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More Properties of A-H Banach Spaces

Parviz Azimi

We continue the study of geometric properties of classes of Banach spaces, constructed by Hagler and the author in particular, we show that (i) every infinite dimensional subspace of a member X of the class $X_{\alpha,1}$ spaces contains asymptotically isometric complemented copies of l_1. Hence the dual X^*, of X contains subspaces isometrically isomorphic to $C[0,1]^*$. (ii) Every member of the class of $X_{\alpha,p}$ $(1 \leq p < \infty)$ fails the Dunford-Pettis property. (iii) We observe that all $X_{\alpha,p}$ spaces are Banach spaces without unconditional basis but all constructed spaces contain a subspace which is weakly sequentially complete with an unconditional basis which is weakly null sequence but not in norm.

1. Introduction

In [2], a class of hereditarily l_1 Banach sequence spaces was studied. Here we continue the study of these spaces and deduce further structural properties of a member X of this class. In particular, we investigate the recent result of J. Dilworth, M. Giradi and J. Hagler [4] for X. In [2] it was shown that X is hereditarily l_1 dual space, Hagler in an unpublished result showed that X is hereditarily, complementably l_1 and its predual contains many subspaces isomorphic to c_0. From theorem 2 of [4] we can deduce many other interesting properties of a member X of A-H Banach spaces. In [4], it is shown that X^* contains an isometric copy of L_1 if and only if X contains asymptotically isometric copies of l_1. we show here that any member X of A-H Banach spaces contains asymptotically isometric copies of l_1. and therefore X^* contains isometric copies of L_1 in particular, by theorem 2 of [4], X^* contains isometric copy of $C[0,1]^*$. We observe that for $p > 1$, $X_{\alpha,p}$ contains reflexive subspaces which are weakly sequentially complete with unconditional basis. Now we go through the construction of a member X of A-H Banach spaces.

A block F is an interval(finite or infinite) of integers. For any block F, and $x = (t_1, t_2, ...)$ a finitely non-zero sequence of scalar, we let $< x, F >= \sum_{j \in F} t_j$. A sequence of blocks $F_1, F_2, ...$ is admissible if $\max F_i < \min F_{i+1}$ for each i. Finally, let $1 = \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq ...$ be a sequence of real numbers with $\lim_{i \to \infty} \alpha_i = 0$ and $\sum_{i=1}^{\infty} \alpha_i = \infty$.

We now define a norm which uses the α_i,s and admissible sequence of blocks in its definition. Let $1 \leq p < \infty$ and $x = (t_1, t_2, ...)$ be finitely non-zero sequence of reals. Define

$$\| x \| = \max \left[\sum_{i=1}^{n} \alpha_i \mid < x, F_i > \mid^p\right]^{\frac{1}{p}}$$

where the max is taken over all n, and admissible sequences F_1, $F_2, ...$. The Banach space X is the completion of the finitely non-zero sequences of scalars in this norm.

The concept of asymptotically isometric copies of l_1 was introduced first by Hagler, and recently is used in fixed point theory and in [4]. Here is the precies definition.

2. Definitions and notation

Definitions and notation are standard, but we give some of these here.

The dual space of X is denoted by X^*. A subspace Y of X is complemented in X if there is a projection $P : X \to X$ such that $P(X) = Y$ and $\| P \| < \infty$.

Let l_1 be the space of absolutely summable sequences and L_1 the space of Lebesgue-integrable functions on $[0, 1]$. c_0 is the space of all null sequences $x = (t_1, t_2, ...)$ with $\| x \| = \max_n \mid t_n \mid$.

A Banach space X is called hereditarily l_1 if every infinite dimensional subspace of X contains a subspace isomorphic to l_1.

Definition 2.1. Let X be a Banach space. We say that X contains asymptotically isometric copies of l_1 if for some sequence $\lambda_0 < \lambda_1 < ...$ with $\lim_n \lambda_n = 1$, there is sequence (x_n) in X such that for all m and scalars $(t_n : 0 \leq n \leq m)$

$$\sum_{n=0}^{m} \lambda_n \mid t_n \mid \leq \| \sum_{n=0}^{m} t_n x_n \| \leq \sum_{n=0}^{m} \mid t_n \mid .$$

Definition 2.2. A Banach space X has the Dunford-Pettis property (DPP) if given weakly null sequences (x_n) and (x_n^*) in X and X^*, respectively, then $\lim_n x_n^* (x_n) = 0$.

3. The results

The key to the analysis of the space X is via the following result(lemma 4 of [2]).

