbf{k})|S(\xi_N; {}_{G,\varphi}wal_{\mathbf{k}})|,$$

where for each group G and a bijection φ, and for each vector $\mathbf{k}$ with non-negative integer coordinates $S(\xi_N; {}_{G,\varphi}wal_{\mathbf{k}}) = \frac{1}{N}\sum_{j=0}^{N-1} {}_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}_j)$ denotes the sum of the net ξ_N with respect to the function ${}_{G,\varphi}wal_{\mathbf{k}}$.

We note that the results, exposed in Theorem 1 were announced in [5].

Let for $1 \leq i \leq s$ $0 \leq \beta_i' \leq \beta_i'' \leq 1$ be rational numbers with denominators b^n and let us signify $J_i = [\beta_i', \beta_i'')$ and $J = \prod_{i=1}^s J_i$. For an arbitrary vector $\mathbf{k} = (k_1,\ldots,k_s)$ with non-negative integer coordinates we define

$$\omega^J_{G,\varphi Haar}(\mathbf{k}) = \prod_{i=1}^s \omega^{J_i}_{G,\varphi Haar}(k_i),$$

where for $1 \leq i \leq s$ the quantities $\omega^{J_i}_{G,\varphi Haar}(k_i)$ are defined as

$$\omega^{J_i}_{G,\varphi Haar}(k_i) = \begin{cases} 1, & \text{if } k_i = 0, \\ 0, & \text{if } k_i(g_i) \notin \{b^{g_i}\beta_i'(g_i), b^{g_i}\beta_i''(g_i)\}, \\ \frac{C+1}{b^{\frac{g}{2}+1}} & \text{if } k_i(g_i) \in \{b^{g_i}\beta_i'(g_i), b^{g_i}\beta_i''(g_i)\}, \\ & b^{g_i} \leq k_i < b^{g_i+1}, g_i \geq 0, g_i \in \mathbf{Z}. \end{cases}$$

We define the set $\Gamma = \frac{\mathbf{Z}^s}{b^n} \bmod 1$. For each vector $\mathbf{p} \in \Gamma$ of the form $\mathbf{p} = (p_1,\ldots,p_s)$ we define $I_{\mathbf{p}} = \prod_{i=1}^s \left[p_i, p_i + \frac{1}{b^n}\right)$. For an arbitrary subinterval J of $[0,1)^s$ we define the sets

$$\underline{J} = \bigcup_{I_{\mathbf{p}}\subseteq J} I_{\mathbf{p}} \quad \text{and} \quad \overline{J} = \bigcup_{I_{\mathbf{p}}\cap J\neq\emptyset} I_{\mathbf{p}} \tag{5}$$

and put $H(J) = \underline{J}$ or $H(J) = \overline{J}$.

Theorem 2. (The inequality of Erdös-Turan-Koksma in the terms of the Haar functions over finite groups) *Let $n \geq 1$ be an arbitrary fixed integer. For an arbitrary net $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ of $N \geq 1$ points in $[0,1)^s$ the inequality*

$$D(\xi_N) \leq 1 - \left(1 - \frac{2}{b^n}\right)^s + 2 \sup_{J \in \mathcal{J}} \sum_{\mathbf{k} \in \Delta_s^*(n)} \omega^{H(J)}_{{}_{G,\varphi}Haar}(\mathbf{k}) |S(\xi_N; {}_{G,\varphi}h_{\mathbf{k}})|,$$

holds, where for each group G and a bijection φ and for each vector $\mathbf{k}$ with non-negative integer coordinates $S(\xi_N; {}_{G,\varphi}h_{\mathbf{k}}) = \frac{1}{N} \sum_{j=0}^{N-1} {}_{G,\varphi}h_{\mathbf{k}}(\mathbf{x}_j)$ denotes the sum of the net ξ_N with respect to the function ${}_{G,\varphi}h_{\mathbf{k}}$.

3. The proof of Theorem 1

For arbitrary integers $g \geq 0$ and a, $0 \leq a < b^g$ an interval of the form $I(a,g) = \left[\frac{a}{b^g}, \frac{a+1}{b^g}\right)$ we will call an elementary interval with length b^{-g}.

We will note some properties of the Walsh functions over finite groups:

1. For an arbitrary integer $k \geq 1$ with the $b-$adic representation (4) the equality ${}_{G,\varphi}wal_k(x) = {}_{G,\varphi}wal_{k(g)}(x) {}_{G,\varphi}wal_{k_g b^g}(x)$ holds for $\forall x \in [0,1)$.

2. Let for integers $g \geq 0$ and $0 \leq a < b^g$ $I(a,g)$ be an arbitrary elementary interval with length b^{-g}. Then, for each integer k, $0 \leq k < b^g$ the function ${}_{G,\varphi}wal_k$ is a constant on the interval $I(a,g)$.

3. For each integers k and a such that $b^g \leq k < b^{g+1}$ and $0 \leq a < b^g$ with $g \geq 0$ the equality $\int_{I(a,g)} {}_{G,\varphi}wal_k(x)dx = 0$ holds.

