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Superconvergence of the Interpolated Quadratic Finite Elements on Triangular Meshes[1]

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In this paper quadratic triangular interpolated finite elements are introduced. They improve the global convergence of the approximate solution as well as of the approximate gradient when an elliptic boundary value problem of second order is solved by the finite element method. It is a variant of a postprocessing procedure and global superconvergence error estimates are derived. We apply also the method interpolated triangular finite elements for an elliptic eigenvalue problem. Finally, a numerical example confirming the theoretical results is presented.

Key Words: interpolated finite elements, superconvergence, postprocessing
AMS Subj. Classification: 65N25, 65N15.

1. Introduction and Preliminaries

In recent years, the methods for accelerating the convergence of finite element solutions have been an active research area in numerical analysis. The main goal in the superconvergence study is to improve the existing approximation accuracy by applying certain postprocessing techniques which are easy to implement (see [1, 2, 3, 4]). This paper derives a general superconvergence result for finite element approximations of the second order elliptic problem by using the so-called interpolated finite elements. The method is proposed and analysed by Q. Lin et al. [5, 6, 7] only for rectangular meshes.

Our aim is to develop further the method of interpolated elements presented in [5]. Namely, we shall prove the applicability of this method on the triangular mesh. This is a highly accurate recovery process with superconver-

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gent properties. We gain one order higher convergence using a little additional work subject to the exact solution is smooth enough. Thus the interpolated finite elements are appropriate for the local application on the subdomain. Then good finite element approximations may be obtained on the coarse mesh.

The patch recovery techniques on triangular meshes are consider by Zienkiewicz and Zhu [3], Lin and Zhang [9] and others. But here we present a substantially different approach. Namely, this paper uses the ideas of interpolated rectangular elements and shows the applicability to the triangular elements. Although the superconvergence analysis is more difficult, the proposed postprocessing algorithm is relatively simple to implementation.

For simplicity, let us consider the model problem

$$-\Delta u = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma, \tag{1}$$

where Ω is a rectangular domain in $\mathbf{R}^2$ with boundary $\Gamma = \partial\Omega$. Let $(\cdot,\cdot)$ be the inner product in $L_2(\Omega)$. By $W^{m,p}(\Omega)$, $m = 0, 1, 2, \ldots$ we denote the usual Sobolev space of order m with norm $\|\cdot\|_{m,p,\Omega}$, see [8]. We will mainly deal with $p = 2$ and then we will use the denotations $H^m(\Omega)$ and $\|\cdot\|_{m,\Omega}$ for $W^{m,2}(\Omega)$ and $\|\cdot\|_{m,2,\Omega}$, respectively. The weak solution of problem (1) is a function u belonging to $H_0^1(\Omega)$,

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v|_\Gamma = 0\}$$

and satisfying

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

We denote by P_k the set of polynomials of degree at most k.

Consider families of regular partitions $\tau_h = \cup_i T_i$ of $\overline{\Omega}$ consisting of triangular finite elements T_i. Let the partition τ_h fulfils standard assumptions (see [8], Chapter 3). If h_i is a diameter of T_i, then $h = \max_i h_i$ is the finite element parameter corresponding to any partition τ_h.

Let $P_h : V \to V_h$ be the elliptic projection operator defined by

$$a(u - P_h u, v) = 0 \quad \forall u \in V, \forall v \in V_h.$$

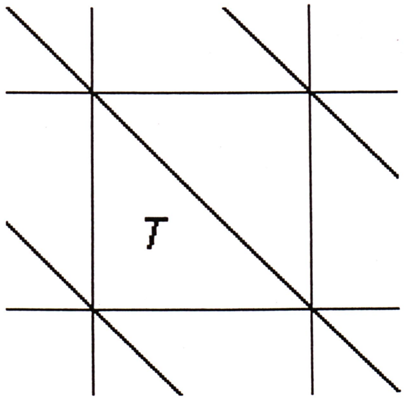

Figure 1:

It is well-known (see [8]) that if the solution $u(x,y)$ of (2) belongs to $H^3(\Omega) \cap V$, then

$$\|u - P_h u\|_{1,\Omega} \leq Ch^2 \|u\|_{3,\Omega}. \tag{3}$$

Our aim is to improve the optimal estimate (3). We need two main ingredients to prove the superconvergence property:

- An interpolation approximating the finite element solution of higher order. Often such interpolation does exist if the inderlying mesh has a special construction (cf. [10]). This special operator is called to satisfy a superclose property [4].
- A higher order interpolation of the original finite element solution to achieve higher order accuracy. The interpolated finite element solution resulting from the postprocessing is called to satisfy a superconvergence property [4].

2. Construction of Superclose Operator $\mathbf{i_h}$

Consider the regular pattern $\tau_h = \cup T$. This partition consists of isosceles right-angled triangles. The length of legs is $2h$ and the middle of hypotenuse is (x_T, y_T) for any $T \in \tau_h$ (see Figure 1).

