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Uniform Ends

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We study the notion of uniform end for non – compact metric space and the related notion of ends. We show the main properties of set and space of uniform ends.

Key Words: end, uniform end, uniform local connectedness, uniformly simple

The notion of end plays a significant role in the study of non-compact spaces. For the first time this notion appeared in the papers of Freudenthal at the end of the first half of twentieth century. Here we will investigate the properties of the set and space of uniform ends, giving two different but equivalent definitions of this notion.

If $X$is connected, the set and the space of uniform ends can be defined by the use of inverse limits. This approach has advantages based on the wide application of inverse limits in topology.

We will consider only metric spaces.

Let A and B be non-empty subsets of a metric space X.

Definition 1. A and B are *uniformly separated* in X if there exists a uniformly continuous function $f : X \to [0, 1]$ *such that* $f(A) = 0$ and $f(B) = 1$.

Definition 2. Let$X$ be a metric space and $Y \subseteq X$. Y is *weakly connected*, if Y is not a union of two uniformly separated subsets.

Immediately, from the definitions we have the it following properties:

1. If A and B are uniformly separated subsets in X then A and B are separated in X.

2. If X is connected, then X is weakly connected.

3. Let A and B be uniformly separated nonempty subsets of X, $X = A \cup B$, and let C be weakly connected in X. Then $C \subseteq A$ or $C \subseteq B$.

4. Under the same condition, if C is connected in X, then $C \subseteq A$ or $C \subseteq B$.

The following propositions and the definition are analog with the theorems for usual connectedness.

Proposition 1. *If X is weakly connected and $f : X \to Y$ is a uniformly continuous function, then $f(X)$ is weakly connected.*

Definition 3. A *component of weak connectedness* $C_w(x)$ of a point x is the biggest weakly connected set containing x.

The proof of propositions 2 and 3 is given in [6].

Proposition 2. *Each component of connectedness of X is contained in a component of weak connectedness. A component of weak connectedness is an union of components of connectedness.*

Corollary 1. *If X is locally connected, then the components of weak connectedness are open sets.*

Definition 4. [8] The metric space X is *uniformly locally connected* , if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any two points x,y satisfying $d(x, y) < \delta$ are contained in a connected set with a diameter less than ε.

Proposition 3. *If X is uniformly locally connected, then X is locally connected.*

Corollary 2. *If X is a uniformly locally connected metric space, then he components of weak connectedness are open sets.*

Now, we will start with definitions of notions of end and uniform end for metric spaces. Let X be a connected, locally connected, locally compact metric space. To define the notion of end, we will need the some facts. The following theorem is known. The proof can be found in [5].

Theorem 1. *Let K be a compact subset of X and V is a neighbourhood of K. Then all but finally many components of $X \backslash K$ are in V.*

Definition 5. $A \subseteq X$ is *essential* in$X$ if A is not contained in any compact of X.

Theorem 2. *Let C be a compact of X. There exists the compact D such that $C \subseteq D$ and $X \backslash D$ has:*

1. only a final number of components;

2. all components are essential.

¿From the proposition 2, it follows that there exists the compact D such that $C \subseteq D$ and $X \backslash D$ has:

1. only a final number of components of weak connectedness;
2. all components of weak connectedness are essential.

Let C be a compact of X. Let $S(X \backslash C)$ denote the set of components of connectedness of $X \backslash C$.

The set of ends is

$$E(X) = \varprojlim_{C} S(X \backslash C),$$

where the inverse limit is taken by all compact subsets of X.There is a cofinal sequence of compacta in X, $C_1 \subseteq C_2 \subseteq ... \subseteq C_n \subseteq$ ...i.e. for any compact set $C, C \subseteq X$, there exists a compact set C_n, such that $C \subseteq C_n$. It is a known fact from the theory of inverse limits that in this case:

$$\varprojlim_{C} S(X \backslash C) = \varprojlim_{n} S(X \backslash C_n).$$

So, the set of ends of X consists of all sequences of components of connectedness