For a function $f \in L_1([0,1)^s)$ and each vector $\mathbf{k}$ with non-negative integer coordinates the Fourier-Walsh coefficient over (G,φ) ${}_{\mathcal{W}_{G,\varphi}}\widehat{f}(\mathbf{k})$ is defined as ${}_{\mathcal{W}_{G,\varphi}}\widehat{f}(\mathbf{k}) = \int_{[0,1)^s} f(\mathbf{x}) {}_{G,\varphi}\overline{wal}_{\mathbf{k}}(\mathbf{x}) d\mathbf{x}$.

Lemma 1. *For an arbitrary β, $0 \leq \beta < 1$ with the $b-$adic representation $\beta = \sum_{j=0}^{\infty} \beta_j b^{-j-1}$, where for $j \geq 0$ $\beta_j \in \{0,1,\ldots,b-1\}$ we put $I = [0,\beta)$. We define the function $f_I(x) = 1_I(x) - \lambda(I)$, $x \in [0,1)$, where $1_I(x)$ denotes the characteristic function on I, and λ is the Lebesgue measure of I.*

Let $k \geq 1$ be an arbitrary integer with the $b-$adic representation (4). Then, the following hold:

(i) The $k-th$ coefficient of Walsh over (G,φ) of the function f satisfies the equalities

$$ {}_{\mathcal{W}_{G,\varphi}}\overline{\widehat{f}}_I(k) = \begin{cases} {}_{G,\varphi}wal_{k(g)}(\beta(g)) \left[\frac{1}{b^{g+1}} \sum_{a=0}^{\beta_g - 1} \chi_{\varphi(k_g)}(\varphi(a)) + \right. \\ \left. +(\beta - \beta(g+1)) \chi_{\varphi(k_g)}(\varphi(\beta_g))\right], & \text{if} \quad \beta_g \neq 0 \\ (\beta - \beta(g+1)) {}_{G,\varphi}wal_{k(g)}(\beta(g)), & \text{if} \quad \beta_g = 0. \end{cases} $$

(ii) The inequality $|\mathcal{W}_{G,\varphi}\widehat{f}_I(k)| \leq \dfrac{C+1}{b^{g+1}}$ *holds.*

P r o o f. For an arbitrary integer $k \geq 1$, $b^g \leq k < b^g$ with the integer $g \geq 0$ the following equalities

$$\mathcal{W}_{G,\varphi}\overline{\widehat{f}}_I(k) = \int_0^{\beta(g)} {}_{G,\varphi}wal_k(x)dx + \int_{\beta(g)}^{\beta} {}_{G,\varphi}wal_k(x)dx$$

hold. From property 3 of the functions ${}_{G,\varphi}wal_k$, we have that $\int_0^{\beta(g)} {}_{G,\varphi}wal_k(x)dx = 0$ and we obtain the equality

$$\mathcal{W}_{G,\varphi}\overline{\widehat{f}}_I(k) = \int_{\beta(g)}^{\beta} {}_{G,\varphi}wal_k(x)dx. \tag{6}$$

In the case, when $\beta_g \neq 0$ from properties 1 and 2 of the functions ${}_{G,\varphi}wal_k$ and (6), we obtain

(7)

$$\mathcal{W}_{G,\varphi}\overline{\widehat{f}}_I(k) = {}_{G,\varphi}wal_{k(g)}(\beta(g)) \left[\sum_{a=0}^{\beta_g-1} \int_{\beta(g)+\frac{a}{b^{g+1}}}^{\beta(g)+\frac{a+1}{b^{g+1}}} + \int_{\beta(g+1)}^{\beta}\right] {}_{G,\varphi}wal_{k_g b^g}(x)dx.$$

For each integer a, $0 \leq a \leq \beta_g - 1$ and for $\forall x \in \left[\beta(g) + \frac{a}{b^{g+1}}, \beta(g) + \frac{a+1}{b^{g+1}}\right)$, we have ${}_{G,\varphi}wal_{k_g b^g}(x) = \chi_{\varphi(k_g)}(\varphi(a))$, and for $\forall x \in [\beta(g+1), \beta)$, ${}_{G,\varphi}wal_{k_g b^g}(x) = \chi_{\varphi(k_g)}(\varphi(\beta_g))$. Then from (7) we obtain

$$\mathcal{W}_{G,\varphi}\overline{\widehat{f}}_I(k) =$$

$$= {}_{G,\varphi}wal_{k(g)}(\beta(g)) \left[\frac{1}{b^{g+1}} \sum_{a=0}^{\beta_g-1} \chi_{\varphi(k_g)}(\varphi(a)) + (\beta - \beta(g+1))\chi_{\varphi(k_g)}(\varphi(\beta_g))\right].$$

In the case, when $\beta_g = 0$ we have that $\beta(g) = \beta(g+1)$, from properties 1 and 2 of the functions ${}_{G,\varphi}wal_k$ and (6), we obtain $\mathcal{W}_{G,\varphi}\overline{\widehat{f}}_I(k) = {}_{G,\varphi}wal_{k(g)}(\beta(g))(\beta - \beta(g+1))$.

The inequality in (ii) is a direct consequence of the equalities in (i). ■

Lemma 2. *Let $n \geq 1$ be an arbitrary fixed integer. For arbitrary integers a, c and d, such that $0 \leq a \leq b^n$, $0 \leq c \leq d \leq b^n$ we introduce the significations $I_{a,n} = \left[0, \frac{a}{b^n}\right)$ and $I_{c,d,n} = \left[\frac{c}{b^n}, \frac{d}{b^n}\right)$. Let $I_n = I_{a,n}$ or $I_n = I_{c,d,n}$. We define the function $f_{I_n}(x) = 1_{I_n}(x) - \lambda(I_n)$, $x \in [0,1)$. Then, we have:*

(i) For each integer $k \geq b^n$ the equality $\mathcal{W}_{G,\varphi}\widehat{f}_{I_n}(k) = 0$ holds.

