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Fixed Point Theorems for Mappings Satisfying Implicit Relations on Two Metric Spaces

V. Popa [1] *and M. Telci* [2]

Presented by V. Kiryakova

In this paper, two fixed point theorems on two metric spaces are proved. These results generalize theorems of Fisher [1] and Popa [2].

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1. Introduction

The following related fixed point theorem was proved by Fisher [1].

Theorem 1. *Let* (X, d) *and* (Y, ρ) *be complete metric spaces. If* T *is a mapping of* X *into* Y *and* S *is a mapping of* Y *into* X *satisfying the inequalities*

$$\rho(Tx, TSy) \leq c\max\{d(x, Sy), \rho(y, Tx), \rho(y, TSy)\}, \quad (1)$$

$$d(Sy, STx) \leq c\max\{\rho(y, Tx), d(x, Sy), d(x, STx)\} \quad (2)$$

for all x *in* X *and* y *in* Y*, where* $0 \leq c < 1$*, then* ST *has a unique fixed point* z *in* X *and* TS *has a unique fixed point* w *in* Y*. Further,* $Tz = w$ *and* $Sw = z$*.*

Recently, by using the functions, Telci [3] proved a generalized of Theorem 1.

Let $\mathcal{F}$ be the set of all real functions $f : R_+^3 \to R_+$ satisfying the following properties:
(i) f is upper semi-continuous in each coordinate variable,

(ii) If either $u \leq f(v,0,u)$ or $u \leq f(v,u,0)$ for all $u, v \geq 0$, then there exists a real constant $0 \leq c < 1$ such that $u \leq cv$.

Theorem 2. *[3] Let (X,d) and (Y,ρ) be complete metric spaces, let T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities*

$$\rho(Tx,TSy) \leq f(d(x,Sy),\rho(y,Tx),\rho(y,TSy)), \tag{3}$$

$$d(Sy,STx) \leq g(\rho(y,Tx),d(x,Sy),d(x,STx)) \tag{4}$$

for all x in X and y in Y, where $f,g \in \mathcal{F}$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

In this paper, two new generalization of Theorems in [1] are proved for mappings satisfying implicit relationson on two metric spaces.

2. Implicit Relations.

We will also denote by $\widetilde{\mathcal{F}}$ the set of all real functions $F : R_+^4 \to R$ such that :
(F_1): F is continuous in each coordinate variable,
(F_2): If either $F(u,v,u,0) \leq 0$ or $F(u,v,0,u) \leq 0$ for all $u,v \geq 0$, then there exists a real constant $0 \leq h < 1$ such that $u \leq hv$.

Example 1. $F(t_1,t_2,t_3,t_4) = t_1 - k.\max\{t_2,t_3,t_4\}$ where $0 \leq k < 1$.
F_1: Obviously.
F_2: Let $u > 0$ and $F(u,v,u,0) = u - k.\max\{v,u,0\} \leq 0$. If $u \geq v$, then $u(1-k) \leq 0$, a contradiction. Thus $u < v$ and $u \leq kv = hv$, where $h = k$. Similarly, if $F(u,v,0,u) \leq 0$, then $u \leq hv$. If $u = 0$, then $u \leq hv$.

Example 2. $F(t_1,t_2,t_3,t_4) = t_1^2 - c.\max\{t_2t_3,t_2t_4,t_3t_4\}$ where $0 \leq c < 1$.
F_1: Obviously.
F_2: Let $u > 0$ and $F(u,v,u,0) = u^2 - c.\max\{vu,0,0\} = u(u-cv) \leq 0$. Then $u \leq cv = hv$, where $h = c$. Similarly, if $F(u,v,0,u) \leq 0$, then $u \leq hv$. If $u = 0$ then $u \leq hv$.

Example 3. $F(t_1,t_2,t_3,t_4) = t_1 \max\{t_3,t_4\} - ct_2 \max\{t_2,t_4\}$ where $0 \leq c < 1$.
F_1: Obviously.

