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Fast Domain Decomposition Solver for Degenerating PDE of 4th Order in 3D Domain

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Introduction

We consider a fast preconditioner-solver for the finite-difference (finite element with the first order elements) approximation of a specific 4-th order PDE in the unit cube. The differential equation has degenerating coefficients and contains only mixed fourth-order derivatives, each with respect to two variables. The fast preconditioner-solver is designed on the basis of a domain decomposition (DD) Dirichlet-Dirichlet type approach by means of a coarse orthogonal non-uniform decomposition mesh. Sizes of its cells decrease towards the plains of coefficients degeneration, so that at approaching these plains we have the cells with the growing aspect ratio, which depends on the mesh parameter of the fine mesh. Such a decomposition allows us to replace coefficients of PDE in each decomposition cell by constants without loss of spectral equivalence of the preconditioner. At that, highly orthotropic local Dirichlet problems, arising on the subdomains of decomposition, can be efficiently solved by optimal multilevel methods or by almost optimal Fast Discrete Fourier Transform (FDFT). The interface problem, arising on the boundary separating the decomposition subdomains, is the most complicated in several aspects and requires efficient preconditioner-solver for the corresponding Schur complement. We justify an almost spectrally equivalent preconditioner-solver for the face problem, which, after transformation to the trigonometric basis, takes the block-diagonal form with tridiagonal blocks.

We consider only the discretizations on the uniform orthogonal meshes, whereas in practice it is often necessary to use nonuniform and locally condensed meshes. The application of the same type algorithm to the class of

the nonuniform meshes, induced by the Gauss-Lobatto-Chebyshev and Gauss-Lobatto-Legendre quadrature rules, was studied in [4]. Similar approach may be used in a more general case. For a discretization with local refinements, one can incorporate in our DD algorithm special algorithms for handling local problems on those decomposition subdomains, which contain condensed meshes.

Numerical methods for PDE with deteriorating and orthotropic coefficients are paid much attention in the literature, see, *e.g.,* [3, 9, 10] on algorithms for solving the corresponding systems of mesh equations. Although we consider a specific problem, a few of our suggestions have more general significance. Among them the choice of decomposition and the Schur complement preconditioning, which, in distinction with a common approach, retains coupling of the opposite faces. Concurrently with other possible applications, the finite-difference problem under consideration, arises also, *e.g.*, in the fast methods for the systems with the internal stiffness matrices of cubic p-finite elements induced the hierarchical coordinate functions, which are the tensor products of the integrated Legendre polynomials [4, 5, 6]. In a whole, the algorithm is a generalization of DD approach, developed in [4] for a finite-difference discretization of a degenerated elliptic PDE of 2nd order in 2D domain. Due to complexity of the 3D problem of a higher order, in its justification we lean sometimes upon numerical experiments, whereas for 2D case such DD algorithm was theoretically substantiated in [4]. It is necessary to note that a fast BPX type algorithm, based on the multiresolution wavelet analysis, was recently obtained for the same discrete problem in [2]. The algorithm under consideration in this paper is quite different, based on different approach and covers some additional important fields of applications, including, *e.g.,* discretzations by the spectral elements.

1. Boundary value problem and its finite-difference approximation

Let us consider the following boundary value problem in the unit cube $\delta = (0,1)^3$:

$$Lu \equiv z^2 u_{xxyy} + y^2 u_{xxzz} + x^2 u_{yyzz} = f(x,y,z), \quad u|_{\partial\delta} = 0. \tag{1}$$

We introduce a uniform *fine mesh* of size $h = 1/(N+1)$ and approximate the differential operator in (1) on the nineteen-point stencil. Let $u_{i,j,k}$ be an approximate solution at the nodes (ih, jh, kh) and $f_{i,j,k} = f(ih, jh, kh)$, for $i,j,k = 0,1,\ldots,N+1$. To obtain $u_{i,j,k}$, we have to solve the finite-difference system of linear algebraic equations

$$
\begin{aligned}
&h^{-2}[\,k^2\,(4u_{i,j,k}-2u_{i,j-1,k}-2u_{i-1,j,k}-2u_{i,j+1,k}-2u_{i+1,j,k}+u_{i-1,j-1,k}\\
&+u_{i-1,j+1,k}+u_{i+1,j+1,k}+u_{i+1,j-1,k})+\,j^2\,(4u_{i,j,k}-2u_{i,j,k-1}-2u_{i-1,j,k}\\
&-2u_{i+1,j,k}-2u_{i,j,k+1}+u_{i-1,j,k-1}+u_{i-1,j,k+1}+u_{i+1,j,k+1}+u_{i+1,j,k-1})\\
&+\,i^2\,(4u_{i,j,k}-2u_{i,j,k-1}-2u_{i,j-1,k}-2u_{i,j+1,k}-2u_{i,j,k+1}+\\
&u_{i,j-1,k-1}+u_{i,j-1,k+1}+u_{i,j+1,k+1}+u_{i,j+1,k-1})]=f_{i,j,k}\,,\\
&1\le i,j,k\le N-1\,,
\end{aligned}
\tag{2}
$$

where $u_{i,j,k}=0$, if at least one of the indices equals 0 or $N+1$. Clearly, the finite-difference problem (2) is not easy for numerical solution, since the differential equation (1) has the degenerated and highly orthotropic coefficients in the vicinity of the coordinate planes. In a matrix form, we write (2) as $\mathbf{\Lambda u}=\mathbf{f},\ \mathbf{u},\mathbf{f}\in U=R^{(N-1)^3}$, and intent to obtain the efficient DD preconditioner for the matrix $\mathbf{\Lambda}$, which assumes a fast solving procedure for (2).

