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Implicit Difference Functional Inequalities and Applications

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The paper deals with initial boundary value problems for first order partial functional differential equations. A general class of implicit difference methods is investigated. A comparison result for functions satisfying implicit difference functional inequalities of the Volterra type is proved. This general idea is used in the investigation of the stability of difference schemes under the assumptions that the right hand sides of equations satisfy the nonlinear estimates of the Perron type with respect to the functional variable. A numerical example is given.

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1. Introduction

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X to Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let $a > 0$, $d_0 \in R_+ = [0,+\infty)$, $b = (b_1,\dots,b_n) \in R^n_+$ and $d = (d_1,\dots,d_n) \in R^n_+$ be given. For each $x = (x_1,\dots,x_n) \in R^n$ we write $x = (x', x'')$ where $x' = (x_1,\dots,x_\kappa)$, $x'' = (x_{\kappa+1},\dots,x_n)$, where $0 \le \kappa \le n$ is fixed. If $\kappa = n$ we have $x' = x$, if $\kappa = 0$ then $x'' = x$. We define the sets

$$E = [0,a] \times [-b', b') \times (-b'', b''], \quad D = [-d_0, 0] \times [0, d'] \times [-d'', 0].$$

Let $c = (c_1,\dots,c_n) = b + d$ and

$$E_0 = [-d_0, 0] \times [-b', c'] \times [-c'', b''],$$

$$\partial_0 E = ((0,a] \times [-b', c'] \times [-c'', b'']) \setminus E, \quad E^* = E_0 \cup E \cup \partial_0 E.$$

We consider a functional differential equation on the set E, initial boundary conditions are given on $E_0 \cup \partial_0 E$. The set D appears in the definition of the functional variable.

Suppose that $z : E^* \to R$ and $(t,x) \in [0,a] \times [-b,b]$ are fixed. We define the function $z_{(t,x)} : D \to R$ as follows

$$z_{(t,x)}(s,y) = z(t+s, x+y), \quad (s,y) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t-d_0, t] \times [x', x'+d'] \times [x''-d'', x'']$ and this restriction is shifted to the set D. For a function $w \in C(D,R)$ we write

$$\| w \|_D = \max \{ |w(t,x)| : (t,x) \in D \}.$$

Put $\Omega = E \times R \times C(D,R) \times R^n$ and suppose that $f : \Omega \to R$ and $\varphi : E_0 \cup \partial_0 E \to R$ are given functions. We consider the differential functional equation

(1.1) $$\partial_t z(t,x) = f(t, x, z(t,x), z_{(t,x)}, \partial_x z(t,x))$$

with the initial boundary condition

(1.2) $$z(t,x) = \varphi(t,x) \ \text{ on } \ E_0 \cup \partial_0 E.$$

A function $v : E^* \to R$ is called a classical solution of problem (1.1), (1.2) if

(i) $v \in C(E^*, R)$ and v is of class C^1 on E,

(ii) v satisfies equation (1.1) on E and condition (1.2) holds.

Differential equations with deviated variables and differential integral problems can be derived from (1.1) by specializing the operator f. Existence results for the mixed problem (1.1), (1.2) can be found in [3], Chapter 5.

The classical theory of differential inequalities generated by partial differential equations or systems was developed in monographs [5], [11], [13]. As it is well-known they found applications in several differential topics, in particular about estimates of solutions of differential problems, estimates of the domain of solutions, estimates of difference between two solutions, criterions of uniqueness and continuous dependence. The monograph [3] contains an exposition of recent developments hyperbolic functional differential inequalities and applications.

In the recent years a number of papers concerned with difference and functional difference inequalities were published [1], [4], [6]-[10], [12]. These inequalities usually are explored for to establish error estimates and convergence results for difference methods generated by differential or functional differential problems.

In the paper we start the investigation of implicit difference functional inequalities generated by mixed problems for nonlinear equations. We prove that under natural assumptions on given functions and on the mesh there is a class of implicit difference schemes for a mixed problem which is convergent.