Denote the vertices of any element $T \in \tau_h$ by p_j and the edges by l_j, $j = 1, 2, 3$, respectively. The operator $i_h : C^0 \to V_h$ can be defined using the following "vertices-edges" conditions as degrees of freedom:

$$\forall v \in C(\Omega),\ \forall T \in \tau_h,\quad i_h v(p_j) = v(p_j);$$

(4)

$$\int_{l_j} i_h v\, dl = \int_{l_j} v\, dl,\ \ j = 1,2,3.$$

It is evident that

$$i_h v \in V_h,\quad \forall v \in C(\Omega),$$

$$i_h v = v,\qquad \forall v \in V_h,$$

and also

$$i_h v|_T = \sum_{i=1}^{6} \hat{v}_i . \varphi_i \left(\frac{x - x_T}{h}, \frac{y - y_T}{h} \right),$$

where $\{\varphi_i\}_{i=1}^{6}$ are the base functions corresponding to the degrees of freedom (4). Note that

$$\hat{v}_i = v(p_i),\ i = 1,2,3 \text{ and } \hat{v}_i = \frac{1}{h} \int_{l_j} v\, dl,\ \ i = j+3,\ \ j = 1,2,3.$$

So, it is easy to obtain for any point $(\hat{x}, \hat{y})$ belonging to the element of reference $\widehat{T}$, i.e. when $-1 \le \hat{x} \le 1;\ \ -1 \le \hat{y} \le -\hat{x}$, that

$$\varphi_1(\hat{x}, \hat{y}) = \frac{1}{4}(4\hat{x} + 4\hat{y} + 6\hat{x}\hat{y} + 3\hat{x}^2 + 3\hat{y}^2);$$

$$\varphi_2(\hat{x}, \hat{y}) = \frac{1}{4}(-1 + 2\hat{x} + 3\hat{x}^2);$$

$$\varphi_3(\hat{x}, \hat{y}) = \frac{1}{4}(-1 + 2\hat{y} + 3\hat{y}^2);$$

$$\varphi_4(\hat{x}, \hat{y}) = \frac{1}{4}(-3\hat{x} - 3\hat{y} - 3\hat{x}\hat{y} - 3\hat{x}^2);$$

$$\varphi_5(\hat{x}, \hat{y}) = \frac{1}{4}(3 + 3\hat{x} + 3\hat{y} + 3\hat{x}\hat{y});$$

$$\varphi_6(\hat{x}, \hat{y}) = \frac{1}{4}(-3\hat{x} - 3\hat{y} - 3\hat{x}\hat{y} - 3\hat{y}^2).$$

Remark 1. Such a basis could be used directly to the finite element implementation and not only to the patch recovery superconvergence method.

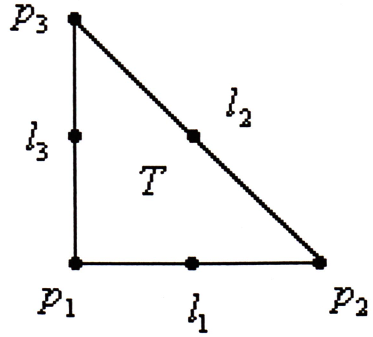

Figure 2:

3. Superclose Property of the Interpolant i_h

Now we shall prove an essential result concerning the superclose property caused by the operator i_h.

For any $T \in \tau_h$ introduce the error functions:

$$E(x) = \frac{1}{2}\left[(x - x_T)^2 - h^2\right],$$

$$F(y) = \frac{1}{2}\left[(y - y_T)^2 - h^2\right].$$

Obviously $E(p_2) = 0$ and $E(x)|_{l_3} = 0$. By analogy, $F(p_3) = 0$ and $F(y)|_{l_1} = 0$ (see Figure 2).

The following error functions are also needed:

(5)
$$G_1(x,y) = \frac{1}{2}(x - x_T + h)(x - x_T + y - y_T),$$

$$G_2(x,y) = \frac{1}{2}(y - y_T + h)(x - x_T + y - y_T).$$

Note that

$$G_1(x,y)|_{l_2} = G_1(x,y)|_{l_3} = 0$$

and

$$G_2(x,y)|_{l_1} = G_2(x,y)|_{l_2} = 0.$$

The main result concerning the superclose property of i_h contains in the following theorem:

Theorem 1. *Let $u(x,y) \in H^4(\Omega)$. Then for any $v \in V_h$ the following inequality holds:*

$$a(i_h u - u, v) \le Ch^3 \|u\|_{4,\Omega} \|v\|_{1,\Omega}. \tag{6}$$

P r o o f. Let us denote $U = i_h - u$. Having in mind that $v \in V_h$, then for $(x,y) \in T$ it follows

$$\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial x}(x_T, y_T) + \frac{\partial^2 v}{\partial x^2}(x,y).(x - x_T) + \frac{\partial^2 v}{\partial x \partial y}(x,y).(y - y_T),$$

$$\frac{\partial v}{\partial y}(x,y) = \frac{\partial v}{\partial y}(x_T, y_T) + \frac{\partial^2 v}{\partial y^2}(x,y).(y - y_T) + \frac{\partial^2 v}{\partial x \partial y}(x,y).(x - x_T).$$

Consequently

$$\begin{aligned} a(U,v) \;&= \int_T \frac{\partial U}{\partial x}\frac{\partial v}{\partial x}(x_T,y_T)\,dx\,dy + \int_T \frac{\partial U}{\partial y}\frac{\partial v}{\partial y}(x_T,y_T)\,dx\,dy \\ &+ \int_T (x - x_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\,dy + \int_T (y - y_T)\frac{\partial U}{\partial y}\frac{\partial^2 v}{\partial y^2}\,dx\,dy \\ &+ \int_T (y - y_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x \partial y}\,dx\,dy + \int_T (x - x_T)\frac{\partial U}{\partial y}\frac{\partial^2 v}{\partial x \partial y}\,dx\,dy. \end{aligned} \tag{7}$$