The same DD preconditioner-solver, which is considered in this paper, with the same efficiency is applicable to the finite element system of algebraic equations, induced on the same mesh by the space $\mathcal{V}_o(\delta)$ of the piece wise trilinear functions vanishing on the boundary $\partial\delta$.

In the respect to p reference element with the tensor products of the integrated Legendre's polynomials for the coordinate functions, it is worth noting the following. According to [4], a more close in the spectrum to its stiffness matrix is the matrix $\mathbf{\Lambda}$ of the finite-difference approximation on the uniform square grid of the more complex differential operator

$$
\begin{aligned}
Lu\equiv\ & z^2\,u_{xxyy}+y^2\,u_{xxzz}+x^2\,u_{yyzz}+\left(\tfrac{y^2}{z^2}+\tfrac{z^2}{y^2}\right)u_{xx}+\left(\tfrac{x^2}{z^2}+\tfrac{z^2}{x^2}\right)u_{yy}\\
&+\left(\tfrac{x^2}{y^2}+\tfrac{y^2}{x^2}\right)u_{zz}+\left(\tfrac{x^2}{y^2z^2}+\tfrac{y^2}{x^2z^2}+\tfrac{z^2}{x^2y^2}\right)u\,.
\end{aligned}
$$

The DD algorithm discussed in the paper is also applicable to such preconditioner $\mathbf{\Lambda}$ and the presence of the minor terms seem serving to the improvement of its efficiency.

2. Preconditioner for the finite-difference matrix

The matrix $\mathbf{B}$ spectrally equivalent to the matrix $\mathbf{\Lambda}$ can be obtained by replacing the coefficients of the differential equation (1) by the piece-wise constant ones on a special *(coarse) decomposition mesh.* Let $\xi_{s,i}=\eta_i=ih$ be the coordinates of the planes of the fine mesh, where $s=1,2,3$ are numbers of the corresponding axes x,y,z, respectively, and $i=0,1,\ldots,N+1$. The

coordinates of the planes of the decomposition mesh, which are denoted by $\overline{\xi}_{s,l} = \zeta_l$ and coincide with some planes of the fine mesh, are defined by two parameters $q > 1$ and $n_0 \geq 1$ according to the formulas

$$\zeta_0 = 0, \quad \zeta_{l_0} = 1, \quad \zeta_l = \eta_i, \text{ for } i = \theta(l) := \text{int}\,[(q^l - 1)n_0], \; l = 1, 2, \ldots, l_0 - 2,$$
$$\zeta_{l_0-1} = \begin{cases} \eta_{\theta(l_0-1)}, & \text{if} \quad \eta_{\theta(l_0)} = 1, \\ \eta_i, \quad i = \text{int}\,[0.5(\theta(l_0 - 2) + N + 1)], & \text{if} \quad \eta_{\theta(l_0)} > 1, \end{cases}$$

where $l_0 = \text{int}\,[\ln(N/n_0+1)\,/\ln q]$ and $\text{int}\,[\rho]$ rounds ρ to the nearest integer greater than or equal to ρ. Cells of the decomposition mesh are parallelepipeds $\delta_{\mathbf{n}}$, $\mathbf{n} = (n_1, n_2, n_3)$, $\quad 1 \leq n_1, n_2, n_3 \leq l_0$,

$$\delta_{\mathbf{n}} = \{(x, y, z) : \zeta_{n_1-1} < x < \zeta_{n_1}, \; \zeta_{n_2-1} < y < \zeta_{n_2}, \; \zeta_{n_3-1} < z < \zeta_{n_3}\},$$

which faces $F_{\mathbf{n},s,t}$ lie in the planes $\xi \equiv \overline{\xi}_{s,n_s+t}$, $t = 0, 1$. Let us introduce the piece-wise constant function

$$\psi(\zeta) = \begin{cases} \psi_l \equiv \frac{1}{2}(\zeta_{l-1}^2 + \zeta_l^2), & \zeta \in (\zeta_{l-1}, \zeta_l), \\ \psi_1 \equiv \frac{1}{2}(h^2 + \zeta_1^2), & \zeta \in (\zeta_0, \zeta_1), \end{cases}$$

and the matrix $\mathbf{B}$ of the similar to (2) finite-difference discretization of the differential equation with these piece-wise constant coefficients

$$\psi(z)u_{xxyy} + \psi(y)u_{xxzz} + \psi(x)u_{yyzz} = f(x, y, z), \quad u\,|_{\partial\delta} = 0\,. \tag{3}$$

The both matrices $\mathbf{B}$ and $\mathbf{\Lambda}$ have a nineteen points stencil and are spectrally equivalent, *i.e.,*

$$\underline{\gamma}\mathbf{B} \leq \mathbf{\Lambda} \leq \overline{\gamma}\mathbf{B} \tag{4}$$

with the constants $\underline{\gamma}, \overline{\gamma}$ depending only on the parameters n_0 and q of the coarse mesh. Alternatively, for our purposes the preconditioner $\mathbf{B}$ satisfying (4) may be defined as the finite element matrix generated for the problem (3) by means of the space $\mathcal{V}_o(\delta)$.