The paper is organized as follows. In Section 2 we construct a general class of implicit difference schemes for (1.1), (1.2). Section 3 deals with a comparison result for difference functional inequalities. A convergence theorem and an error estimate for implicit difference methods are presented in Section 4. A numerical examples is given in the last part of the paper.

2. Discretization

Let us denote by $F(X,Y)$ the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. Let $\mathbf{N}$ and $\mathbf{Z}$ be the sets of natural numbers and integers, respectively. For $x, y \in R^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ we put

$$\| x \| = \sum_{i=1}^{n} |x_i| \ \text{ and } \ x \diamond y = (x_1 y_1, \dots, x_n y_n) \in R^n.$$

We define a mesh on the set E^* and D in the following way. Let (h_0, h'), $h' = (h_1, \dots, h_n)$, stand for steps of the mesh. For $h = (h_0, h')$ and $(r, m) \in \mathbf{Z}^{1+r}$, where $m = (m_1, \dots, m_n)$, we define nodal points as follows

$$t^{(r)} = r * h_0, \ \ x^{(m)} = m \diamond h', \ \ x^{(m)} = (x_1^{(m_1)}, \dots, x_n^{(m_n)}).$$

Let us denote by H the set of all $h = (h_0, h')$ such that there are $K_0 \in \mathbf{Z}$ and $N = (N_1, \dots, N_n) \in \mathbf{N}^n$ with the properties $K_0 h_0 = d_0$ and $N \diamond h' = d$. We assume that $H \neq \emptyset$. Let $K \in \mathbf{N}$ be defined by the relations $K h_0 \le a < (K+1) h_0$. Write

$$R_h^{1+n} = \{ (t^{(r)}, x^{(m)}) : (r, m) \in \mathbf{Z}^{1+n} \}$$

and

$$E_h = E \cap R_h^{1+n}, \ \ E_{h.0} = E_0 \cap R_h^{1+n}, \ \ D_h = D \cap R_h^{1+n},$$
$$\partial_0 E_h = \partial_0 E \cap R_h^{1+n}, \ \ E_h^* = E_{h.0} \cup E_h \cup \partial_0 E_h.$$

We consider a functional difference equation on the set E_h, initial boundary conditions are given on $E_{0.h} \cup \partial_0 E_h$. The set D_h appears in the definition of the functional variable. Moreover we put

$$E_{h.r} = E_h^* \cap ([-d_0, t^{(r)}] \times R^n)$$

where $0 \le r \le K$ and

$$I_h = \{t^{(r)} : 0 \le r \le K\}, \;\; I'_h = I_h \setminus \{t^{(K)}\}.$$

For functions $z : E^*_h \to R$ and $w : I_h \to R$ we write

$$z^{(r,m)} = z(t^{(r)}, x^{(m)}), \;\; w^{(r)} = w(t^{(r)}).$$

For the above z and for a point $(t^{(r)}, x^{(m)}) \in E_h$ we define the function $z_{[r,m]} : D_h \to R$ by the formula

$$z_{[r,m]}(s, y) = z(t^{(r)} + s, x^{(m)} + y), \;\; (s, y) \in D_h.$$

The function $z_{[r,m]}$ is the restriction of z to the set

$$([t^{(r)} - d_0] \times [x^{(m')}, x^{(m')} + d'] \times [x^{(m'')} - d'', x^{(m'')}]) \cap R^{1+n}_h$$

and this restriction is shifted to the set D_h. For a function $w : D_h \to R$ we write

$$\| w \|_{D_h} = \max\{|w^{(r,m)}| : (t^{(r)}, x^{(m)}) \in D_h\}.$$

Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$, $1 \le j \le n$, where 1 is the j-th coordinate. We consider difference operators δ_0 and $\delta = (\delta_1, \ldots, \delta_n)$ defined in the following way

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}\Big(z^{(r+1,m)} - z^{(r,m)}\Big)$$

and

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\Big(z^{(r,m+e_i)} - z^{(r,m)}\Big), \;\; 1 \le i \le \kappa, \tag{2.1}$$

$$\delta_i z^{(r,m)} = \frac{1}{h_i}\Big(z^{(r,m)} - z^{(r,m-e_i)}\Big), \;\; \kappa + 1 \le i \le n. \tag{2.2}$$

Set $\delta z^{(r,m)} = (\delta_1 z^{(r,m)}, \ldots, \delta_n z^{(r,m)})$. Note that $\delta z^{(r,m)}$ is given by (2.2) if $\kappa = 0$ and $\delta z^{(r,m)}$ is defined by (2.1) for $\kappa = n$.