We estimate each term in the right-hand side of (7). For both the first and the second terms the following equalities hold respectively:

$$\int_T \frac{\partial U}{\partial x}\frac{\partial v}{\partial x}(x_T,y_T)\,dx\,dy = \left(\int_{l_2} - \int_{l_3}\right) U \frac{\partial v}{\partial x}(x_T,y_T)\,dy = 0; \tag{8}$$

$$\int_T \frac{\partial U}{\partial y}\frac{\partial v}{\partial y}(x_T,y_T)\,dx\,dy = \left(\int_{l_2} - \int_{l_1}\right) U \frac{\partial v}{\partial y}(x_T,y_T)\,dy = 0. \tag{9}$$

Integrating by parts and using (5) it follows for the third term of (7):

$$\int_T (x - x_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\,dy = 2\int_T \frac{\partial G_1}{\partial y}(x,y)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\,dy$$

$$= 2\left(\int_{l_2} - \int_{l_1}\right) G_1(x,y)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx - 2\int_T G_1(x,y)\frac{\partial^2 U}{\partial x \partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy.$$

According to the definition of $G_1(x,y)$ we have

$$\int_{l_2} G_1(x,y)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx = 0.$$

On the other hand, the sum of integrals

$$\int_{l_1} G_1(x,y)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx$$

over all $T \in \tau_h$ disappears. This is due to the fact that $G_1(x,y)|_{l_1} = E(x)$ and the integrand is a continuous function on any edge parallel to the x−axis. For integrals over any parts of Γ one can integrate by parts twice and use that the function v vanishes on Γ. Thus we get:

$$\int_{l_1} E(x)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx = \int_{l_1}\frac{\partial^2}{\partial x^2}\left[E(x)\frac{\partial U}{\partial x}\right] v\,dx = 0.$$

Therefore, the following relation could be written:

$$\text{(10)}\qquad \int_T (x-x_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\,dy = -2\int_T G_1(x,y)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\,dy.$$

Using that

$$2G_1(x,y) = 2\left[\frac{\partial G_1}{\partial x}(x,y)\right]^2 + \frac{\partial^2}{\partial x^2}\left[G_1^2(x,y)\right],$$

and the identity $\dfrac{\partial^2 G_1}{\partial x^2} = 1$, after integrating by parts we obtain:

$$\int_T (x-x_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\,dy = -2\int_T \left[\frac{\partial G_1}{\partial x}\right]^2 \frac{\partial^2 U}{\partial x\partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy$$

$$-\int_T \frac{\partial^2}{\partial x^2}\left[G_1^2\right]\frac{\partial^2 U}{\partial x\partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy = 2\int_T G_1(x,y)\frac{\partial G_1}{\partial x}\frac{\partial^3 U}{\partial x^2\partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy$$

$$+2\int_T G_1(x,y)\frac{\partial^2 U}{\partial x\partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy - \int_T G_1^2(x,y)\frac{\partial^4 U}{\partial x^3\partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy.$$

This equality in combination with (10) gives

$$\int_T (x-x_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\,dy = \int_T \frac{\partial G_1^2}{\partial x}\frac{\partial^3 U}{\partial x^2\partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy$$

$$-\int_T (x-x_T)\frac{\partial^2 U}{\partial x\partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy - \int_T G_1^2(x,y)\frac{\partial^4 U}{\partial x^3\partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy.$$

Consequently

$$\int_T (x-x_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x^2}\,dx\,dy = -\int_T G_1^2(x,y)\frac{\partial^4 U}{\partial x^3\partial y}\frac{\partial^2 v}{\partial x^2}\,dx\,dy. \tag{11}$$

By analogy for the fourth ingredient in the right-hand side of (7) the following equality holds:

$$\int_T (y-y_T)\frac{\partial U}{\partial y}\frac{\partial^2 v}{\partial y^2}\,dx\,dy = -\int_T G_2^2(x,y)\frac{\partial^4 U}{\partial x\partial y^3}\frac{\partial^2 v}{\partial y^2}\,dx\,dy. \tag{12}$$

Now, let us consider the sum of the fifth and sixth terms in (7):

$$\int_T (y-y_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy + \int_T (x-x_T)\frac{\partial U}{\partial y}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy$$

$$= \left(\int_{l_2} - \int_{l_3}\right)(y-y_T)U\frac{\partial^2 v}{\partial x\partial y}\,dy + \left(\int_{l_2} - \int_{l_1}\right)(x-x_T)U\frac{\partial^2 v}{\partial x\partial y}\,dx$$

$$= -\int_{y_T-h}^{y_T+h} (y-y_T)U(x_T-h,y)\frac{\partial^2 v}{\partial x\partial y}\,dy$$

$$-\int_{x_T-h}^{x_T+h} (x-x_T)U(x,y_T-h)\frac{\partial^2 v}{\partial x\partial y}\,dy$$

$$+\int_{y_T-h}^{y_T+h} \left[(y-y_T+x-x_T)U(x,y)\right]|_{x-x_T+y-y_T=0}\frac{\partial^2 v}{\partial x\partial y}\,dy.$$

Obviously, the last integral vanishes. The remaining ones in the right-hand side will be transformed by using that

$$x-x_T = \frac{1}{6}\left[E^2(x)\right]''',$$

$$y - y_T = \frac{1}{6}\left[F^2(y)\right]'''.$$

Then, integration by parts by parts reveals that

$$\begin{aligned} &-\int_{y_T-h}^{y_T+h} (y - y_T)U(x_T - h, y)\frac{\partial^2 v}{\partial x \partial y}\,dy \\ &= \frac{1}{6}\int_{y_T-h}^{y_T+h} F^2(y)\frac{\partial^3 U}{\partial y^3}(x_T - h, y)\frac{\partial^2 v}{\partial x \partial y}\,dy, \end{aligned} \tag{13}$$

we we took into consideration that the functions $U(x_T - h, y)$, $[F^2(y)]'$ and $F(^2(y)$ vanish when $y = y_T \pm h$.