The proof of spectral equivalence is similar to the one for 2D case [4] and is done by the transformation of the quadratic forms corresponding to $\mathbf{B}$ and $\mathbf{\Lambda}$ to the forms convenient for the comparison.

3. Main components of DD algorithm

At each step of some preconditioned iterative process, for instance PCG (preconditioned conjugate gradient method) for solving a system with the matrix $\mathbf{\Lambda}$ we have to solve a system with the preconditioner

$$\mathbf{B}\mathbf{v} = \mathbf{d}\,, \qquad \mathbf{v}, \mathbf{d} \in U. \tag{5}$$

The vector space $U = R^{N^3}$ of all unknowns can be decomposed in the sum $U = U_I \oplus U_F \oplus U_W$ of three subspaces, where

$U_I = \biguplus_{s,n_s=1}^{3,\,l_0} U_{I,\mathbf{n}}$ is the subspace of the internal unknowns for the subdomains $\delta_{\mathbf{n}}$ of decomposition,

U_F is the subspace of the internal unknowns for the faces of decomposition,

U_W is the subspace of the wire-basket unknowns.

With respect to this decomposition, matrix and right-hand side vector of the system (5) can be written in the block form

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_I & \mathbf{B}_{IF} & \mathbf{B}_{IW} \\ \mathbf{B}_{FI} & \mathbf{B}_F & \mathbf{B}_{FW} \\ \mathbf{B}_{WI} & \mathbf{B}_{WF} & \mathbf{B}_W \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} \mathbf{v}_I \\ \mathbf{v}_F \\ \mathbf{v}_W \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} \mathbf{d}_I \\ \mathbf{d}_F \\ \mathbf{d}_W \end{pmatrix}, \tag{6}$$

where $\mathbf{B}_I$ is the block-diagonal matrix with each independent block $\mathbf{B}_{I,\mathbf{n}}$ on the diagonal corresponding to the subdomain $\delta_{\mathbf{n}}$. We stress several important facts that should be taken into consideration at solving system (5),(6).

1) The matrices of local discrete Dirichelet problems have the form

$$\mathbf{B}_{I,\mathbf{n}} = \psi_{n_1}\mathbf{I}_{n_1} \otimes \mathbf{\Delta}_{n_2} \otimes \mathbf{\Delta}_{n_2} + \psi_{n_2}\mathbf{\Delta}_{n_1} \otimes \mathbf{I}_{n_2} \otimes \mathbf{\Delta}_{n_3} + \psi_{n_3}\mathbf{\Delta}_{n_1} \otimes \mathbf{\Delta}_{n_2} \otimes \mathbf{I}_{n_3}\,,$$

where $\mathbf{\Delta}_{n_s} = \mathrm{tridiag}\,[-1, 2, -1]$ and $\mathbf{I}_{n_s} = \mathrm{diag}\,[1]$ are the $(m_s - 1) \times (m_s - 1)$ matrices, $m_s = m_s(n_s)$ is the number of fine mesh intervals on the edge of $\delta_{\mathbf{n}}$ parallel to the axis ξ_s. Clearly, the systems with these matrices can be efficiently solved by FDFT for $\mathcal{O}(N^3 \log N)$ arithmetic operations. The structure of $\mathbf{B}_{I,\mathbf{n}}$ allows one also to implement optimal multilevel solvers with the computational cost $\mathcal{O}(N^3)$, based on several robust BPX type preconditioners. They may be designed following the approach developed for anisotropic discretizations, e.g., in [3].

2) There is such a basis in U that $U = U_I \oplus U_{F,\mathrm{tr}} \oplus U_W$ and the following statements are true:

i) $U_I \perp_{\mathbf{B}} U_{F,\mathrm{tr}}$, *i.e.* subspaces U_I and $U_{F,\mathrm{tr}}$ are orthogonal in a sense of the scalar product $(\cdot,\cdot)_{\mathbf{B}} := (\cdot, \mathbf{B}\cdot)$.

ii) The basis vectors of $U_{\mathrm{f,trig}}$ corresponding to any one face are orthogonal in the sense of the same scalar product, but are not orthogonal for adjacent faces. Apart from that, each face basis vector is coupled with a single and having the same local indices (see iii)) basis vector of the opposite face.