Put $\Omega_h = E'_h \times R \times F(D_h, R) \times R^n$ where

$$E'_h = \{(t^{(r)}, x^{(m)}) \in E_h : (t^{(r+1)}, x^{(m)}) \in E_h\}$$

and suppose that

$$f_h : \Omega_h \to R, \;\; \varphi_h : E_{h.0} \cup \partial_0 E_h \to R$$

are given functions. We will approximate classical solutions of (1.1), (1.2) with solutions of the functional difference equation

$$\delta_0 z^{(r,m)} = f_h(t^{(r)}, x^{(m)}, z^{(r,m)}, z_{[r,m]}, \delta z^{(r+1,m)}) \tag{2.3}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \text{ on } E_{h.0} \cup \partial_0 E_h. \tag{2.4}$$

Let us denote by F_h the Niemycki operator corresponding to (2.3) i. e.

$$F_h[z]^{(r,m)} = f_h(t^{(r)}, x^{(m)}, z^{(r,m)}, z_{[r,m]}, \delta z^{(r+1,m)}).$$

Problem (2.3), (2.4) has the following property: the unknown function is a functional variable and the difference operators appear in a classical sense.

3. Difference functional inequalities

We begin with a maximum principle for implicit difference functional inequalities generated by (2.3), (2.4). Suppose that the function

$$\lambda_n = (\lambda_{h.1}, \ldots, \lambda_{h.n}) : E'_h \times F(D_h, R) \to R^n$$

is given. Consider the implicit functional difference equation

$$z^{(r+1,m)} = h_0 \sum_{i=1}^{n} \lambda_{h.i}(t^{(r)}, x^{(m)}, z_{[r,m]}) \delta_i z^{(r+1,m)}.$$

The maximum principle asserts that solutions of difference functional inequalities generated by the above equation cannot have a positive maximum (or a negative minimum) on the set E_h.

Write

$$B = [-b', b') \times (-b'', b''], \quad B^* = [-b', c'] \times [-c'', b'']$$

and

$$R_h^n = \{x^{(m)} : m \in \mathbf{Z}^n\}$$

where $h \in H$. We consider the sets

$$B_h = B \cap R_h^n, \quad B_h^* = B^* \cap R_h^n$$

and $\partial_0 B_h = B_h^* \setminus B_h$. Define $\theta = (\theta_1, \ldots, \theta_n)$ where

$$\theta_i = 1 \text{ for } 1 \le i \le \kappa \text{ and } \theta_i = -1 \text{ for } \kappa + 1 \le i \le n.$$

Theorem 3.1. *Suppose that* $0 \le r \le K - 1$ *is fixed and the function* $\lambda_h = (\lambda_{h.1}, \ldots, \lambda_{h.n}) : E_{h.r} \times F(D_h, R) \to R^n$ *is such that*

$$\lambda_h(t, x, w) \diamond \theta \ge 0 \text{ on } E_{h.r} \times F(D_h, R).$$

(I) *If* $w : E_{h.r+1} \to R$ *satisfies the implicit difference inequality*

$$w^{(r+1,m)} \le h_0 \sum_{i=1}^{n} \lambda_{h.i}(t^{(r)}, x^{(m)}, w_{[r,m]}) \delta_i w^{(r+1,m)}$$

for $x^{(m)} \in B_h$ *and* $\mu \in \mathbf{Z}^n$, $\mu = (\mu_1, \dots, \mu_n)$, *is such that*

$$w^{(r+1,\mu)} = M$$

where

$$M = \max\{w^{(r+1,m)} : x^{(m)} \in B_h^*\} \ \text{and} \ M > 0 \tag{3.1}$$

then $x^{(\mu)} \in \partial_0 B_h$.