By analogy

$$\begin{aligned} &-\int_{x_T-h}^{x_T+h} (x - x_T)U(x, y_T - h)\frac{\partial^2 v}{\partial x \partial y}\,dx \\ &= \frac{1}{6}\int_{x_T-h}^{x_T+h} E^2(x)\frac{\partial^3 U}{\partial x^3}(x, y_T - h)\frac{\partial^2 v}{\partial x \partial y}\,dx. \end{aligned} \tag{14}$$

Now, let us take also account of

$$\int_{y_T-h}^{y_T+h} F^2(y)\frac{\partial^3 U}{\partial y^3}(x_T - h, y)\frac{\partial^2 v}{\partial x \partial y}\,dy$$

$$-\int_{y_T-h}^{y_T+h} F^2(y)\frac{\partial^3 U}{\partial y^3}(x_T + y_T - y, y)\frac{\partial^2 v}{\partial x \partial y}\,dy$$

$$= \left(\int_{l_1} - \int_{l_2}\right) F^2(y)\frac{\partial^3 U}{\partial y^3}\frac{\partial^2 v}{\partial x \partial y}\,dy = -\int_T F^2(y)\frac{\partial^4 U}{\partial x \partial y^3}\frac{\partial^2 v}{\partial x \partial y}\,dx\,dy,$$

and correspondingly

$$\int_{x_T-h}^{x_T+h} E^2(x)\frac{\partial^3 U}{\partial x^3}(x, y_T - h)\frac{\partial^2 v}{\partial x \partial y}\,dx$$

$$-\int_{x_T-h}^{x_T+h} E^2(x)\frac{\partial^3 U}{\partial x^3}(x, x_T + y_T - x)\frac{\partial^2 v}{\partial x \partial y}\,dx$$

$$= \left(\int_{l_3} - \int_{l_2}\right) E^2(x)\frac{\partial^3 U}{\partial x^3}\frac{\partial^2 v}{\partial x \partial y}\,dx = -\int_T E^2(x)\frac{\partial^4 U}{\partial x^3 \partial y}\frac{\partial^2 v}{\partial x \partial y}\,dx\,dy.$$

The last two relations on combination with (13) and (14) give:

$$\int_T (y-y_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy + \int_T (x-x_T)\frac{\partial U}{\partial y}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy$$

$$(15)\qquad \begin{aligned} &= -\frac{1}{6}\int_T F^2(y)\frac{\partial^4 U}{\partial x\partial y^3}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy - \frac{1}{6}\int_T E^2(x)\frac{\partial^4 U}{\partial x^3\partial y}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy \\ &+\frac{1}{6}\int_{y_T-h}^{y_T+h} F^2(y)\frac{\partial^3 U}{\partial y^3}(x_T+y_T-y,y)\frac{\partial^2 v}{\partial x\partial y}\,dy \\ &+\frac{1}{6}\int_{x_T-h}^{x_T+h} E^2(x)\frac{\partial^3 U}{\partial x^3}(x,x_T+y_T-x)\frac{\partial^2 v}{\partial x\partial y}\,dx. \end{aligned}$$

Consider both the line integrals on l_2 in the right-hand side of (15). For any point $(x,y)\in l_2$ equality $E(x)=F(y)$ holds. Then

$$(16)\qquad \begin{aligned} &\frac{1}{6}\int_{y_T-h}^{y_T+h} F^2(y)\frac{\partial^3 U}{\partial y^3}(x_T+y_T-y,y)\frac{\partial^2 v}{\partial x\partial y}\,dy \\ &+\frac{1}{6}\int_{x_T-h}^{x_T+h} E^2(x)\frac{\partial^3 U}{\partial x^3}(x,x_T+y_T-x)\frac{\partial^2 v}{\partial x\partial y}\,dx. \\ &= \frac{1}{6}\int_{y_T-h}^{y_T+h} F^2(y)\left[\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial x^3}\right]\Bigg|_{x-x_T+y-y_T=0}\frac{\partial^2 v}{\partial x\partial y}\,dy \\ &= \frac{1}{12}\int_{y_T-h}^{y_T+h} F^2(y)\left[\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial x^3}\right]\Bigg|_{x-x_T+y-y_T=0}\left[2\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 v}{\partial y^2}\right]dy \\ &+\frac{1}{12}\int_{y_T-h}^{y_T+h} F^2(y)\left[\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial x^3}\right]\Bigg|_{x-x_T+y-y_T=0}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]dy. \end{aligned}$$

The first term in the right-hand side of (16) is cancelled. This fact follows from the definition of $U(x,y)$ and

$$\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial x^3} = -\left(\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 u}{\partial x^3}\right).$$

This function is continuous and the same is true for the function

$$2\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 v}{\partial y^2}$$

on the edge l_2. Consequently, this integral vanishes after summation over all $T \in \tau_h$. If l_2 coincides with any part of Γ, the function v vanishes.