For arbitrary integers g, $0 \leq g < n$ and k, $b^g \leq k < b^{g+1}$ we have the following:

(ii^*) For each integer a, $0 \leq a \leq b^n$ the inequality $|\mathcal{W}_{G,\varphi}\widehat{f}_{I_{a,n}}(k)| \leq \frac{C+1}{b^{g+1}}$ holds.

(ii) For each integers c and d, $0 \le c \le d \le b^n$ the inequality $|\mathcal{W}_{G,\varphi}\widehat{f}_{I_{c,d,n}}(k)| \le \frac{2(C+1)}{b^{g+1}}$ *holds.*

P r o o f. For each integers $k \ge 1$, a, c and d, defined in the condition of the Lemma the equalities

$$\mathcal{W}_{G,\varphi}\widehat{f}_{I_{c,d,n}}(k) = \mathcal{W}_{G,\varphi}\widehat{f}_{I_{d,n}}(k) - \mathcal{W}_{G,\varphi}\widehat{f}_{I_{c,n}}(k) \tag{8}$$

and

$$\mathcal{W}_{G,\varphi}\widehat{f}_{I_{a,n}}(k) = \mathcal{W}_{G,\varphi}\widehat{1}_{I_{a,n}}(k) \tag{9}$$

hold.

(i) Let $k \ge b^n$ be an arbitrary integer. We will prove that for each integer a, $0 \le a \le b^n$ the equality

$$\mathcal{W}_{G,\varphi}\widehat{1}_{I_{a,n}}(k) = 0 \tag{10}$$

holds. Let the integer $h \ge n$ be defined by the condition $b^h \le k < b^{h+1}$. Then, from property 3 of the functions ${}_{G,\varphi}wal_k$ we have

$$\mathcal{W}_{G,\varphi}\widehat{\widehat{1}}_{I_{a,n}}(k) = \int_0^1 1_{\left[0,\frac{a}{b^n}\right)}(x)_{G,\varphi}wal_k(x)dx$$

$$= \int_0^{\frac{a}{b^n}} {}_{G,\varphi}wal_k(x)dx = \sum_{m=0}^{a.b^{h-n}-1} \int_{\frac{m}{b^h}}^{\frac{m+1}{b^h}} {}_{G,\varphi}wal_k(x)dx = 0.$$

From (8), (9) and (10) for each integer $k \ge b^n$, we obtain $\mathcal{W}_{G,\varphi}\widehat{f}_{I_n}(k) = 0$.

(ii^*) For arbitrary integers g, $0 \le g < n$ and k, $b^g \le k < b^{g+1}$ according to Lemma 1 (ii), we obtain that for each integer a, $0 \le a \le b^n$, $\left|\mathcal{W}_{G,\varphi}\widehat{f}_{I_{a,n}}(k)\right| \le \frac{C+1}{b^{g+1}}$.

(ii) For arbitrary integers c and d, $0 \le c \le d \le b^n$ from (8) and the last inequality, we obtain $\left|\mathcal{W}_{G,\varphi}\widehat{f}_{I_{c,d,n}}(k)\right| \le \frac{2(C+1)}{b^{g+1}}$. ■

Let us introduce the signification $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

Lemma 3. *Let $n \ge 1$ be an arbitrary fixed integer. Let for $1 \le i \le s$ a_i, c_i and d_i be arbitrary integers, such that $0 \le a_i \le b^n$ and $0 \le c_i \le d_i \le b^n$. Let $\mathbf{a} = (a_1, \ldots, a_s)$, $\mathbf{c} = (c_1, \ldots, c_s)$ and $\mathbf{d} = (d_1, \ldots, d_s)$. We define the sets $I_{\mathbf{a},n} = \prod_{i=1}^s \left[0, \frac{a_i}{b^n}\right)$, $I_{\mathbf{c},\mathbf{d},n} = \prod_{i=1}^s \left[\frac{c_i}{b^n}, \frac{d_i}{b^n}\right)$ and put $I_n = I_{\mathbf{a},n}$ or $I_n = I_{\mathbf{c},\mathbf{d},n}$. We define the function $f_{I_n}(\mathbf{x}) = 1_{I_n}(\mathbf{x}) - \lambda_s(I_n)$, $\mathbf{x} \in [0,1)^s$. Then:*

(i) For each vector $\mathbf{k} \in \Delta_s^*(n)$ we have $\left|\mathcal{W}_{G,\varphi}\widehat{f}_{I_{\mathbf{c},\mathbf{d},n}}(\mathbf{k})\right| \le \omega_{G,\varphi walsh}(\mathbf{k})$;

(i^*) For each vector $\mathbf{k} \in \Delta_s^*(n)$ we have $\left|\mathcal{W}_{G,\varphi}\widehat{f}_{I_{\mathbf{a},n}}(\mathbf{k})\right| \le \omega'_{G,\varphi walsh}(\mathbf{k})$;

(ii) For each vector $\mathbf{k} \in \mathbf{N}_0^s \setminus \Delta_s^*(n)$ *we have* $\mathcal{W}_{G,\varphi}\widehat{f}_{I_n}(\mathbf{k}) = 0$.