$$(K_1, K_2, ..., K_n, ...) \in \prod_{n=}^{\infty} S(X \backslash C_n)$$

such that $K_1 \supseteq K_2 \supseteq ... \supseteq K_n \supseteq ...$

The sequence $C_1 \subseteq C_2 \subseteq ... \subseteq C_n \subseteq ...$ can be chosen in a such way that the sets $S(X \backslash C_n)$ to be finite and to contain only essential components of connectedness. Now, by analogy, we will introduce the notion of uniform end. Let $S_w(X \backslash C)$ denote the set of components of weak connectedness of $X \backslash C$. We define the set of *uniform ends* with:

$$UE\ (X) = \varprojlim_{C} S_w(X \backslash C) = \varprojlim_{n} S_w(X \backslash C_n).$$

where the inverse limit is taken by all compact subsets of X.

So, the set of uniform ends of X consists of all sequences of components of weak connectedness

$$(Q_1, Q_2, ..., Q_n, ...) \in \prod_{n=1}^{\infty} S_w(X \backslash C_n)$$

such that $Q_1 \supseteq Q_2 \supseteq ... \supseteq Q_n \supseteq ...$.

By proposition 2, the set $S_w(X\backslash C_n)$ is finite and contains only essential components of weak connectedness.

We put discrete topology on each $S_w(X\backslash C_n)$. Then the set $UE(X)$ becomes a compact topological space as an inverse limit of compact spaces.

The sets (and spaces) $UE(X)$ and $E(X)$ are related in the following way: Having in mind the fact that each component of weak connectedness is a union of components of connectedness and taking a cofinal sequence of compacta in X, $C_1 \subseteq C_2 \subseteq ... \subseteq C_n \subseteq ...$, there exist surjections $w_{c_i}, i \in N$, such that the next diagrams commute:

$$\begin{array}{lcl}
S_w(X\backslash C_1) & \underset{w_{c_1}}{\longleftarrow} & S(X\backslash C_1) \\
\uparrow & & \uparrow \\
S_w(X\backslash C_2) & \underset{w_{c_2}}{\longleftarrow} & S(X\backslash C_2) \\
. & & . \\
. & & . \\
\uparrow & & \uparrow \\
S_w(X\backslash C_n) & \underset{w_{c_n}}{\longleftarrow} & S(X\backslash C_n) \\
. & & . \\
. & & .
\end{array}$$

The surjections $w_{c_i}, i \in N$, induce a map $w : E(X) \to UE(X)$ and w is surjection[3] .

Now, we will give another definition of topological space of uniform ends by use of weakly admissible sequences. Next facts will be needed:

Definition 6. Y *uniformly separates* X if and only if $X\backslash Y$ is not weakly connected.

Definition 7. A sequence (a_n) in X is ***weakly admissible*** if:

(1)no subsequences of (a_n) converge to a point of X;

(2)no compact subset of X separates uniformly (in X) two subsequences of (a_n).

Let $\mathcal{A}_{UX}$ be the set of all weakly admissible sequences in X. We will define a relation of equivalence " $\sim$" in $\mathcal{A}_{UX}$ in this way:

$(a_n) \sim (b_n)$if and only if no compact subset of X, uniformly separates an infinite subsequence of (a_n) from an infinite subsequence of (b_n).

Proposition 4. *The relation " $\sim$" is an equivalence relation on* $\mathcal{A}_{UX}$*.*

P r o o f. Reflexivity and symmetry are obvious. We proof transitivity. Let $(a_n) \sim (b_n)$, $(b_n) \sim (c_n)$ and suppose that (a_n) is not in a relation with (c_n). It means that there exists a compact set C in X such that $X\backslash C = A \cup B$ and A

and B are uniformly separated, i.e. $d(A,B) > 0$ and there exists a subsequence (a'_n) of (a_n) in A and a subsequence (c'_n) of (c_n) in B. Because of $(a_n) \sim (b_n)$, it follows that at most a finite number of members of (b_n) are in B, which is not true because of the condition$(b_n) \sim (c_n)$. So, $(a_n) \sim (c_n)$. ■

For each sequence $(a_n) \in \mathcal{A}_{UX}$, $[(a_n)]$ will denote the equivalence class containing (a_n). We define the set of ***uniform ends*** with:

$$UE(X) = \{[(a_n)] \,|[(a_n)] \in \mathcal{A}_{UX}\}$$

We will give another definition of the topology of the space of uniform ends.