F_2: Let $u > 0$ and $F(u, v, u, 0) = u \max\{u, 0\} - cv \max\{v, 0\} \leq 0$. Then $u \leq vc^{1/2} = hv$, where $h = c^{1/2}$. Now let $F(u, v, 0, u) = u \max\{0, u\} - cv \max\{v, u\} \leq 0$. If $u \geq v$, then $u^2 \leq cvu \leq cu^2$, a contradiction. Thus $u < v$ which implies $u \leq vc^{1/2} = hv$. If $u = 0$, then $u \leq hv$.

Example 4. $F(t_1, t_2, t_3, t_4) = t_1^2 - [at_1t_2 + bt_1t_3 + ct_1t_4 + dt_2^2]$ where $a, b, c \geq 0$,
$0 < d$ and $a + b + c + d < 1$.
F_1: Obviously.
F_2: Let $u > 0$ and $F(u, v, u, 0) \leq 0$. Then $(1 - b)u^2 - auv - dv^2 \leq 0$. If $v = 0$, then $(1 - b)u^2 \leq 0$, a contradiction. Let $f(t) = (1 - b)t^2 - at - d$, where $t = u/v$. Then $f(0) < 0$ and $f(1) = 1 - (a + b + d) > 0$. If $0 < h_1 < 1$ is a root of the equation $f(t) = 0$, then $f(t) \leq 0$ for $t \leq h_1$ and $u \leq h_1v$. Similarly, $F(u, v, 0, u) \leq 0$ implies $u \leq h_2v$, where $0 < h_2 < 1$. Thus $u \leq hv$, where $h = \max\{h_1, h_2\}$. If $u = 0$, then $u \leq hv$.

Example 5. $F(t_1, t_2, t_3, t_4) = t_1^3 - [at_1^2t_2 + bt_1t_3t_4 + ct_2t_3t_4]$ where $0 \leq a < 1$.
F_1: Obviously.
F_2: Let $u > 0$ and $F(u, v, u, 0) = u^3 - au^2v \leq 0$, then $u^2(u - av) \leq 0$ which implies $u \leq av = hv$, where $0 \leq h < 1$. If $F(u, v, 0, u) \leq 0$, then $u \leq hv$. If $u = 0$, then $u \leq hv$.

Remark 1. The condition (ii) implies condition (F_2) for $F(t_1, t_2, t_3, t_4) = t_1 - f(t_2, t_3, t_4)$. Indeed if $u, v \geq 0$ and $F(u, v, u, 0) = u - f(v, u, 0) \leq 0$, then $u \leq f(v, u, 0)$ which implies, by (ii), that $u \leq cv = hv$, where $h = c$ and $0 \leq h < 1$. Similarly, $F(u, v, 0, u) \leq 0$ implies $u \leq hv$.

3. Fixed points on two complete metric spaces.

Theorem 3. *Let (X, d) and (Y, ρ) be complete metric spaces, let T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities*

$$F(\rho(Tx, TSy), d(x, Sy), \rho(y, Tx), \rho(y, TSy)) \leq 0, \tag{5}$$

$$G(d(Sy, STx), \rho(y, Tx), d(x, Sy), d(x, STx)) \leq 0 \tag{6}$$

for all x in X and y in Y, where $F, G \in \widetilde{\mathcal{F}}$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

P r o o f. Suppose x is an arbitrary point in X. We define sequences $\{x_n\}$ in X and $\{y_n\}$ in Y by

$$x_n = (ST)^n x, \quad y_n = T(ST)^{n-1} x$$

for $n = 1, 2, ...$

We note that if $d(x_n, x_{n+1}) = 0$ and $\rho(y_n, y_{n+1}) = 0$ for some n, then $x_n = z$ is a fixed point of ST and $y_n = w$ is a fixed point of TS. We therefore suppose that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n.

Let $h = \max\{h_1, h_2\}$ where h_1 and h_2 are real constants satisfying the condition (F_2) for F and G respectively.