iii) The trace of each basis vector on the interface boundary is nonzero only on one face, and on each face $F_{\mathbf{n},s,t}$ these traces are the vectors, which are denoted as

$$\boldsymbol{\mu}_{s,\alpha,\beta} = (\mu_{s,\alpha,\beta}(i,j))_{i,j=0}^{m_{\widehat{s}},m_{\overline{s}}}, \qquad \alpha = 1,..,m_{\widehat{s}}-1\,, \qquad \beta = 1,..,m_{\overline{s}}-1\,,$$

and have for the components the numbers

$$\mu_{s,\alpha,\beta}(i,j) \;=\; \sin(\alpha\, i\pi/m_{\widehat{s}})\sin(\beta j\pi/m_{\overline{s}})\,.$$

Here $s,\widehat{s},\overline{s} = 1,2,3$ and are all distinct, the numbers α,β,i,j are local indices of the face basis vectors and the face nodes, respectively, increasing towards the positive directions of the corresponding axes, and $\alpha = 1,..,m_{\widehat{s}}-1$, $\beta = 1,..,m_{\overline{s}}-1$. In the third direction, the face coordinate vectors may be continued inside $\delta_{\mathbf{n}}$ linearly.

After the transformation to the basis described above, equations (5), (6) get the following form

$$\widetilde{\mathbf{B}}\,\widetilde{\mathbf{v}} = \widetilde{\mathbf{d}}\,, \tag{7}$$

$$\widetilde{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_I & \mathbf{O} & \mathbf{B}_{IW} \\ \mathbf{O} & \mathbf{S}_{F,\mathrm{tr}} & \mathbf{B}_{FW,\mathrm{tr}} \\ \mathbf{B}_{WI} & \mathbf{B}_{WF,\mathrm{tr}} & \mathbf{B}_W \end{pmatrix}, \quad \widetilde{\mathbf{v}} = \begin{pmatrix} \mathbf{v}_I \\ \mathbf{v}_{F,\mathrm{tr}} \\ \mathbf{v}_W \end{pmatrix}, \quad \widetilde{\mathbf{d}} = \begin{pmatrix} \mathbf{d}_I \\ \mathbf{d}_{F,\mathrm{tr}} \\ \mathbf{d}_W \end{pmatrix}.$$

iv) It is important that, similarly to 2-D case, there are easily derived explicit expressions for the entries of the block

$$\widetilde{\mathbf{B}}_\Gamma = \begin{pmatrix} \mathbf{S}_{F,\mathrm{tr}} & \mathbf{B}_{FW,\mathrm{tr}} \\ \mathbf{B}_{WF,\mathrm{tr}} & \mathbf{B}_W \end{pmatrix},$$

which is the restriction of the matrix $\widetilde{\mathbf{B}}$ to the subspace $U_\Gamma := U_{F,\mathrm{tr}} \oplus U_W$. Additionally, we can show that the computation of the entries by means of these explicit expressions costs $\mathcal{O}(N^3)$ arithmetic operations. Another important for the algorithm fact is that, due to the property 1) and to the sparse pattern of $\mathbf{B}_{I,W}$, the arithmetic cost of the matrix-vector multiplication by $\mathbf{B}_I^{-1}$ – *i.e.,* solving systems with the matrix $\mathbf{B}_I$ – and the matrix-vector-multiplication by $\mathbf{B}_{I,W}, \mathbf{B}_{W,I}$ is $\mathcal{O}(N^3 \log N)$, *e.g.,* assuming FDFT for the multiplications by $\mathbf{B}_I^{-1}$. Now, looking at the factorization

$$\widetilde{\mathbf{B}}^{-1} \;=\; \begin{pmatrix} \mathbf{I}_I & -\mathbf{B}_I^{-1}\mathbf{B}_{I\Gamma} \\ \mathbf{O} & \mathbf{I}_\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{B}_I^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{S}_\Gamma^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_I & \mathbf{O} \\ -\mathbf{B}_{\Gamma I}\mathbf{B}_I^{-1} & \mathbf{I}_\Gamma \end{pmatrix},$$

$$\mathbf{B}_{\Gamma,I} \;=\; \begin{pmatrix} \mathbf{O} \\ \mathbf{B}_{WI} \end{pmatrix},$$

we conclude that all operations, except for solving the system with the matrix $\mathbf{S}_\Gamma$, may be performed for at least $\mathcal{O}(N^3 \log N)$ or less arithmetic operations, and deriving a fast DD solver is reduced to designing a fast solver for the matrix $\mathbf{S}_\Gamma$,

$$\mathbf{S}_\Gamma = \begin{pmatrix} \mathbf{S}_{F,\mathrm{tr}} & \mathbf{B}_{FW,\mathrm{tr}} \\ \mathbf{B}_{WF,\mathrm{tr}} & \mathbf{B}_\Delta \end{pmatrix}, \qquad \mathbf{B}_\Delta = \mathbf{B}_W - \mathbf{B}_{WI}\mathbf{B}_I^{-1}\mathbf{B}_{IW}\,.$$