P r o o f. We note that $\mathcal{W}_{G,\varphi}\widehat{f}_{I_n}(\mathbf{0}) = 0$.

For each vector $\mathbf{k} \neq \mathbf{0}$ the following equality

$$\mathcal{W}_{G,\varphi}\widehat{f}_{I_n}(\mathbf{k}) = \mathcal{W}_{G,\varphi}\widehat{1}_{I_n}(\mathbf{k}) = \prod_{i=1}^{s} \mathcal{W}_{G,\varphi}\widehat{1}_{I_n^{(i)}}(k_i) \tag{11}$$

holds, where for $1 \leq i \leq s$ $I_n^{(i)} = \left[0, \frac{a_i}{b^n}\right)$ or $I_n^{(i)} = \left[\frac{c_i}{b^n}, \frac{d_i}{b^n}\right)$.

(i) Let $\mathbf{k} \in \Delta_s^*(n)$. For each i, $1 \leq i \leq s$ we have the next conditions: If $k_i \neq 0$, then we choose the integer g_i, by the conditions $0 \leq g_i < n$ and $b^{g_i} \leq k_i < b^{g_i+1}$. Then, from Lemma 2 (ii) we have the inequality

$$\left|\mathcal{W}_{G,\varphi}\widehat{1}_{I^{(i)}_{c_i,d_i,n}}(k_i)\right| \leq \frac{2(C+1)}{b^{g_i}}. \tag{12}$$

If $k_i = 0$ we have that

$$\left|\mathcal{W}_{G,\varphi}\widehat{1}_{I^{(i)}_{c_i,d_i,n}}(k_i)\right| = \lambda\left(I^{(i)}_{c_i,d_i,n}\right) \leq 1. \tag{13}$$

From (12) and (13) we obtain that for each i, $1 \leq i \leq s$ $\left|\mathcal{W}_{G,\varphi}\widehat{1}_{I^{(i)}_{c_i,d_i,n}}(k_i)\right| \leq \omega_{G,\varphi walsh}(k_i)$. From the last inequality and (11) we obtain the equality

$$\left|\mathcal{W}_{G,\varphi}\widehat{f}_{I_{\mathbf{c},\mathbf{d},n}}(\mathbf{k})\right| \leq \omega_{G,\varphi walsh}(\mathbf{k}).$$

Using Lemma 2 (ii^*) by analogy, we prove (i^*).

(ii) Let $\mathbf{k} \in \mathbf{N}_0^s \setminus \Delta_s^*(n)$. Then, there is an index t, $1 \leq t \leq s$ such that $k_t \geq b^n$. According to Lemma 2 (i) we have $\mathcal{W}_{G,\varphi}\widehat{f}_{I_n^{(t)}}(k_t) = 0$ and from (11) we obtain $\mathcal{W}_{G,\varphi}\widehat{f}_{I_n}(\mathbf{k}) = 0$. ∎

Proof of Theorem 1. Following Niederreiter [19], we will realize the proof in three steps:

Step 1-A discretization of the problem;

Step 2-An estimation of the discretization error;

Step 3-An estimation of the discrete discrepancy.

Step 1. Let $n \geq 1$ be an arbitrary fixed integer. For an arbitrary subinterval $J = \prod_{i=1}^{s}[u_i, v_i)$ or $J = \prod_{i=1}^{s}[0, v_i)$, where for $1 \leq i \leq s$ $0 \leq u_i \leq v_i \leq 1$ let $\underline{J}$ and $\overline{J}$ be the intervals, defined by (5).

Let $\xi_N = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ be an arbitrary net of $N \geq 1$ points in $[0,1)^s$. For an arbitrary subinterval $J \in \mathcal{J}$ we signify $R(\xi_N; J) = N^{-1}A(\xi_N; J) - \lambda_s(J)$.

From the inequalities $\lambda_s(\underline{J}) \leq \lambda_s(J) \leq \lambda_s(\overline{J})$ and $A(\xi_N; \underline{J}) \leq A(\xi_N; J) \leq A(\xi_N; \overline{J})$ we consecutively have

$$R(\xi_N; J) = N^{-1}A(\xi_N; J) - \lambda_s(J) - (N^{-1}A(\xi_N; \underline{J}) - \lambda_s(\underline{J})) + R(\xi_N; \underline{J});$$

$$R(\xi_N; J) = N^{-1}A(\xi_N; J) - \lambda_s(J) - (N^{-1}A(\xi_N; \overline{J}) - \lambda_s(\overline{J})) + R(\xi_N; \overline{J});$$

$$|R(\xi_N; J)| \leq N^{-1}(A(\xi_N; J) - A(\xi_N; \underline{J})) + (\lambda_s(J) - \lambda_s(\underline{J})) + |R(\xi_N; \underline{J})|;$$

$$|R(\xi_N; J)| \leq N^{-1}(A(\xi_N; \overline{J}) - A(\xi_N; J)) + (\lambda_s(\overline{J}) - \lambda_s(J)) + |R(\xi_N; \overline{J})|;$$

$$2|R(\xi_N; J)| \leq$$

$$\leq N^{-1}A(\xi_N; \overline{J}) - N^{-1}A(\xi_N; \underline{J}) + (\lambda_s(\overline{J}) - \lambda_s(\underline{J})) + |R(\xi_N; \underline{J})| + |R(\xi_N; \overline{J})|;$$

$$|R(\xi_N; J)| \leq \lambda_s(\overline{J}) - \lambda_s(\underline{J}) + |R(\xi_N; \underline{J})| + |R(\xi_N; \overline{J})|, \tag{14}$$

which is an expression of the discretization of the problem and gives in an explicit form the error, which is obtained when we replace the interval J with the intervals $\underline{J}$ and $\overline{J}$.