Lemma 3.1. *Let the sequence (α_i) be as above. For the given integer N and given $\varepsilon > 0$, there exist a $\delta > 0$ such that, if $b_1, b_2, ..., b_n$ are ≥ 0, $b_i < \delta$ for all i, and $\sum_{i=1}^{n} \alpha_i b_i = 1$, then $\sum_{i=1}^{n} \alpha_{i+N} b_i \geq 1 - \varepsilon$*

Theorem 3.2. *The Banach space $X_{\alpha,1}$ contains asymptotically isometric copies of l_1.*

Proof. Let (u_i) be a sequence of norm one vectors in $X_{\alpha,1}$ and (G_i) an admissible sequence of blocks such that $\{j : u_i(j) \neq 0\} \subset G_i$. For each i, put $s_i = s(u_i)$ where $s(u_i) = \max_G |< u_i, G >|$. If $\lim_{i\to\infty} s_i = 0$, then a subsequence (v_j) of (u_j) satisfies

$$\| \sum_{j=1}^{n} t_j v_j \| \geq \sum_{j=1}^{n} (1 - \varepsilon_j) \mid t_j \mid$$

where (ε_j) is a decreasing sequence, $\varepsilon_i < 1$ and (t_j) is a sequence of scalars.

We select (v_j) by induction. Let $v_1 = u_1$. Pick n_1 and $F_1, F_2, ..., F_{n_1}$ satisfying $\max F_{n_1} = \max G_1$ and $\sum_{i=1}^{n_1} \alpha_i |< v_1, F_i >| = \| v_1 \| = 1$. Let δ_1 be any δ guaranteed by lemma 3.1 for integer n_1 and $\varepsilon_1 (< 1)$. We let $n_0 = 0$. Assume now that we have selected for $k = 1, ..., p-1$

(1) an integer $m_k (> m_{k-1})$ so that $v_k = u_{m_k}$.

(2) an integer $n_k (> n_{k-1})$, blocks $F_{n_{k-1}+1}, ..., F_{n_k}$ and $\delta_k > 0$ such that

(a) $\max F_{n_k} = \max G_{m_k}$

(b) the sequence $F_1, F_2, ..., F_{n_1}, ..., F_{n_2}, ..., F_{n_k}$ is admissible.

(c) $\sum_{i=1}^{n_k - n_{k-1}} \alpha_i |< v_k, F_i >| = \| v_k \| = 1$

(d) δ_k is any δ guaranteed by lemma 2.1 for integer n_{k-1} and $\varepsilon_k (< \varepsilon_{k-1})$.

Now let $\delta_p > 0$ be any δ guaranteed by lemma 2.1 for integer n_{p-1} and $\varepsilon_p (< \varepsilon_{p-1})$. Pick $m_p (> m_{p-1})$ so that $s_{m_p} < \delta_p$ and $v_p = u_{m_p}$. Finally, pick blocks $F_{n_{p-1}}, ..., F_{n_p}$ such that (a), (b) and (c) above are satisfied for v_p and G_{m_p}. This completes the induction process.

Observe that $|< v_k, F_{i+n_{k-1}} >| < s_{n_k} < \delta_k$ for $i = 1, ..., n_k - n_{k-1}$. By lemma 3.1

$$\sum_{i=1}^{n_k - n_{k-1}} \alpha_{i+n_{k-1}} |< v_k, F_{i+n_{k-1}} >| > 1 - \varepsilon_k.$$

This inequality can be rewritten as

$$\sum_{i=n_{k-1}+1}^{n_k} \alpha_i \mid < v_k, F_i > \mid > 1 - \varepsilon_k.$$

Now, let scalars $t_1, t_2, ..., t_k$ be given. Since the sequence $F_1, ..., F_{n_k}$ is admissible, it follows from the observation above that

$$\begin{aligned}
\| \sum_{j=1}^{n} t_j v_j \| &\geq \sum_{i=1}^{n_k} \alpha_i \mid < \sum_{j=1}^{n} t_j v_j, F_i > \mid \\
&= \sum_{j=1}^{n} \mid t_j \mid \left(\sum_{i=1}^{n_k} \alpha_i \mid < v_j, F_i > \right) \\
&= \sum_{j=1}^{n} \mid t_j \mid \left(\sum_{i=n_{j-1}+1}^{n_j} \alpha_i \mid < v_j, F_i > \right) \geq \sum_{j=1}^{n} (1 - \varepsilon_j) \mid t_j \mid .
\end{aligned}$$

To complete the proof we need establish the result for norm norm one vectors (u_i) and blocks (G_i) with $\max G_i < \min G_{i+1}$ such that $\{j : u_i (j) \neq 0\} \subset G_i$ if some subsequence of $(s_i) \to 0$, then we are done . If not we use the argument similar to the proof of theorem 1(1) of [2]. ∎

Theorem 3.2 and theorem 2 of [4] have the following corollary.