As far as the second integral on l_2 in presentation (16) the following equality holds:

$$\frac{1}{12}\int_{y_T-h}^{y_T+h} F^2(y)\left[\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial x^3}\right]\Bigg|_{x-x_T+y-y_T=0}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]dy$$

$$=\frac{1}{12}\int_{y_T-h}^{y_T+h} F^2(y)\left[\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial x^3}\right]\frac{\partial^2 v}{\partial y^2}\Bigg|_{x=x_T-h}^{x=x_T+y_T-y} dy$$

$$+\frac{1}{12}\int_{x_T-h}^{x_T+h} E^2(x)\left[\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial x^3}\right]\frac{\partial^2 v}{\partial x^2}\Bigg|_{y=y_T-h}^{y=x_T+y_T-x} dx$$

$$+\frac{1}{12}\int_{y_T-h}^{y_T+h} F^2(y)\left[\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial x^3}\right]\frac{\partial^2 v}{\partial y^2}\Bigg|_{x=x_T-h} dy$$

$$+\frac{1}{12}\int_{x_T-h}^{x_T+h} E^2(x)\left[\frac{\partial^3 U}{\partial y^3}+\frac{\partial^3 U}{\partial x^3}\right]\frac{\partial^2 v}{\partial x^2}\Bigg|_{y=y_T-h} dx.$$

The last two term are integrals on the edges l_3 and l_1, respectively. They are cancelled after summation over the legs of all elements $T \in \tau_h$. If any leg is a part of Γ, the $v = 0$ and its tangential derivatives vanish, too.

Hence, from the presentation above, using (15) and (16) it follows:

$$\int_T (y-y_T)\frac{\partial U}{\partial x}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy+\int_T (x-x_T)\frac{\partial U}{\partial y}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy$$

$$\begin{aligned}&=-\frac{1}{6}\int_T F^2(y)\frac{\partial^4 U}{\partial x\partial y^3}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy-\frac{1}{6}\int_T E^2(x)\frac{\partial^4 U}{\partial x^3\partial y}\frac{\partial^2 v}{\partial x\partial y}\,dx\,dy\\&+\frac{1}{12}\int_T F^2(y)\left[\frac{\partial^4 U}{\partial x\partial y^3}+\frac{\partial^4 U}{\partial x^4}\right]\frac{\partial^2 v}{\partial y^2}\,dx\,dy\\&+\frac{1}{12}\int_T E^2(x)\left[\frac{\partial^4 U}{\partial y^4}+\frac{\partial^4 U}{\partial x^3\partial y}\right]\frac{\partial^2 v}{\partial x^2}\,dx\,dy.\end{aligned}\tag{17}$$

Finally, taking into account that $\dfrac{\partial^{i+j}U}{\partial x^i \partial y^j} = -\dfrac{\partial^{i+j}u}{\partial x^i \partial y^j}$ for $i+j \geq 3$ and using the inverse inequality, from (7), (8), (9), (11), (12) and (17) we obtain

$$\int_T \nabla U \cdot \nabla v \, dx \, dy \leq Ch^3 \|u\|_{4,T} \|v\|_{1,T},$$

and consequently

$$a(i_h u - u, v) \leq Ch^3 \|u\|_{4,\Omega} \|v\|_{1,\Omega}.$$

■

As a consequence of Theorem it is easy to prove the main result of this section:

Theorem 2. *Under the assumptions of Theorem the following superclose estimation holds:*

(18) $$\|i_h u - P_h u\|_{1,\Omega} \leq Ch^3 \|u\|_{4,\Omega}.$$

Proof. It follows that ($\alpha = const > 0$):

$$\begin{aligned} \alpha \|i_h u - P_h u\|^2_{1,\Omega} &\leq a(i_h u - P_h u, i_h u - P_h u) && (V_h - \text{ellipticity}) \\ &= a(i_h u - u, i_h u - P_h u) && (P_h - \text{elliptic projector}) \\ &\leq Ch^3 \|u\|_{4,\Omega} \|i_h u - P_h u\|_{1,\Omega}. && (\text{from (6)}) \end{aligned}$$

■

4. High Interpolation Operator $\mathbf{I_{2h}}$ – Construction and Estimations

In this section we present a patch-recovery construction for regular pattern. This construction is related to the interpolated finite element approach.

Consider an element patch on τ_h containing four congruent right-angled isosceles triangles (see Figure 3).

So, the patch recovery finite element strategy is fulfilled by coupling every time four triangles as it is designed in Figure 3. This finite element partition is denoted by $\widetilde{\tau}_{2h}$. Consequently, the mesh $\widetilde{\tau}_{2h}$ of size $2h$ is obtained as a result of arainging in groups of adjacent elements $T_s \in \tau_h$, $s = 1, \ldots, 4$. Thus, the fifteen degrees of freedom are: *(i)* the values of the function v on the vertices of the

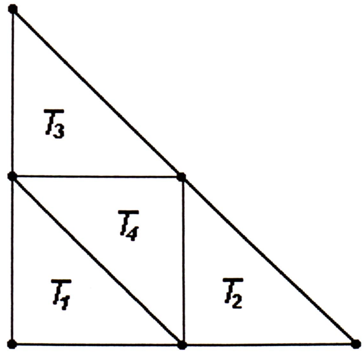

Figure 3:

four subtriangles $T_s \in \tau_h,\ s = 1, \dots, 4$ such that $\bigcup_{s=1}^4 T_s \in \widetilde{\tau}_{2h}$; *(ii)* the integral values of v on the edges l_j of $T_s \in \tau_h,\ s = 1, \dots, 4$.