Using inequality (5), we have

$$\begin{aligned} F(\rho(Tx_{n-1}, TSy_n), d(x_{n-1}, x_n), 0, \rho(y_n, y_{n+1})) &= \\ F(\rho(y_n, y_{n+1}), d(x_{n-1}, x_n), 0, \rho(y_n, y_{n+1})) &\leq 0, \end{aligned}$$

which implies, by (F_2), that

$$\rho(y_n, y_{n+1}) \leq hd(x_{n-1}, x_n). \tag{7}$$

Similarly, using inequality (6), we have

$$\begin{aligned} G(d(Sy_n, STx_n), \rho(y_n, y_{n+1}), 0, d(x_n, x_{n+1})) &= \\ G(d(x_n, x_{n+1}), \rho(y_n, y_{n+1}), 0, d(x_n, x_{n+1})) &\leq 0, \end{aligned}$$

which implies, by (F_2), that

$$d(x_n, x_{n+1}) \leq h\rho(y_n, y_{n+1}). \tag{8}$$

It now follows from inequalities (7) and (8) that

$$d(x_n, x_{n+1}) \leq h\rho(y_n, y_{n+1}) \leq h^2 d(x_{n-1}, x_n).$$

Hence, by induction we get

$$d(x_n, x_{n+1}) \leq h\rho(y_n, y_{n+1}) \leq h^{2n} d(x, x_1),$$

for $n = 1, 2, ...$ Since $c < 1$, it follows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with limits z in X and w in Y.

Using inequality (5), we have

$$\begin{aligned} F(\rho(Tz, TSy_{n-1}), d(z, x_{n-1}), \rho(y_{n-1}, Tz), \rho(y_{n-1}, y_n)) &= \\ F(\rho(Tz, y_n), d(z, x_{n-1}), \rho(y_{n-1}, Tz), \rho(y_{n-1}, y_n)) &\leq 0. \end{aligned}$$

Letting n tend to infinity and using (F_1), we have

$$F(\rho(Tz,w),0,\rho(w,Tz),0) \leq 0,$$

and it follows that $w = Tz$ by (F_2).

Using inequality (6), we have

$$\begin{aligned} G(d(Sw,STx_{n-1}),\rho(w,y_n),d(x_{n-1},Sw),d(x_{n-1},x_n)) &= \\ G(d(Sw,x_n),\rho(w,y_n),d(x_{n-1},Sw),d(x_{n-1},x_n)) &\leq 0. \end{aligned}$$

Letting n tend to infinity and using (F_1), we have

$$G(d(Sw,z),0,d(z,Sw),0) \leq 0,$$

and it follows that $z = Sw$ by (F_2).

Thus $STz = Sw = z, TSw = Tz = w$, and so ST has a fixed point z and TS has a fixed point w.

To prove uniqueness, suppose that ST has a second fixed point z' and TS has a second fixed point w'. Then using inequality (5) , we have

$$\begin{aligned} F(\rho(Tz,TSw'),d(z,Sw'),\rho(w',w),0) &= \\ F(\rho(w,w'),d(Sw,Sw'),\rho(w',w),0) &\leq 0, \end{aligned}$$

which implies, by (F_2), that

(9) $$\rho(w,w') \leq hd(Sw,Sw').$$

Further, applying inequality (6), we have

$$\begin{aligned} G(d(STSw,STSw'),\rho(TSw,TSw'),d(Sw',Sw),0) &= \\ G(d(Sw,Sw'),\rho(w,w'),d(Sw',Sw),0) &\leq 0, \end{aligned}$$

which implies, by (F_2), that

(10) $$d(Sw,Sw') \leq h\rho(w,w').$$

It now follows from inequalities (9) and (10) that

$$\rho(w,w') \leq hd(Sw,Sw') \leq h^2\rho(w,w')$$

and so $w = w'$ since $h < 1$. The fixed point w of TS must therefore be unique.

Now $STz' = z'$ implies $TSTz' = Tz'$ and so $Tz' = w$. Thus

$$z = STz = Sw = STz' = z',$$

proving that z is the unique fixed point of ST. This completes the proof of the theorem.