3) By zeroing in $\mathbf{S}_{F,\mathrm{tr}}$ all the non-diagonal blocks coupling the unknowns of adjacent faces, we obtain the preconditioner, which is denoted by $\boldsymbol{\mathcal{S}}_{F,\mathrm{tr}}$. The system with this preconditioner can be solved by a direct method for $\mathcal{O}(N^2 \log N)$ arithmetic operations. The reason for this is that up to the perturbation, the matrix $\boldsymbol{\mathcal{S}}_{F,\mathrm{tr}}$ is the block-diagonal matrix with $3(N^2 - 2N(l_0 - 1) + (l_0 - 1)^2)$ *independent tridiagonal* blocks of the size $l_0 - 1$. Results of our numerical experiments support the estimate $\mathrm{cond}\,[\boldsymbol{\mathcal{S}}_{F,\mathrm{tr}}^{-1}\mathbf{S}_{F,\mathrm{tr}}] \le c(\log N)^2$, see Fig. 1, which is the same with the estimate for 2-d approved by numerical experiments and the theoretical analysis [4, 1]. We implement secondary inexact preconditioned two-stage Chebyshev iterative processes to obtain the preconditioner-solver $\boldsymbol{\mathcal{S}}_\Gamma$ for the matrix $\mathbf{S}_\Gamma$. Formally it is defined as

$$\boldsymbol{\mathcal{S}}_\Gamma^{-1} = \begin{pmatrix} \mathbf{I}_F & -\mathbf{S}_{F,\mathrm{it}}^{-1}\mathbf{B}_{tr,FW} \\ \mathbf{O} & \mathbf{I}_W \end{pmatrix} \begin{pmatrix} \mathbf{S}_{F,\mathrm{it}}^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_\delta^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_F & \mathbf{O} \\ -\mathbf{B}_{\mathrm{tr},WF}\mathbf{S}_{F,\mathrm{it}}^{-1} & \mathbf{I}_W \end{pmatrix}, \tag{8}$$

where

$$\mathbf{S}_{F,\mathrm{it}} = \mathbf{S}_{F,\mathrm{it}}(\mathbf{S}_{F,\mathrm{tr}}, \boldsymbol{\mathcal{S}}_{F,\mathrm{tr}}, k_1)\,, \qquad \mathbb{S}_{F,\mathrm{it}} = \mathbf{S}_{F,\mathrm{it}}(\mathbf{S}_{F,\mathrm{tr}}, \boldsymbol{\mathcal{S}}_{F,\mathrm{tr}}, k_2)\,,$$

$$\mathbf{B}_\delta = \mathbf{B}_\Delta - \mathbf{B}_{WF,\mathrm{tr}}\mathbb{S}_{F,\mathrm{it}}^{-1}\mathbf{B}_{FW,\mathrm{tr}}\,.$$

In general, for a given nonegative matrix $\mathbb{A}$, we denote by $\mathbb{A}_{\mathrm{it}}(\mathbb{A}, \boldsymbol{\mathcal{A}}, \nu)$ the matrix which is implicitly defined by the iteration process

$$\mathbb{A}_{\mathrm{it}}^{-1} = [\mathbf{I} - \prod_{k=1}^{\nu}(\mathbf{I} - \sigma_k \boldsymbol{\mathcal{A}}^{-1}\mathbb{A})]\,\mathbb{A}^{-1} \tag{9}$$

with some preconditioner $\boldsymbol{\mathcal{A}}$ and Chebyshev iteration parameters σ_k. Let us remind that, if $\mathbf{u} := \mathbb{A}_{\mathrm{it}}^{-1}\mathbf{f}$, then $\mathbf{u} = \mathbf{u}^k$ is produced by ν iterations

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \sigma_k \boldsymbol{\mathcal{A}}^{-1}(\mathbb{A}\mathbf{u}^k - \mathbf{f})\,, \qquad \mathbf{u}^0 = \mathbf{0}\,. \tag{10}$$

4) We see that the matrices $\mathbf{S}_{F,\mathrm{it}}$ and $\mathbb{S}_{F,\mathrm{it}}$ are defined similarly, but the numbers of iterations k_1 and k_2 may be different. The pointed out above

computational cost of solving systems with the matrix $\boldsymbol{\mathcal{S}}_{F,\mathrm{tr}}$ and the dependence of the condition number of $\boldsymbol{\mathcal{S}}_{F,\mathrm{tr}}^{-1}\mathbf{S}_{F,\mathrm{tr}}$ on N allow us to estimate the number of iterations and the computational work for providing the spectral equivalence of $\mathbf{S}_{F,\mathrm{it}}$ and $\mathbf{S}_{F,\mathrm{tr}}$, which may be desirable for $\mathbf{S}_{F,\mathrm{it}}$ in the second parenthesis of (8). This estimation can be carried out with the use of known techniques, see, *e.g.*, [5], and results in the bound $\mathcal{O}(\log N)$ and $\mathcal{O}(N^2(\log N)^2)$, respectively. In the latter estimate we did not take into account the cost of the multiplications by $\mathbf{S}_{F,\mathrm{tr}}$, which can be avoided in DD algorithm, see below.

The analysis of the analytic expressions of the elements of $\mathbf{S}_{F,\mathrm{tr}}$ and its fill in as well as the results of our numerical experiments show that the number of its non-zero elements is $\mathcal{O}(N^3)$. Hence, the computational cost of the operation $\mathbf{S}_{F,\mathrm{it}}^{-1}\mathbf{v}_F$ in the *Schur complement algorithm* is $\mathcal{O}(N^3 \log N)$ for all $\mathbf{v}_F \in U_F$.