Corollary 3.3. *(i) The dual $X^*_{\alpha,1}$ of $X_{\alpha,1}$ contains subspaces isometrically isomorphic to $C[0,1]^*$, (ii) $C(\Delta)$ is isometric to a quotient space of $X_{\alpha,1}$ where Δ is the Cantor set and (iii) L_1 is linearly isometric to a subspace of $X^*_{\alpha,1}$.*

The following theorem shows that the predual of $X_{\alpha,1}$ contains asymptotically isometrically copies of c_0. The proof is via lemma 3.1.

Theorem 3.4. *The Banach spaces $X_{\alpha,p}$ $(1 \leq p < \infty)$ fail the DPP.*

Proof. Let $u_i = e_{2i} - e_{2i-1}$ and $f_i : X_{\alpha,p} \to R$ such that for any $x = (t_1, t_2, ...) \in X_{\alpha,p}$, we have $f_i (x) = t_i$. for integers i. Then for $g_n = f_{2n} - f_{2n-1}$, we have $g_n (u_n) = 2$. To complete the proof we need to show that $u_n \to 0$ weakly, and $g_n \to 0$ weakly. The first one follows from the fact that, for every increasing sequence (n_k) of integers, we have

$$\lim_{k\to\infty} \frac{\| u_{n_1} + u_{n_2} + ... + u_k \|}{k} = \frac{(\sum_{i=1}^{2k} \alpha_i)^{\frac{1}{P}}}{k} = 0$$

It remains to show that $g_n \to 0$ weakly. If not there are $F \in X_{\alpha,p}^{**}$ with $\| F \|= 1$, $\delta > 0$ and a subsequence (g_{n_k}) such that $F(g_{n_k}) > \delta$ for all integers k. So for integer N, we have $\sum_{k=1}^{N} F(g_{n_k}) > N\delta$ and hence

$$\frac{\| \sum_{k=1}^{N} g_{n_k} \|}{N} > \delta.$$

This implies that for any integer N, there exist $x = (t_1, t_2, ...) \in X_{\alpha,p}$ such that

$$\frac{1}{N} \sum_{k=1}^{N} g_{n_k}(x) > \delta.$$

Then $\lim_{n\to\infty} t_n = 0$ for integers N and corresponding $x = (t_1, t_2, ...)$, since $\sum_{i=1}^{\infty} \alpha_i = \infty$. Therefore,

$$\begin{aligned} \left| \frac{1}{N} \sum_{k=1}^{N} g_{n_k}(x) \right| &= \frac{1}{N} \left| \sum_{k=1}^{N} (t_{2n_k} - t_{2n_k-1}) \right| \\ &\leq \frac{1}{N} \sum_{k=1}^{N} | t_{2n_k} | + \frac{1}{N} \sum_{k=1}^{N} | t_{2n_k-1} | \to 0 \end{aligned}$$

as $N \to \infty$ which is a contradiction. ∎

R e m a r k . It is known that if X^* has the DPP, then so does X. This implies $X_{\alpha,p}^*$ also fails the DPP.

It is known that if an infinite-dimensional Banach space has no normalized weakly null sequence then it contains infinite unconditional basic sequence, in fact it contains a subsequence isomorphic to l_1.In $[A]$ we proved that $X_{\alpha,p}$ is a class of hereditarily complementably l_p Banach spaces. Here is some other properties of these spaces.

Theorem 3.5. *(i) For $p > 1$ the Banach space $X_{\alpha,p}$ is without unconditional basis. (ii) Let $u_i = e_{2i} - e_{2i-1}$ $(i \in N)$ and Y be the closed subspace of an specific $X_{\alpha,p}$ generated by u_i, i.e. $Y = [u_i]$. Then the sequence (u_i) is an unconditional basis of Y. (iii) Y is weakly sequentially complete and (iv) $u_i \to 0$ weakly, but not in norm.*

P r o o f. Part (i) follows from the fact that for $p > 1$, $X_{\alpha,p}$ does not contain l_1 and is not reflexive(theorem 1.c.12[LT])

Part(ii) is a consequence of the fact that for any sequence (t_i), and any j, we have $\| \sum_{i\neq j} t_i u_i \| \leq \| \sum_i t_i u_i \|$, See[6](proposition 1.c.6.page 18).

For part(iii), since (u_i) is unconditional basis for $[u_i]$ and since $[u_i]$ does not contain a copy of c_0, it follows from [3](theorem 2 page 74) that $[u_i]$ is weakly sequentially complete.

Part(iv) follows from theorem 3.4 and the fact that $\| u_i \|= (1+\alpha_2)^{\frac{1}{p}}$. ■

R e m a r k . A result of James [5] asserts that a Banach space with and unconditional basis is either reflexive or has a subspace isomorphic to c_0 or l_1. This implies that the Banach spaces $y = [u_i]$ for $p > 1$ is reflexive.

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Department of Mathematics
University of Sistan and Baluchestan,
Zahedan, Iran
E-mail: azimi@hamoon.usb.ac.ir

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