For instance, it is possible to present the following basic functions defined on a reference element, i.e. for the element determined by $0 \le \hat{x} \le 2$ and $0 \le \hat{y} \le 2 - \hat{x}$:

$$\psi_1(\hat{x},\hat{y}) = \frac{1}{4}\Big(4 - 24\hat{x} - 24\hat{y} + 78\hat{x}\hat{y} + 39\hat{x}^2 + 39\hat{y}^2 - 72\hat{x}^2\hat{y} - 72\hat{x}\hat{y}^2$$
$$-24\hat{x}^3 - 24\hat{y}^3 + 20\hat{x}^3\hat{y} + 20\hat{x}\hat{y}^3 + 30\hat{x}^2\hat{y}^2 + 5\hat{x}^4 + 5\hat{y}^4\Big)\,;$$

$$\psi_2(\hat{x},\hat{y}) = \frac{1}{4}\left(-4\hat{x} + 15\hat{x}^2 - 16\hat{x}^3 + 5\hat{x}^4\right);$$

$$\psi_3(\hat{x},\hat{y}) = \frac{1}{4}\left(-4\hat{y} + 15\hat{y}^2 - 16\hat{y}^3 + 5\hat{y}^4\right);$$

$$\psi_4(\hat{x},\hat{y}) = 6\hat{x}\hat{y} - 9\hat{x}^2\hat{y} - 9\hat{x}\hat{y}^2 + 2\hat{x}^3\hat{y} + 2\hat{x}\hat{y}^3 + 9\hat{x}^2\hat{y}^2;$$

$$\psi_5(\hat{x},\hat{y}) = -8\hat{y} + 6\hat{x}\hat{y} + 24\hat{y}^2 + 3\hat{x}^2\hat{y} - 21\hat{x}\hat{y}^2 - 20\hat{y}^3 - 2\hat{x}^3\hat{y} + 10\hat{x}\hat{y}^3$$
$$+3\hat{x}^2\hat{y}^2 + 5\hat{y}^4;$$

$\psi_6(\hat{x},\hat{y}) = -8\hat{x} + 6\hat{x}\hat{y} + 24\hat{x}^2 - 21\hat{x}^2\hat{y} + 3\hat{x}\hat{y}^2 - 20\hat{x}^3 + 10\hat{x}^3\hat{y} - 2\hat{x}\hat{y}^3$
$+3\hat{x}^2\hat{y}^2 + 5\hat{x}^4;$

$\psi_7(\hat{x},\hat{y}) = \frac{1}{4}\Big(46\hat{x} - 99\hat{x}\hat{y} - 99\hat{x}^2 + 138\hat{x}^2\hat{y} + 66\hat{x}\hat{y}^2 + 68\hat{x}^3 - 46\hat{x}^3\hat{y}$
$-14\hat{x}\hat{y}^3 - 4\hat{5}x^2\hat{y}^2 - 15\hat{x}^4\Big)\,;$

$\psi_8(\hat{x},\hat{y}) = \frac{1}{4}\left(14\hat{x} - 51\hat{x}^2 + 52\hat{x}^3 - 15\hat{x}^4 - 3\hat{x}\hat{y} + 18\hat{x}^2\hat{y} - 14\hat{x}^3\hat{y}\right.$
$\left.-6\hat{x}\hat{y}^2 + 3\hat{x}^2\hat{y}^2 + 2\hat{x}\hat{y}^3\right)\,;$

$\psi_9(\hat{x},\hat{y}) = \frac{1}{4}\left(3\hat{x}\hat{y} - 18\hat{x}^2\hat{y} + 14\hat{x}^3\hat{y} + 6\hat{x}\hat{y}^2 - 3\hat{x}^2\hat{y}^2 - 2\hat{x}\hat{y}^3\right);$

$\psi_{10}(\hat{x},\hat{y}) = \frac{1}{4}\left(3\hat{x}\hat{y} + 6\hat{x}^2\hat{y} - 2\hat{x}^3\hat{y} - 18\hat{x}\hat{y}^2 - 3\hat{x}^2\hat{y}^2 + 14\hat{x}\hat{y}^3\right);$

$\psi_{11}(\hat{x},\hat{y}) = \frac{1}{4}\left(14\hat{y} - 3\hat{x}\hat{y} - 6\hat{x}^2\hat{y} + 2\hat{x}^3\hat{y} - 51\hat{y}^2 + 18\hat{x}\hat{y}^2 + 3\hat{x}^2\hat{y}^2 + 52\hat{y}^3\right.$
$\left.-14\hat{x}\hat{y}^3 - 15\hat{y}^4\right)\,;$

$\psi_{12}(\hat{x},\hat{y}) = \frac{1}{4}\left(46\hat{y} - 99\hat{x}\hat{y} + 66\hat{x}^2\hat{y} - 14\hat{x}^3\hat{y} - 99\hat{y}^2 + 138\hat{x}\hat{y}^2 - 45\hat{x}^2\hat{y}^2\right.$
$\left.+68\hat{y}^3 - 46\hat{x}\hat{y}^3 - 15\hat{y}^4\right)\,;$