According to the above discussion, for $\mathbf{S}_{F,\mathrm{it}}$ in the second parenthesis we could set $k_1 = \mathcal{O}(\log N)$. However, we note that the multiplication of a vector $\mathbf{v}_{\boldsymbol{\Gamma}}$ by the product of the second and third factors in (8) assumes two multiplications $\mathbf{S}_{F,\mathrm{it}}\mathbf{v}_F$ by the same subvector $\mathbf{v}_F$ of the vector $\mathbf{v}_{\boldsymbol{\Gamma}}$. Clearly, it is reasonable to perform the multiplication $\mathbf{S}_{F,\mathrm{it}}\mathbf{v}_F$ only once, but to choose for k_1 maximal of the two necessary numbers of iterations. As usual, in the case of the restriction-prolongation operations (performed in the first and and third factors in (8)) by iterative solution of the Dirichlet problems, the number of iterations should be greater. The vector $\mathbf{v}_\Gamma$ of the prolongation on the interface, produced by the second column of the first factor in (8), is uniquely defined as having for the subvectors $\mathbf{v}_F := -\boldsymbol{\mathcal{S}}_{F,\mathrm{it}}^{-1}\mathbf{B}_{tr,FW}\mathbf{v}_W$ and $\mathbf{v}_W$. By the theory of DD [7], instead of the spectral equivalence of $\mathbf{S}_{F,\mathrm{it}}$ and $\mathbf{S}_{F,\mathrm{tr}}$, the decisive factor for choosing the number of iterations in this case is the inequality

$$\|\mathbf{v}_\Gamma\|_{\mathbf{S}_\Gamma} \le \gamma_{W\to\Gamma}\|\mathbf{v}_W\|_{\mathbf{B}_\Delta}\,,$$

with sufficiently good $\gamma_{W\to\Gamma}$. As it was mentioned above, the necessary number of iterations for providing the above inequality is greater than for providing the spectral equivalence of the preconditioner $\mathbf{S}_{F,\mathrm{it}}$, [5, 7]. This difference is caused by the bad initial approximation $\mathbf{u}^0 = \mathbf{0}$ in (10) for the interior component of the low energy vector to be obtained in the result of the prolongation. However, the ratio of the low energy prolongation to the initial approximation of the order N^{-s} with some fixed s causes the increase in the number of iterations only by the factor of the order $s \log N$, see [5, 7], which retains the algorithm in the range of almost optimal ones.

It is also worth noting that the techniques of [3] for obtaining BPX type preconditioners for the orthotropic problems in rectangles can be generalized on the matrices $\mathbf{B}_\mathbf{n}$ of the 3-d Neuman problems on the subdomins $\delta_\mathbf{n}$. As it

is known, such preconditioners can directly lead to the preconditioners for the Schur complement $\mathbf{S}_B$ and to the prolongation operators, see, *e.g.,* [8]. Such Schur complement preconditioners and prolongation operators can be more efficient in comparison with the considered above. However, discussion of this topic is out of the scope of this paper.

Since $\mathbf{B}_\delta$ has the size $\mathcal{O}(N(\log N)^2)$, solving linear systems with this matrix even by Gauss elimination preserves almost optimality of DD algorithm. For this reason, we do not add details on the wire basket component and only point out some opportunities for the algorithm speed up. The edge coordinate vectors may be also transformed to the "trigonometric" ones, the entries of which living on the edge of $\delta_{\mathbf{n}}$ parallel to the axis ξ_s are $\sin(\alpha\, i\pi/m_s)$ and are continued inside $\delta_{\mathbf{n}}$ linearly in each of the two rest variables. At that, the matrices $\mathbf{B}_W$ and $\mathbf{B}_\Delta$ are transformed to the simpler ones $\mathbf{B}_{W,\mathrm{tr}}$ and $\mathbf{B}_{\Delta,\mathrm{tr}}$. This simplifies obtaining the preconditioners (*e.g.,* explicitly given) for $\mathbf{B}_{\Delta,\mathrm{tr}}$, which in turn may allow to avoid calculation of $\mathbf{B}_\delta$.