Definition 8. A sequence (a_n) of points of X is *eventually* in $A \subseteq X$ if only a finite number of members of the sequence are outside of A i.e. if there exists $m_0 \in N$,such that for all $m \geq m_0$, $a_m \in A$.

Let $B = \{G \,|G \subseteq X \text{ be open in } X \text{ and } \partial G \text{ is compact}\}$.

For each $G \in B$, let $G^* = \{[(a_m)] \in UE(X) \,|(a_m) \text{ be eventually in } G\}$ and $B^* = \{G^* \,|G \in B\}$.

If $G_1, G_2 \in B$, then:

$$(G_1 \cap G_2)^* = \{[(a_m)] \in UE(X) \,|(a_m) \text{ eventually in } G_1 \cap G_2\} =$$
$$\{[(a_m)] \in UE(X) \,|(a_m) \text{ eventually} \, G_1\} \cap$$
$$\{[(a_m)] \in UE(X) \,|(a_m) \text{ eventually } G_2\} = G_1^* \cap G_2^*.$$

Hence, B^* is a basis for a topology of the set $UE\ (X)$.

Theorem 3. *$UE(X)$ is T_2-space.*

Proof. Let $[(a'_n)]$, $[(a_n")] \in UE(X)$ and $[(a'_n)] \neq [(a_n")]$.It means that $(a'_n) \sim (a_n")$ i.e. there exists compact set C such that $X \backslash C = A \cup B$, where A and B are uniformly separated in $X \backslash C$ i.e. $d(A,\, B) > 0$ and there exists a subsequence (b'_n) of (a'_n) in A and a subsequence $(b_n")$ of $(a_n")$ in B.

Since $X = IntX \cup \partial X \cup ExtX$, and the union is disjoint, and A,B are disjoint open sets, it follows that $\partial A \subseteq C$. So, ∂A is compact.

By analogy , ∂B is compact, so $A, B \in B$ and $(a'_n) \in A^*$, $(a_n") \in B^*$.

Now, we will prove that $A^* \cap B^* = \emptyset$.

Suppose that there exists $[(a_n)] \in A^* \cap B^*$. Then (a_n) is eventually in A and in B which is not possible because $A \cap B = \emptyset$.

Hence, $UE(X)$ is T_2 - space. ■

Theorem 4. *Let X be a connected, locally connected, locally compact metric space. There exists a homeomorphism between the topological space*

$\lim\limits_{\overleftarrow{n}} S_w\left(X\backslash C_n\right)$ *and* $UE\left(X\right)$ *i.e. the above two definitions of the space of uniform ends are equivalent.*

P r o o f. We define a map $f:\lim\limits_{\overleftarrow{n}} S_w\left(X\backslash C_n\right)\rightarrow\left\{\left[\left(a_n\right)\right]\left|a_n\in A_{UX}\right.\right\}$ in this way:

$f\left(S_1,\, S_2,...\right)=\left[\left(a_n\right)\right]$ if $\left(a_n\right)$ is eventually in S_i, for every $i\in$.

We will show that f is homeomorphism. Because the space $\lim\limits_{\overleftarrow{n}} S_w\left(X\backslash C_n\right)$ is compact and $\left\{\left[\left(a_n\right)\right]\left|a_n\in A_{UX}\right.\right\}$ is T_2-space , it is sufficient to show that f is continuous bijection.

1)The map f *is well defined;*

Let $f\left(S_1,\, S_2,...\right)=\left[\left(a_n\right)\right]$ and let $f\left(S_1,\, S_2,...\right)=\left[\left(b_n\right)\right]$.

Then, from the definition of f, $\left(a_n\right)$ is eventually in S_i, $\forall_i\in N$ and $\left(b_n\right)$ is eventually in S_i, $\forall_i\in N$ i.e. $\left(a_n\right)$ and $\left(b_n\right)$ are eventually in S_i, $\forall_i\in N$.