$\psi_{13}(\hat{x},\hat{y}) = 6\left(6\hat{x}\hat{y} - 7\hat{x}^2\hat{y} + 2\hat{x}^3\hat{y} - 7\hat{x}\hat{y}^2 + 4\hat{x}^2\hat{y}^2 + 2\hat{x}\hat{y}^3\right)\,;$

$\psi_{14}(\hat{x},\hat{y}) = -6\left(2\hat{x}\hat{y} - 5\hat{x}^2\hat{y} + 2\hat{x}^3\hat{y} - \hat{x}\hat{y}^2 + 2\hat{x}^2\hat{y}^2\right)\,;$

$\psi_{15}(\hat{x},\hat{y}) = -6\left(2\hat{x}\hat{y} - \hat{x}^2\hat{y} - 5\hat{x}\hat{y}^2 + 2\hat{x}^2\hat{y}^2 + 2\hat{x}\hat{y}^3\right)\,.$

The interpolation operator I_{2h} verifies vertex-edge conditions determined by the degree of freedom and the basic functions.

Let $\widetilde{V}_{2h} \subset V$ be finite element spaces associated with $\widetilde{\tau}_{2h}$. Then $\widetilde{V}_{2h}$ consists of piecewise polynomials from P_4.

I_{2h} is constructed in such a way that the following properties are valid:

(19) $$I_{2h} \circ i_h = I_{2h},$$

(20) $$\|I_{2h}v\|_{r,\Omega} \le C\|v\|_{r,\Omega} \quad \forall v \in V_h, \quad r = 0,1,$$

because the interpolation operator $I_{2h}:\ V_h \to \widetilde{V}_{2h}$ is bounded.

Finally, taking into account that the interpolation polynomial $I_{2h}v$ is of degree 4, it follows

$$\|I_{2h}v - v\|_{1,\Omega} \le Ch^4 \|v\|_{5,\Omega}. \tag{21}$$

The main estimation is contained in the next theorem: **Theorem 3.** *Let $u \in H^5(\Omega) \cap H_0^1(\Omega)$. Then the following estimate holds:*

$$\|I_{2h} \circ P_h u - u\|_{1,\Omega} \le Ch^3 \|u\|_{5,\Omega}. \tag{22}$$

Proof. Applying (19), we decompose

$$I_{2h} \circ P_h u - u = I_{2h} \circ (P_h u - i_h u) + (I_{2h} u - u)\,.$$

Inequality (20) leads to

$$\|I_{2h} \circ P_h u - u\|_{1,\Omega} \le \|I_{2h}\|\ \|P_h u - u\|_{1,\Omega} + \|I_{2h} u - u\|_{1,\Omega}.$$

The estimates (18) and (21) complete the proof.

■

The main estimation (22) is of type superconvergence. It is obtained using small additional computation.

5. Superconvergence Eigenvalue Problem

Here we present a direct application of the result obtained in the previous section. Consider the following model eigenvalue problem: find $\lambda \in \mathbf{R}$ and $u(x,y) \in V$, $u \ne 0$ such that

$$a(u,v) = \lambda(u,v), \quad \forall v \in V. \tag{23}$$

Let us introduce the finite element approximate eigenvalue problem corresponding to (23): find $\lambda_h \in \mathbf{R}$, $u_h(x,y) \in V_h$, $u_h \ne 0$ such that

$$a(u_h,v) = \lambda_h(u_h,v), \quad \forall v \in V_h. \tag{24}$$

Our assumption here is that FE space V_h uses polynomials of degree two and the partition τ_h consists of triangular finite elements. Such being the case, it is well known (see [11, 12]) that the rate of convergence of FE approximation to the eigenvalues and eigenfunctions is given by the following estimations:

$$|\lambda - \lambda_h| \le C(\lambda) h^4 \|u\|^2_{3,\Omega}, \tag{25}$$

(26) $$|u - u_h| \le C(\lambda)h^2\|u\|_{3,\Omega}.$$

The solutions of (23) and (24) are related to the Rayleigh quotient

$$\lambda = \frac{a(u,u)}{(u,u)} \quad \text{and} \quad \lambda_h = \frac{a(u_h,u_h)}{(u_h,u_h)}.$$

First, we estimate the difference $u_h - P_h u$ in H^1−norm of higher order of accuracy as compared to (3) and (26). This special feature of both the functions is called to satisfy a superclose property (cf. [4]). The following lemma is an important connection between the FE eigenvalue approximation and the patch recovery technique:

Lemma 1. *Let the eigenfunction $u(x,y)$ belongs to $H^3(\Omega) \cap V$ and let u_h be the corresponding FE approximation obtained by (24) and using quadratic triangular elements. Then*

(27) $$\|u_h - P_h u\|_{1,\Omega} \le Ch^4\|u\|_{3,\Omega}.$$

Proof. From the ellipticity on the FE space it follows ($\rho = const$):

$$\rho\|u_h - P_h u\|^2_{1,\Omega} \le a(u_h - P_h u, u_h - P_h u).$$

Let us denote $u_h - P_h u = z_h \in V_h$. Using the orthogonality of P_h we obtain:

$$\begin{aligned}
\rho\|z_h\|^2_{1,\Omega} &\le \lambda_h(u_h, z_h) - a(P_h u, z_h)\\
&= (\lambda_h - \lambda)(u_h, z_h) + \lambda(u_h, z_h) - a(u, z_h)\\
&= (\lambda_h - \lambda)(u_h, z_h) + \lambda(u_h - u, z_h) + \lambda(u, z_h) - a(u, z_h)\\
&= (\lambda_h - \lambda)(u_h, z_h) + \lambda(u_h - u, z_h)\\
&\le |\lambda - \lambda_h|\|u_h\|_{0,\Omega}\|z_h\|_{1,\Omega} + \lambda\|u_h - u\|_{-1,\Omega}\|z_h\|_{1,\Omega}.
\end{aligned}$$