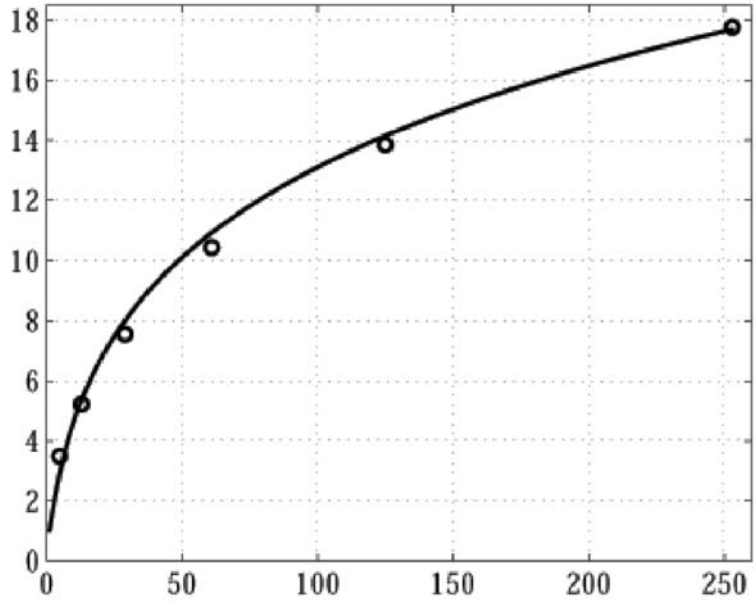

Figure 1: Dependence of $1/\lambda_{min}$ on N

5) The version of the DD algorithm, in which all Schur complements are not calculated explicitly, has a close computational complexity. Which of

the two versions is advantageous may be decided on the basis of numerical experimentation. If, *e.g.*, the Schur complement $\mathbf{S}_\Gamma$ is not calculated explicitly, the number of multiplications by $\mathbf{B}_I^{-1}$ in the DD algorithm increases, but in a fixed number of times. One possible domain decomposition preconditioner $\boldsymbol{\mathcal{B}}$ of this type for the matrices $\boldsymbol{\Lambda}$ and $\mathbf{B}$ is defined by the following expressions

$$\boldsymbol{\mathcal{B}}^{-1} = \begin{pmatrix} \mathbf{I}_I & \mathbf{O} \\ -\mathbf{B}_{\Gamma I}\mathbf{B}_I^{-1} & \mathbf{I}_\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{B}_I^{-1} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\mathcal{S}}_\Gamma^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_I & -\mathbf{B}_I^{-1}\mathbf{B}_{I\Gamma} \\ \mathbf{O} & \mathbf{I}_\Gamma \end{pmatrix},$$

where

$$\boldsymbol{\mathcal{S}}_\Gamma^{-1} = \begin{pmatrix} \mathbf{I}_F & \mathbf{O} \\ -\mathbf{S}_{WF}\boldsymbol{\mathcal{S}}_F^{-1} & \mathbf{I}_W \end{pmatrix} \begin{pmatrix} \boldsymbol{\mathcal{S}}_F^{-1} & \mathbf{O} \\ \mathbf{O} & \mathbb{S}_W^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_I & -\boldsymbol{\mathcal{S}}_F^{-1}\mathbf{S}_{FW} \\ \mathbf{O} & \mathbf{I}_W \end{pmatrix},$$

$$\mathbf{S}_F = \mathbf{B}_F - \mathbf{B}_{FI}\mathbf{B}_I^{-1}\mathbf{B}_{IF}, \qquad \mathbf{S}_W = \mathbf{B}_W - \mathbf{B}_{WI}\mathbf{B}_I^{-1}\mathbf{B}_{IW},$$

$$\mathbf{S}_{FW} = \mathbf{B}_{FW} - \mathbf{B}_{FI}\mathbf{B}_I^{-1}\mathbf{B}_{IW}, \qquad \mathbb{S}_W = \mathbf{S}_W - \mathbf{S}_{WF}\mathbb{S}_F^{-1}\mathbf{S}_{FW},$$

$$\boldsymbol{\mathcal{S}}_F^{-1} = \boldsymbol{\mathcal{F}}_F^{-1}\mathbf{S}_{F,\mathrm{it}}^{-1}\boldsymbol{\mathcal{F}}_F, \qquad \mathbb{S}_F^{-1} = \boldsymbol{\mathcal{F}}_F^{-1}\mathbb{S}_{F,\mathrm{it}}^{-1}\boldsymbol{\mathcal{F}}_F\,.$$

Here $\boldsymbol{\mathcal{F}}_F$ is a matrix of 2D DFFT to the trigonometric basis on the faces of decomposition and $\mathbf{S}_{F,\mathrm{it}}^{-1}, \mathbb{S}_{F,\mathrm{it}}^{-1}$ were defined earlier. Matrices $\mathbf{S}_F, \mathbf{S}_W$, $\mathbf{S}_{FW}$, $\mathbf{S}_{FW}$, $\mathbb{S}_W$, $\boldsymbol{\mathcal{S}}_F^{-1}$ and $\mathbb{S}_F^{-1}$ are not calculated explicity and they are not stored in the memory. For instance, the product $\mathbf{S}_{FW}\mathbf{v}_W$ is calculated as $\mathbf{S}_{FW}\mathbf{v}_W = \mathbf{B}_{FW}\mathbf{v}_W - \mathbf{B}_{FI}(\mathbf{B}_I^{-1}(\mathbf{B}_{IW}\mathbf{v}_W))$.