Corollary 1. *[1] Theorem 1.*

P r o o f. Follows from Theorem 3 and Example 1.

Corollary 2. *[2] Let (X,d) and (Y,ρ) be complete metric spaces, let T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities*

$$\begin{aligned} d(Sy,STx)\max\{d(x,Sy),d(x,STx)\} &\le c_1\rho(y,Tx)\max\{\rho(y,Tx),d(x,STx)\},\\ \rho(Tx,TSy)\max\{\rho(y,Tx),\rho(y,TSy)\} &\le c_2 d(x,Sy)\max\{d(x,Sy),\rho(y,TSy)\} \end{aligned}$$

for all x in X and y in Y, where $0 \le c_1, c_2 < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

P r o o f. Follows from Theorem 3 and Example 3.

Corollary 3. *[2] Let (X,d) and (Y,ρ) be complete metric spaces, let T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities*

$$d^2(Sy,STx) \le c_1\max\{\rho(y,Tx)d(x,Sy),\rho(y,Tx)d(x,STx),d(x,Sy)d(x,STx)\},$$

$$\rho^2(Tx,TSy) \le c_2\max\{d(x,Sy)\rho(y,Tx),d(x,Sy)\rho(y,TSy),\rho(y,Tx)\rho(y,TSy)\}$$

for all x in X and y in Y, where $0 \le c_1, c_2 < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

P r o o f. Follows from Theorem 3 and Example 2.

Corollary 4. *Let (X,d) be a complete metric space. If S and T are mappings of X into itself satisfying the inequalities*

$$F(d(Tx,TSy),d(x,Sy),d(y,Tx),d(y,TSy)) \le 0, \tag{11}$$

$$G(d(Sy,STx),d(y,Tx),d(x,Sy),d(x,STx)) \le 0 \tag{12}$$

for all x, y in X, where $F, G \in \widetilde{\mathcal{F}}$, then ST has a unique fixed point z and TS has a unique fixed point w. Further, $Tz = w$ and $Sw = z$ and if $z = w$, then z is the unique fixed point of S and T.

P r o o f. The existence of z and w follows from the Theorem 3. If $z = w$, then z is of course a common fixed point of S and T.

Now suppose that T has a second fixed point z'. Than applying inequality (11) we have

$$\begin{aligned} F(d(Tz', TSz), d(z', Sz), d(z, Tz'), d(z, TSz)) &= \\ F(d(z', z), d(z', z), d(z', z), 0) &\leq 0. \end{aligned}$$

which implies, by (F_2), that $d(z', z) \leq hd(z', z)$ and then $d(z', z) = 0$. Similarly z is the unique fixed point of S.

4. Fixed points on two compact metric spaces.

Now let $\widetilde{\mathcal{F}^*}$ denote the set of all functions $F : R_+^4 \to R_+$ satisfying the following condition;
$(F_2)^*$: If either $F(u, v, u, 0) < 0$ or $F(u, v, 0, u) < 0$ for all $u, v \geq 0$, then $u < v$.

Theorem 4. *Let (X, d) and (Y, ρ) be compact metric spaces, let T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities*

$$F(\rho(Tx, TSy), d(x, Sy), \rho(y, Tx), \rho(y, TSy)) < 0 \tag{13}$$

for all x in X and y in Y with $x \neq Sy$, where $F \in \widetilde{\mathcal{F}^*}$ and

$$G(d(Sy, STx), \rho(y, Tx), d(x, Sy), d(x, STx)) < 0 \tag{14}$$

for all x in X and y in Y with $y \neq Tx$, where $G \in \widetilde{\mathcal{F}^*}$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

P r o o f. The function $\psi : X \to R_+$ defined by $\psi(x) = d(x, STx)$ is continuous on X. Since X is compact, there exists a point u in X such that

$$\psi(u) = d(u, STu) = \min\{d(x, STx); x \in X\}.$$

Now suppose that $Tu \neq TSTu$. Then $u \neq STu$.