In the last step we used the duality in negative norms. The estimate (27) follows from (25) and the inequality in negative norm [13]

$$\|u_h - u\|_{-1,\Omega} \le Ch^4\|u\|_{3,\Omega}.$$

■

In order to apply the patch recovery superconvergence procedure to the eigenvalue problem we need an another lemma:

Lemma 2. *Let (λ, u) be any exact solution obtained by (23). Then for every $w \in V$ and $w \neq 0$, the following inequality holds:*

$$\left|\frac{a(w,w)}{(w,w)} - \lambda\right| \leq C\frac{\|w-u\|_{1,\Omega}^2}{(w,w)}. \tag{28}$$

The main result of this section is contained in the next theorem:

Theorem 4. *Let (λ, u) be an exact eigenpair and (λ_h, u_h) be its finite element approximation using triangular quadratic elements. Assume that the conditions of Theorem are fulfilled. Then*

$$\|I_{2h}u_h - u\|_{1,\Omega} \leq Ch^3\|u\|_{5,\Omega}, \tag{29}$$

$$\left|\frac{a(I_{2h}u_h, I_{2h}u_h)}{(I_{2h}u_h, I_{2h}u_h)} - \lambda\right| \leq Ch^6\|u\|_{5,\Omega}^2. \tag{30}$$

Proof. First we consider the patch recovery superconvergence for the eigen- functions. From Theorem and Lemma it follows:

$$\begin{aligned}\|I_{2h}u_h - u\|_{1,\Omega} &\leq \|I_{2h}u_h - I_{2h}\circ P_h u\|_{1,\Omega} + \|I_{2h}\circ P_h u - u\|_{1,\Omega}\\ &\leq \|I_{2h}\|\|u_h - P_h u\|_{1,\Omega} + \|I_{2h}\circ P_h u - u\|_{1,\Omega}.\\ &\leq C_1h^4\|u\|_{3,\Omega} + C_2h^3\|u\|_{5,\Omega}.\end{aligned}$$

Using the inequalities (28) and (29) we easily get

$$\begin{aligned}\left|\frac{a(I_{2h}u_h, I_{2h}u_h)}{(I_{2h}u_h, I_{2h}u_h)} - \lambda\right| &\leq C\frac{\|I_{2h}u_h - u\|_{1,\Omega}^2}{\|I_{2h}u_h\|_{0,\Omega}^2}\\ &\leq Ch^6\|u\|_{5,\Omega}^2.\end{aligned}$$

■

6. Numerical Results

The presented theory is illustrated by the example given below. Consider the problem

Table 1:

N_e		$\|\lambda_1 - \lambda_{h,1}\|$	$\|\lambda_2 - \lambda_{h,2}\|$	$\|\lambda_3 - \lambda_{h,3}\|$	$\|\lambda_4 - \lambda_{h,4}\|$
32	FE	6.6×10^{-2}	0.53	1.04	3.21
32	PPR	5.7×10^{-3}	8×10^{-2}	0.23	1.21
128	FE	4.4×10^{-3}	4×10^{-2}	7.4×10^{-2}	0.28
128	PPR	9.4×10^{-5}	1.5×10^{-3}	3.3×10^{-3}	4.7×10^{-2}
512	FE	2.8×10^{-4}	2.6×10^{-3}	4.8×10^{-3}	2.8×10^{-2}
512	PPR	1.5×10^{-6}	2.6×10^{-5}	9.2×10^{-5}	9×10^{-3}

$$-\Delta u = \lambda u, \quad x \in \Omega,$$

$$u_{|\Gamma} = 0,$$

where $\Omega = (0,l) \times (0,l)$ and $\Gamma = \partial\Omega$.

The first exact eigenpairs of this problem are:

$$\lambda_1 = \frac{2\pi^2}{l^2}, \quad u_1(x,y) = C \sin\frac{\pi}{l}x \sin\frac{\pi}{l}y;$$

$$\lambda_2 = \frac{5\pi^2}{l^2}, \quad u_2(x,y) = C \sin\frac{\pi}{l}x \sin\frac{2\pi}{l}y;$$

$$\lambda_3 = \frac{5\pi^2}{l^2}, \quad u_3(x,y) = C \sin\frac{2\pi}{l}x \sin\frac{\pi}{l}y;$$

$$\lambda_4 = \frac{8\pi^2}{l^2}, \quad u_4(x,y) = C \sin\frac{2\pi}{l}x \sin\frac{2\pi}{l}y.$$

Table 1 gives the results when $l = 1$ for quadratic meshes (i.e. $n = 2$ above) of N_e identical 6-nodes triangular elements ($N_e = 32; 128; 512$). We present the finite element (FE) approximation $\lambda_{h,j}$ of the eigenvalues

$$\lambda_1 = 19.7392088022, \lambda_{2,3} = 49.3480220054, \lambda_4 = 78.9568352087,$$

and their improvements by postprocessing patch recoveries (PPR). The values of PPR have been obtained by using interpolated finite elements unifying four elements in the patch. Let us note that it is reasonable to use a postprocessing patch recovery technique on a coarse mesh.

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