Suppose that $\left(a_n\right)\sim\left(b_n\right)$. It means that there exists a compact subset C of X , that separates uniformly a subsequence of $\left(a_n\right)$ from a subsequence of $\left(b_n\right)$. There exists a cofinal sequence of compacta $\left\{C_n\left|n\in N\right.\right\}$ in X and $\exists n_0\in N$, such that $C\subseteq C_{n_0}$. Hence, C_{n_0} separates uniformly a subsequence of $\left(a_n\right)$ from a subsequence of $\left(b_n\right)$ which is a contradiction with the condition: $\left(a_n\right)$ and $\left(b_n\right)$ are eventually in the component S_{n_0} of $X\backslash C_{n_0}$.

It follows that $\left(a_n\right)\sim\left(b_n\right)$ i.e. $\left[\left(a_n\right)\right]=\left[\left(b_n\right)\right]$.

2)f *is injective:*

Let $f\left(S_1,\, S_2,...\right)=f\left(L_1,L_2,\, ...\right)=\left[\left(a_n\right)\right]$.

Then, $\left(a_n\right)$ is eventually in S_i, $\forall_i\in N$ and $\left(a_n\right)$ is eventually in L_i, $\forall_i\in N$. Because S_i and L_i are components of weakly connectedness of $X\backslash C_i$, $\forall i\in N$, and $S_i\cap L_i\neq\emptyset$, it follows that $S_i=L_i$, $\forall i\in N$. Hence, $\left(S_1,\, S_2,...\right)=\left(L_1,L_2,\, ...\right)$ i.e.f is injective.

3)f *is surjective:*

Let $\left[\left(a_n\right)\right]\in\left\{\left[\left(a_n\right)\right]\left|\left(a_n\right)\in UA\left(X\right)\right.\right\}$ and $n\in N$. Since the compact set $C_n$contains only a most finite members of the sequence $\left(a_n\right)$, it follows that $\left(a_n\right)$ is eventually in $X\backslash C_n$.

We will show that $\left(a_n\right)$ is eventually in some component of weak connectedness S^j of $X\backslash C_n$. It means that $\left(a_n\right)$ is eventually in some component of connectedness.

Suppose that it is not true. Let the components of $X\backslash C_n$ be $\left\{S_1,S_2,\, ...\, S_k\right\}$. Then exists a component S^j of $X\backslash C_n$ which contains a subsequence $\left(a_n'\right)$ of $\left(a_n\right)$ and there is another subsequence $\left(a_n"\right)$ of $\left(a_n\right)$ contained in $\bigcup\limits_{\substack{i=1\\ i\neq j}}^{k} S^j$.

Because X is locally connected, then $X \backslash C_n$ e locally connected and it follows that each component of weak connectedness of $X \backslash C_n$ is open. Then S^j and $\underset{\substack{i=1 \\ i \neq j}}{\overset{k}{\cup}} S^j$ are open.

We obtain that C_n separates uniformly in X two open subsets, S^j and $\underset{\substack{i=1 \\ i \neq j}}{\overset{k}{\cup}} S^j$, and the sequence (a_n) contain subsequence in S^j and in $\underset{\substack{i=1 \\ i \neq j}}{\overset{k}{\cup}} S^j$.
It follows that $(a_n)$is not weakly admissible, which is a contradiction.

4) *$f$is continuous:*

Let $f(S_1, S_2, ...) = [(a_n)]$.

Let G be an open set such that ∂G is compact and(a_n) is eventually in G. Then $W = \{[(b_n)] \in UE(X) \mid (b_n)$ eventually in $G\}$ ia a basic neighbourhood of $[(a_n)]$.

We will show that $\exists n_0 \in N$ such that $S_{n_0} \subseteq G$.

Because (C_n) is a cofinal sequence of compacta , $\exists n_0 \in N$ such that

$C_{n_0} \supseteq \partial G$ and $X \backslash C_{n_0} \subseteq X \backslash \partial G = G \cup Ext\ G$.

Then for the component S_{n_0} of $X \backslash C_{n_0}$ one has

$S_{n_0} \subseteq G \cup Ext, G$ i.e. $(G \cap, S_{n_0}) \cup (Ext\ G \cap S_{n_0}) = S_{n_0}$.Since$S_{n_0}$ is weakly connected and $G \cap S_{n_0} \neq \emptyset$, it follows that $S_{n_0} \subseteq G \cap S_{n_0}$.