Step 2. According to Niederreiter [19, Lemma 3.9] the following estimations hold:

If $J = \prod_{i=1}^{s}[u_i, v_i)$, then

$$\lambda_s(\overline{J}) - \lambda_s(\underline{J}) \leq 1 - \left(1 - \frac{2}{b^n}\right)^s. \tag{15}$$

If $J = \prod_{i=1}^{s}[0, v_i)$, then $\lambda_s(\overline{J}) - \lambda_s(\underline{J}) \leq 1 - \left(1 - \frac{1}{b^n}\right)^s$.

Step 3. The estimation in step 3 will be made only for extreme discrepancy $D(\xi_N)$. The $s-$dimensional intervals $\underline{J}$ and $\overline{J}$ are in the form $\prod_{i=1}^{s}\left[\frac{c_i}{b^n}, \frac{d_i}{b^n}\right)$, where for $1 \leq i \leq s$ $0 \leq c_i \leq d_i \leq b^n$. We put $H(J) = \underline{J}$ or $H(J) = \overline{J}$. For an arbitrary net ξ_N we have the equality

$$R(\xi_N; H(J)) = \frac{1}{N}\sum_{j=0}^{N-1} f_{H(J)}(\mathbf{x}_j). \tag{16}$$

According to Lemma 3 the function $f_{H(J)}(\mathbf{x})$, $\mathbf{x} \in [0,1)^s$ is a Walsh function over (G, φ) polynomial. We use the representation

$$f_{H(J)}(\mathbf{x}) = \sum_{\mathbf{k}\in\Delta_s^*(n)} \mathcal{W}_{G,\varphi}\widehat{f}_{H(J)}(\mathbf{k})_{G,\varphi} wal_{\mathbf{k}}(\mathbf{x}), \quad \forall \mathbf{x} \in [0,1)^s. \tag{17}$$

From (16) and (17) we have that

$$R(\xi_N; H(J)) = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{\mathbf{k}\in\Delta_s^*(n)} \mathcal{W}_{G,\varphi}\widehat{f}_{H(J)}(\mathbf{k})_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}_j)$$

$$= \sum_{\mathbf{k}\in\Delta_s^*(n)} \mathcal{W}_{G,\varphi}\widehat{f}_{H(J)}(\mathbf{k})\frac{1}{N}\sum_{j=0}^{N-1} {}_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}_j),$$

and we obtain the inequality

$$|R(\xi_N; H(J))| \leq \sum_{\mathbf{k}\in\Delta_s^*(n)} \left|\mathcal{W}_{G,\varphi}\widehat{f}_{H(J)}(\mathbf{k})\right| |S(\xi_N; {}_{G,\varphi}wal_{\mathbf{k}})|,$$

with the introduced signification $S(\xi_N; {}_{G,\varphi}wal_{\mathbf{k}}) = \frac{1}{N}\sum_{j=0}^{N-1} {}_{G,\varphi}wal_{\mathbf{k}}(\mathbf{x}_j)$. Using the estimation exposed in Lemma 3 (i), we obtain that

$$|R(\xi_N; H(J))| \leq \sum_{\mathbf{k}\in\Delta_s^*(n)} \omega_{G,\varphi walsh}(\mathbf{k})|S(\xi_N; {}_{G,\varphi}wal_{\mathbf{k}})|. \tag{18}$$

From (14), (15) and (18) we obtain that

$$D(\xi_N) = \sup_{J\in\mathcal{J}} |R(\xi_N; J)| \leq$$

$$\leq 1 - \left(1 - \frac{2}{b^n}\right)^s + 2\sum_{\mathbf{k}\in\Delta_s^*(n)} \omega_{G,\varphi walsh}(\mathbf{k})|S(\xi_N; {}_{G,\varphi}wal_{\mathbf{k}})|.$$

The estimation for $D^*(\xi_N)$ can be obtained by analogous way. The proof of Theorem 1 is completely realized.

4. The proof of Theorem 2

For a function $f \in L_1([0,1)^s)$ and each vector $\mathbf{k}$ with non-negative integer coordinates the Fourier-Haar coefficient over (G,φ) $\mathcal{H}_{G,\varphi}\widehat{f}(\mathbf{k})$ is defined as $\mathcal{H}_{G,\varphi}\widehat{f}(\mathbf{k}) = \int_{[0,1)^s} f(\mathbf{x})_{G,\varphi}\overline{h}_{\mathbf{k}}(\mathbf{x})d\mathbf{x}$.

We note the next property of the Haar functions over (G,φ) : For each integer k and each integer a such that $b^g \leq k < b^{g+1}$ and $0 \leq a < b^g$ with $g \geq 0$ the equality $\int_{I(a,g)} {}_{G,\varphi}h_k(x)dx = 0$ holds. In the special case for each integer $k \geq 1$ we have the equality $\int_{D_k} {}_{G,\varphi}h_k(x)dx = 0$.