4. Numerical experiment

The main advantage of the considered algorithm is the efficient and simple preconditioners-solvers for the two basic DD components: i) the internal subproblems on subdomains of decomposition and ii) the internal subproblems on the faces of the subdomains of decomposition. These preconditioners-solvers deal with the major bulk of unknowns, whereas the number of the remaining unknowns does not exceed $3N(\log N)^2$. The almost optimality of the first preconditioner-solver is evident. The definition of the second substantially differs from the common way of designing face preconditioners, *e.g.,* in the wire basket algorithms [11]. To take into account the properties of our discrete problem more completely, we retain coupling of the opposite faces of subdomains of decomposition in the Schur complement preconditioner. This increases the computational complexity of solvers for Schur complement preconditioner only by a constant, but prevents losses in the condition due to the high orthotropy of the

problems on the rectangular subdomains with the deteriorating aspect ratios. The theoretical substantiation of the almost optimality in the condition of such a face preconditioner requires a deeper study. Here, we present the results of the numerical experiment which validates effectiveness of the preconditioner $\boldsymbol{\mathcal{S}}_F$ for $\mathbf{S}_F$ (and equivalently $\boldsymbol{\mathcal{S}}_{F,\mathrm{tr}}$ for $\mathbf{S}_{F,\mathrm{tr}}$). A way of construction of $\boldsymbol{\mathcal{S}}_F$ implies that $\lambda_{\max}[\boldsymbol{\mathcal{S}}_F^{-1}\mathbf{S}_F] \le 6$. To estimate $\lambda_{\min}[\boldsymbol{\mathcal{S}}_F^{-1}\mathbf{S}_F]$ we note, that the matrices $\boldsymbol{\mathcal{S}}_F, \mathbf{S}_F$ are obtained by assembling the matrices $\boldsymbol{\mathcal{S}}_{F_{\mathbf{n}}}, \mathbf{S}_{F_{\mathbf{n}}}$ for sumdomains $\delta_{\mathbf{n}}$ of decomposition. Hence $\lambda_{\min}[\boldsymbol{\mathcal{S}}_F^{-1}\mathbf{S}_F] \ge \min_{\mathbf{n}} \lambda_{\min}[\boldsymbol{\mathcal{S}}_{F_{\mathbf{n}}}^{-1}\mathbf{S}_{F_{\mathbf{n}}}]$.

The dependence of $1/\min_{\mathbf{n}} \lambda_{\min}[\boldsymbol{\mathcal{S}}_{F_{\mathbf{n}}}^{-1}\mathbf{S}_{F_{\mathbf{n}}}]$ on N, obtained numerically, is shown in the Fig.1 by circles. The solid line is the graph of $1+0.8(\ln N)^{1.75}$. Thus, the generalized condition number has asymptotic behavior as $\mathcal{O}((\ln N)^{1.75})$.

References

[1] I. Anoufriev, V. Korneev, A. Rytov, Computer testing of fast domain decomposition algorithms for hp discretizations of 2-nd order elliptic equations. In: *Proc of the Sixth Hellenic-European Conference on Computer Mathematics and its Applications*, Vol. 1, E. A. Lipitakis, ed., Sept. 25-27, 2003, Athens. LEA-Publishers, 2003, 99-108

[2] S. Beuchler, R. Schneider and C. Schwab, Multiresolution weighted norm equivalence and applications. *Numerische Mathematik*, **98**, 2004, No. 1, 67-97.

[3] Griebel M. and Oswald P., Tensor product type subspace splittings and multilevel iterative methods for anisothropic problems. *Advances Comput. Math.*, **4**, 1995, 171–201.

[4] V.G. Korneev, Local Dirichlet problems on subdomains of decomposition in *hp* discretizations, and optimal algorithms for their solution. *Matematicheskoe Modelirovanie (Mathematical Modelling)*, **14**, 2002, No. 5, 51–74.

[5] V. Korneev, U. Langer and L.S. Xanthis, On fast domain decomposition solving procedures for *hp*-discretizations of 3-d elliptic problems. *Computational Methods in Applied Mathematics*, **3**, 2003, No. 4, 536–559.

[6] Korneev V, Langer U. and Xanthis L., Fast adaptive domain decomposition algorithms for *hp*-discretizations of 2-*d* and 3-*d* elliptic equations: recent advances. *Hermis J. Comput. Math. Appl.*, **4**, 2003, 27–44.

[7] V. Korneev and U. Langer, Domain Decomposition Methods and Preconditioning. In: *Encyclopedia of Computational Mechanics. V. I: Fundamentals,* E. Stein, R. de Borst and T.J.R. Hughes eds. John Wiley and Sons, Ltd., 2004, 617–647.

[8] Oswald P., Interface preconditioners and multilevel extension operators. In: *Proc. 11th Int. Conf. on Domain Decomposition Methods in Science and Engineering,* C.-H. Lai et al., eds., London, 1998, published by ddm.org, 1999, 96-103.

[9] Pflaum Ch., Robust confergence of multilevel algorithms for convection-difusion equations. *SIAM Journal on Numerical Analysis*, **37**, 2000, No. 2, 443 – 469.

[10] Schieweck N., A multigrid convergence proof by a strengthened Caushy inequality for symmetric elliptic boundary value problems. In: *Second Multigrid Seminar,* G. Telschow, ed., Garzau, 1985, 289–309.

[11] A. Toselli and O. Widlund, *Domain Decomposition Methods – Algorithms and Theory.* Springer, Berlin-Heidelberg, 2005.

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