We obtain $S_{n_0} \subseteq G$, which proves the above assertion.

In this way we get that $V = \{ [(c_n)] \in UE\ (X) \mid (c_n)$ *eventually in* $S_{n_0}\}$ $\subseteq W$.

We choose a neighboorhood $U$of $(S_1, S_2, ...)$, $U = \prod_{n \in N} U_n$ where

$$U_n = \begin{cases} S(X \backslash C_n), & n > n_0 \\ S_n, & n \leq n_0 \end{cases}$$

We claim that $f(U) \subseteq V \subseteq W$. Really, if $(S_1, S_2, ..., S'_{n_0}, S'_{n_0+1}) \in U$ and $(S_1, ..., S'_{n_0}, S'_{n_0+1}) = [(d_n)]$, then (d_n) is eventually in S_{n_0} and it follows that $(d_n) \in V$. ■

At the end, we compare the uniformly locally connected spaces ([8]), with uniformly simple spaces - a notion introduced in [9] .

Definition 9. If $w : E(X) \to UE(X)$ is a bijection, then X *is uniformly simple.*

Theorem 5. *If X is uniformly locally connected space, then X is uniformly simple.*

P r o o f. Suppose X is not uniformly simple.

Then the induced map $u:\ E(X) \rightarrow UE(X)$ is not a bijection i.e. there exist $(K_1,\ K_2,\ K_3, ...) \in E(X)$ and $\left(K_1',\ K_2',\ K_3', ...\right) \in E(X)$ such that

$$u\ (K_1,\ K_2,\ K_3,\ ...) = (Q_1,\ Q_2,\ Q_{3,.,.}) \quad ,$$

$$u\ \left(K_1',\ K_2',\ K_3', ...\right) = (Q_1,\ Q_2,\ Q_{3,.,.})$$

and $K_1,\ K_1' \subseteq Q_1,\ K_2,\ K_2' \subseteq Q_2$ ...

We can suppose that $K_1 \neq K_1'$.Then, because the sequences $K_1 \supseteq K_2 \supseteq ... \supseteq K_n \supseteq ...$ and $K_1' \supseteq K_2' \supseteq ... \supseteq K_n' \supseteq ...$ are cofinal, we can suppose that $K_n \neq K_n'$, for each natural number n.There exist sequences (x_n), $x_n \in K_n$ and (y_n) , $y_n \in K_n'$, i.e. these two sequences are in different components of connectedness of X/C_n and $\lim\limits_{n\rightarrow\infty} d(x_n,\ y_n) = 0$. Such sequences must exist because, conversely, $\lim\limits_{n\rightarrow\infty} d\left(K_n,\ K_n'\right) > 0$ which is a contradiction in the condition that $K_n,\ K_n' \subseteq Q_n$, for each natural umber n and the condition that $(Q_1,\ Q_2, ...,\ Q_n, ...)$ is a uniform end.

Let K be a maximal connected set in X such that $x_n,\ y_n \in K$ and we choose $\varepsilon > 0$ such that $diam\, K > \varepsilon$

Then, for $x_n,\ y_n \in C$ and C is a component of connectedness, it follows that :

$$diam\, C \geq diam\, K > \varepsilon,$$

i.e. X is not uniformly locally connected, which is a contradiction.

Hence, X is uniformly simple i.e. number of ends and uniform ends of X is equal. ■

E x a m p l e 1. Let X be subset of plane defined with

$$X = \{\,(0, y)\ |\, y\in R\} \cup \left\{\left(x,\ \frac{1}{x}\right)\ |\, x > 0\right\} \cup \left\{(x,\ n)\ \middle|\, 0 \leq x \leq \frac{1}{n}, n \in N\right\}.$$

Then X is uniformly simple, but it is not uniformly locally connected.

We choose two sequences:

$(x_n)\,,\ \ x_n = (0,\ n + \frac{1}{2}\,)$ and

$(y_n)\,, y_n = (\ \frac{2}{2n+1},\ \frac{2n+1}{2}\ )$.

Then any connected set K, such that $x_n,\ y_n \in K$ has $diam\, K > 1$. Hence, X is not uniformly locally connected.

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