Lemma 4. *For an arbitrary subinterval I of $[0,1)$ we define the function $f_I(x) = 1_I(x) - \lambda(I)$, $x \in [0,1)$. Then, we have the following:*

1. $\mathcal{H}_{G,\varphi}\widehat{f}_I(0) = 0$.

2. Let $n \geq 1$ be a fixed integer and I be an elementary $b-$adic interval with length b^{-n}. Then, for each integer $k \geq b^n$ $\mathcal{H}_{G,\varphi}\widehat{f}_I(k) = 0$.

3. For an arbitrary real β, $0 \leq \beta \leq 1$ with the $b-$adic representation $\beta = \sum_{j=0}^{\infty} \beta_j b^{-j-1}$, where for $j \geq 0$ $\beta_j \in \{0, 1, \dots, b-1\}$ we put $I = [0, \beta)$. For an arbitrary integer $k \geq 1$, having the $b-$adic representation (4) the following hold:

(3i) If $k(g) \neq b^g \beta(g)$, then ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(k) = 0$;

(3ii) If $k(g) = b^g \beta(g)$, then

$$
{}_{\mathcal{H}_{G,\varphi}}\widehat{\overline{f}}_I(k) = \begin{cases} b^{\frac{g}{2}} \left[\frac{1}{b^{g+1}} \sum_{a=0}^{\beta_g - 1} \chi_{\varphi(k_g)}(\varphi(a)) + \right. \\ \left. + \chi_{\varphi(k_g)}(\varphi(\beta_g))(\beta - \beta(g+1)) \right], & \text{if } \beta_g \neq 0 \\ b^{\frac{g}{2}} (\beta - \beta(g+1)), & \text{if } \beta_g = 0. \end{cases}
$$

P r o o f. 1. For an arbitrary subinterval I of $[0, 1)$ we have that ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(0) = \int_0^1 (1_I(x) - \lambda(I))dx = 0$.

2. Let I be an elementary $b-$adic interval with length b^{-n} and $k \geq b^n$ be an arbitrary integer. Let D_k be the fundamental domain of the function ${}_{G,\varphi}h_k$. Then, from Definition 2 we have

$$
{}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(k) = \int_0^1 1_I(x)_{G,\varphi}\overline{h}_k(x)dx = \int_{D_k} 1_I(x)_{G,\varphi}\overline{h}_k(x)dx. \tag{19}
$$

Let $D_k \subseteq I$. Then $1_I(x) = 1$ for $\forall x \in D_k$ and from (19) and exposed property of the function ${}_{G,\varphi}h_k$ we obtain ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(k) = \int_{D_k} {}_{G,\varphi}\overline{h}_k(x)dx = 0$. Let $D_k \cap I = \emptyset$. Then, from Definition 2 and (19) we directly obtain that ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(k) = 0$.

3. Let $k \geq 1$ be an arbitrary integer and $b^g \leq k < b^{g+1}$ with $g \geq 0$. Then, from exposed property of the Haar function ${}_{G,\varphi}h_k$ we have the equality

$$
{}_{\mathcal{H}_{G,\varphi}}\widehat{\overline{f}}_I(k) = \int_0^{\beta(g)} {}_{G,\varphi}h_k(x)dx + \int_{\beta(g)}^{\beta} {}_{G,\varphi}h_k(x)dx = \int_{\beta(g)}^{\beta} {}_{G,\varphi}h_k(x)dx. \tag{20}
$$

(3i) Let $k(g) \neq b^g\beta(g)$. Then ${}_{G,\varphi}h_k(x) = 0$ for $\forall x \in [\beta(g), \beta)$ and from (20) we obtain ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(k) = 0$.

(3ii) Let $k(g) = b^g\beta(g)$. Then, we have that $[\beta(g), \beta) \subset D_k$. If $\beta_g \neq 0$ from Definition 2 and (20) we obtain

$$
{}_{\mathcal{H}_{G,\varphi}}\widehat{\overline{f}}_I(k) = b^{\frac{g}{2}} \sum_{a=0}^{\beta_g - 1} \int_{\beta(g)+\frac{a}{b^{g+1}}}^{\beta(g)+\frac{a+1}{b^{g+1}}} \chi_{\varphi(k_g)}(\varphi(a))dx + b^{\frac{g}{2}} \int_{\beta(g+1)}^{\beta} \chi_{\varphi(k_g)}(\varphi(\beta_g))dx
$$

$$
= b^{\frac{g}{2}} \left[\frac{1}{b^{g+1}} \sum_{a=0}^{\beta_g - 1} \chi_{\varphi(k_g)}(\varphi(a)) + \chi_{\varphi(k_g)}(\varphi(\beta_g))(\beta - \beta(g+1)) \right].
$$

If $\beta_g = 0$ then $\beta(g) = \beta(g+1)$ and from Definition 2 and (20) we obtain ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(k) = b^{\frac{g}{2}}(\beta - \beta(g+1))$. Lemma 4 is completely proved. ■

Lemma 5. *Let $n \geq 1$ be an arbitrary fixed integer and $I_{c,d,n} = \left[\frac{c}{b^n}, \frac{d}{b^n}\right)$ with integers $0 \leq c \leq d \leq b^n$. We define the function $f_{I_{c,d,n}}(x) = 1_{I_{c,d,n}}(x) - \lambda(I_{c,d,n})$, $x \in [0,1)$. We put $\beta' = \frac{c}{b^n}$ and $\beta'' = \frac{d}{b^n}$. Then, the following hold:*

(i) ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_{I_{c,d,n}}(0) = 0$.