Using inequality (14) and condition $(F_2)^*$ we have, since $Tu \neq TSTu$,

$$G(d(STu, STSTu), \rho(Tu, TSTu), 0, d(STu, STSTu)) < 0,$$

and it follovs that

$$d(STu, STSTu) < \rho(Tu, TSTu). \tag{15}$$

But using inequality (13) and condition $(F_2)^*$ we have, since $u \neq STu$,

$$F(\rho(Tu, TSTu), d(u, STu), 0, \rho(Tu, TSTu)) < 0,$$

and it follows that

$$\rho(Tu, TSTu) < d(u, STu). \tag{16}$$

We now deduce from inequalities (15) and (16) that

$$\psi(STu) = d(STu, STSTu) < d(u, STu) = \psi(u),$$

and this gives us a contradiction. So $Tu = TSTu$. If putting $Tu = w$ and $Sw = z$, then we get

$$ST(STu) = S(TSTu) = STu = Sw = z$$

and

$$w = Tu = TS(Tu) = T(STu) = Tz$$

Thus, $Sw = z$ is a fixed point of ST and $Tz = w$ is a fixed point of TS.

To prove uniqueness, suppose that ST has a second distinct fixed point z'. Then applying inequality (14) and we have, since $Tz \neq Tz'$,

$$\begin{aligned} G(d(STz, STz'), \rho(Tz, Tz'), d(z', z), 0) &= \\ G(d(z, z'), \rho(Tz, Tz'), d(z', z), 0) &\leq 0, \end{aligned}$$

which implies, by $(F_2)^*$ that

$$d(z, z') < \rho(Tz, Tz'). \tag{17}$$

Further, applying inequality (13) we have, since $z \neq z' = STz'$,

$$F(\rho(Tz, TSTz'), d(z, z'), \rho(Tz', Tz), 0) < 0,$$

which implies, by $(F_2)^*$ that

$$\rho(Tz, Tz') < d(z, z'). \tag{18}$$

It now follows from inequalities (17) and (18) that

$$d(z, z') < \rho(Tz, Tz') < d(z, z'),$$

which is a contradiction and so the fixed point z must be unique.

Similarly, w is the unique fixed point of TS. This completes the proof of the theorem.

From Theorem 4, we obtain the following corollary which was proved by Fisher [1].

Corollary 5. *[1] Let (X, d) and (Y, ρ) be compact metric spaces, let T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities*

$$\rho(Tx, TSy) < \max\{d(x, Sy), \rho(y, Tx), \rho(y, TSy)\},$$

for all x in X and y in Y with $x \neq Sy$ and

$$d(Sy, STx) < \max\{\rho(y, Tx), d(x, Sy), d(x, STx)\}$$

for all x in X and y in Y with $y \neq Tx$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Corollary 6 . *Let (X, d) be a compact metric space. If S and T are continuous mappings of X into itself satisfying the inequalities*

$$F(\rho(Tx, TSy), d(x, Sy), \rho(y, Tx), \rho(y, TSy)) < 0 \tag{19}$$

for all x, y in X with $x \neq Sy$, where $F \in \widetilde{\mathcal{F}^*}$ and

$$G(d(Sy, STx), \rho(y, Tx), d(x, Sy), d(x, STx)) < 0 \tag{20}$$

for all x, y in X with $y \neq Tx$, where $G \in \widetilde{\mathcal{F}^*}$, then ST has a unique fixed point z in X and TS has a unique fixed point w in X. Further, $Tz = w$ and $Sw = z$ and if $z = w$, then z is the unique fixed point of S and T.

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[1] *Valeriu Popa,*
Department of Mathematics and Physics,
University of Bacău,
5500 Bacău, Romania.
e-mail: vpopa@stiinte.ub.ro

[2] *Mustafa Telci,*
Department of Mathematics,
Trakya University,
22030, Edirne, Turkey.
e-mail: mtelci@trakya.edu.tr