For each integer k, $b^g \leq k < b^{g+1}$ with the conditions $0 \leq g < n$ we have:

(iia) If $k(g) \notin \{b^g\beta'(g), b^g\beta''(g)\}$, then ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_{I_{c,d,n}}(k) = 0$.

(iib) If $k(g) \in \{b^g\beta'(g), b^g\beta''(g)\}$, then $|{}_{\mathcal{H}_{G,\varphi}}\widehat{f}_{I_{c,d,n}}(k)| \leq b^{-\frac{g}{2}-1}(C+1)$.

(iii) For each integer $k \geq b^n$ the equality ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_{I_{c,d,n}}(k) = 0$ *holds.*

The proof of Lemma 5 is similar to the proof of Lemma 2 and it is based on Lemma 4.

Lemma 6. *Let $n \geq 1$ be a fixed integer. Let for $1 \leq i \leq s$ $0 \leq \beta'_i \leq \beta''_i \leq 1$ are rational numbers with denominators b^n and $I = \prod_{i=1}^{s}[\beta'_i, \beta''_i)$. We define $f_I(\mathbf{x}) = 1_I(\mathbf{x}) - \lambda_s(I)$, $\mathbf{x} \in [0,1)^s$. Then, the following hold:*

(i) ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(\mathbf{0}) = 0$.

(ii) For each vector $\mathbf{k} \in \Delta_s^*(n)$ we have $\left|{}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(\mathbf{k})\right| \leq \omega^I_{G,\varphi Haar}(\mathbf{k})$.

(iii) For each vector $\mathbf{k} \in \mathbf{N}_0^s \setminus \Delta_s^*(n)$ we have ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(\mathbf{k}) = 0$.

Proof. (i) It is obvious that ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(\mathbf{0}) = 0$.

Let $\mathbf{k} \neq \mathbf{0}$, $\mathbf{k} = (k_1, \ldots, k_s)$ be an arbitrary vector with non-negative integer coordinates. The following equality

(21)
$$ {}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(\mathbf{k}) = \prod_{i=1}^{s} {}_{\mathcal{H}_{G,\varphi}}\widehat{1}_{I_i}(k_i) $$

holds, and where for $1 \leq i \leq s$ we signify $I_i = [\beta'_i, \beta''_i)$.

(ii) Let $\mathbf{k} \in \Delta_s^*(n)$. For each i, $1 \leq i \leq s$ we have the next inequalities: If $k_i = 0$, then

(22)
$$ {}_{\mathcal{H}_{G,\varphi}}\widehat{1}_{I_i}(0) = \lambda(I_i) \leq 1. $$

If $k_i \neq 0$ then, according to Lemma 5 (iia) and (iib) we obtain

(23)
$$ \left|{}_{\mathcal{H}_{G,\varphi}}\widehat{1}_{I_i}(k_i)\right| \leq \omega^{I_i}_{G,\varphi Haar}(k_i), $$

with defined coordinate quantity $\omega^{I_i}_{G,\varphi Haar}(k_i)$. From (21), (22) and (23) we obtain

$$ \left|{}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(\mathbf{k})\right| \leq \omega^I_{G,\varphi Haar}(\mathbf{k}). $$

(iii) Let $\mathbf{k} \in \mathbf{N}_0^s \setminus \Delta_s^*(n)$. Then, there is an index t, $1 \leq t \leq s$ such that $k_t \geq b^n$. According to Lemma 5 (iii) we have that ${}_{\mathcal{H}_{G,\varphi}}\widehat{1}_{I_t}(k_t) = {}_{\mathcal{H}_{G,\varphi}}\widehat{f}_{I_t}(k_t) = 0$ and from (21) we obtain that ${}_{\mathcal{H}_{G,\varphi}}\widehat{f}_I(\mathbf{k}) = 0$. ∎

We can prove Theorem 2 in the same way we used for a demonstration of Theorem 1. The proof is based on the results, obtained in Lemma 6 and follow the method of the proof of Theorem 1.

Let in the condition of Lemma 5 we replace an interval of the form $I_{c,d,n} = \left[\frac{c}{b^n}, \frac{d}{b^n}\right)$, where $0 \le c \le d \le b^n$ are arbitrary integers with an interval of the form $I_{a,n} = \left[0, \frac{a}{b^n}\right)$, where $0 \le a \le b^n$ is an arbitrary integer. Let $k \ge 1$ be an arbitrary integer such that $b^g \le k < b^{g+1}$ with $0 \le g < n$. Then the estimation $|_{\mathcal{H}_{G,\varphi}}\widehat{f}_{I_{c,d,n}}(k)| \le b^{-\frac{g}{2}-1}(C+1)$, obtained in (iib) of Lemma 5 will give again the inequality $|_{\mathcal{H}_{G,\varphi}}\widehat{f}_{I_{a,n}}(k)| \le b^{-\frac{g}{2}-1}(C+1)$. We will only specify the kind of the set $\{b^g\beta'(g), b^g\beta''(g)\}$, which in this case will be $\{0, b^g\beta''(g)\}$. In this sense in our paper we are not going to expose an estimation of the star-discrepancy in the terms of the Haar functions over finite groups.

5. Conclusion

The obtained estimations of the extreme and the star-discrepancy in Theorems 1 and 2 show the form of the inequality of Erdös-Turan-Koksma, respectively in the terms of Walsh and Haar functions over finite groups. For an arbitrary chosen parameter $n \ge 1$ the form of the domain $\Delta_s^*(n)$, the values of the coefficients $\rho_{\mathcal{F}}(\mathbf{k})$ and $\delta_{\mathcal{F}}(M)$, depending on the functional system $\mathcal{F}$, where $\mathcal{F} = \mathcal{W}_{G,\varphi}(b)$ or $\mathcal{F} = \mathcal{H}_{G,\varphi}(b)$, are shown.

The main concept of the inequality of Erdös-Turan-Koksma, obtained in Theorems 1 and 2, is that the estimation of the discrepancy in practice is reduced to an non-trivial estimation of the sum of the corresponding net with respect to Walsh and Haar functions over finite groups.

The estimations of discrepancy are important to solve many practical and theoretical problems. The inequality of Koksma-Hlawka (see Kuipers and Niederreiter [13]) shows that the order of the error of the quadrature formula with an arbitrary $s-$dimensional net is determined by the order of the discrepancy of this net.

The trigonometric system $\mathcal{T}$ has been used as a means of solving the problems of irregularity of the distribution of sequences and nets for a long time.

The link, which is realized in the process of studying sequences and nets in $[0,1)^s$, constructed in $b-$adic number system ($b \ge 2$-integer) and some orthonormal functional systems on $[0,1)^s$, constructed in $b-$adic number system, too, is quite natural. In this sense, some authors have recently used complete orthonormal functional systems, constructed in $b-$adic number system, for example the Walsh and Haar systems as a means of solving the problems of irregularity of the distrbution of sequences and nets. The obtained results, as in many other papers, and in our paper too, show that this means is suitable and reflects the character of the construction of the sequences and nets.

The authors will present possible variants to extend this study.

1. To investigate the possibility to use an arbitrary complete orthonormal system on $[0,1)^s$, constructed in $b-$adic number system to obtain the general form of the inequality of Erdös-Turan-Koksma. Our hypothesis is that this problem might be solved with minimal additional conditions of the functions of the system. The obtained results will have general character. We think that the developed method of investigation of the dicrepancy with some modifigations can be used to solve this problem.

2. Stegbuchner [23] using the inequality of Erdös-Turan-Koksma, exposed in (2) as a base obtains the multidimensional variant of the inequality of LeVeque [17]. This inequality gives an relationship between the discrepancy and diaphony (see Zinterhof [27]) of an arbitrary $s-$dimensional net.

The so called generalized $b-$adic diaphony, as a measure of the irregularity of the distribution of sequences and nets, which is based on the Walsh functions over finite groups has been introduced in [4]. In this sense the problem of the multidimensional variant of the inequality of LeVeque, in which the discrepancy and the generalized $b-$adic diaphony of an arbitrary $s-$dimensional nets to be connected is open. The authors consider that the comparatively simple method, developed by Stegbuchner can be used to solve the exposed problem.

3. We noticed in Introduction that Niederreiter and Philipp [20] and [21] give further generalization of the inequality of Erdös-Turan-Koksma. The authors assume that it is sensible to obtain in this direction an generalization of the inequality of Erdös-Turan-Koksma in terms of Walsh and Haar functions over finite groups.

4. Having in mind the fact that non-trivial estimation of $S(\xi_N;\psi_{\mathbf{k}})$ of the net ξ_N with respect to the function $\psi_{\mathbf{k}}$ gives the corresponding order of the discrepancy $D(\xi_N)$, then the problem of obtaining the estimations of $S(\xi_N;{}_{G,\varphi}wal_{\mathbf{k}})$ and $S(\xi_N;{}_{G,\varphi}h_{\mathbf{k}})$ of some special nets, as they are the paralepipedal nets (see Hlawka [10]), (t,m,s)-nets (see Niederreiter [18], Faure [3]) and others is still open. Some non-trivial estimations of the sum $S(\xi_N;{}_{G,\varphi}wal_{\mathbf{k}})$ have been obtained (see Larcher and Traunfellner [16], Larcher, Niederreiter and W. Ch. Schmid [14]). Only some partial results are obtained in the direction of investigation of the sum $S(\xi_N;{}_{G,\varphi}h_{\mathbf{k}})$. In [24] is given an estimation with respect to the lattice point set. But the problem of the investigation of the sum $S(\xi_N;{}_{G,\varphi}h_{\mathbf{k}})$, when ξ_N is the paralepipedal nets, some classes of (t,m,s)-net, is open. The solving of these problems will have a link with other problems,

such as a numerical integration of functions, represented by rapidly convergent Fourier- Haar series (see [24]).

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Department of Mathematics,
South West University "Neophit Rilski",
66 Ivan Mihailov str.,
2700 Blagoevgrad, Bulgaria
[1] *e-mail: vassgrozdanov@yahoo.com*
[2] *e-mail: stanislavast@yahoo.com*

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