(II) *If* $w : E_{h.r+1} \to R$ *satisfies the implicit difference inequality*

$$w^{(r+1,m)} \ge h_0 \sum_{i=1}^{n} \lambda_{h.i}(t^{(r)}, x^{(m)}, w) \delta_i w^{(r+1,m)}$$

for $x^{(m)} \in B_h$ *and* $\mu \in \mathbf{Z}^n$ *is such that*

$$w^{(r+1,\mu)} = \widetilde{M}$$

where

$$\widetilde{M} = \min\{w^{(r+1,m)} : x^{(m)} \in B_h^*\} \ \text{and} \ \widetilde{M} < 0$$

then $x^{(\mu)} \in \partial_0 B_h$.

P r o o f. Consider the case (I). Suppose that $x^{(\mu)} \in B_h$. Then

$$w^{(r+1,\mu)} \le h_0 \sum_{i=1}^{\kappa} \frac{1}{h_i} \lambda_{h.i}(t^{(r)}, x^{(\mu)}, w_{[r,\mu]}) \Big[w^{(r+1,\mu+e_i)} - w^{(r+1,\mu)} \Big]$$

$$+ h_0 \sum_{i=\kappa+1}^{n} \frac{1}{h_i} \lambda_{h.i}(t^{(r)}, x^{(\mu)}, w_{[r,\mu]}) \Big[w^{(r+1,\mu)} - w^{(r+1,\mu-e_i)} \Big].$$

This gives

$$w^{(r+1,\mu)} \Big[1 + h_0 \sum_{i=1}^{n} \frac{1}{h_i} |\lambda_{h.i}(t^{(r)}, x^{(\mu)}, w_{[r,\mu]})| \Big]$$

$$\le h_0 \sum_{i=1}^{\kappa} \frac{1}{h_i} \lambda_{h.i}(t^{(r)}, x^{(\mu)}, w_{[r,\mu]}) w^{(r+1,\mu+e_i)}$$

$$-h_0 \sum_{i=\kappa+1}^{n} \frac{1}{h_i} \lambda_{h.i}(t^{(r)}, x^{(\mu)}, w_{[r,\mu]}) w^{(r+1,\mu-e_i)}$$

$$\leq h_0 M \sum_{i=1}^{n} \frac{1}{h_i} |\lambda_{h.i}(t^{(r)}, x^{(\mu)}, w_{[r,\mu]})|.$$

We thus get $w^{(r+1,\mu)} \leq 0$ which contradicts (3.1). Then $x^{(\mu)} \in \partial_0 B_h$ which is our claim. In a similar way we prove that $x^{(\mu)} \in \partial_0 B_h$ in the case (II). This completes the proof. ∎

Assumption $H[f_h]$. Suppose that the function $f_h : \Omega_h \to R$ of the variables (t, x, p, w, q) is such that

1) f_h is nondecreasing with respect to the functional variable and for each $P = (t, x, p, w, q) \in \Omega_h$ there exist the partial derivatives

$$\partial_p f_h(P),\ \partial_q f_h(P) = (\partial_{q_1} f_h(P), \ldots, \partial_{q_n} f_h(P))$$

and

$$\partial_p f_h(t, x, \cdot, w, \cdot) \in C(R^{1+n}, R),\ \partial_q f_h(t, x, \cdot, w, \cdot) \in C(R^{1+n}, R^n),$$

2) for each $P \in \Omega_h$ we have

$$\partial_q f_h(P) \diamond \theta \geq 0$$

and

$$1 + h_0 \partial_p f_h(P) \geq 0.$$

Now we formulate a theorem on implicit functional difference inequalities.

Theorem 3.2. *Suppose that Assumption $H[f_h]$ is satisfied and the functions $u, v : E_h^* \to R$ are such that the implicit difference inequality*

$$\delta_0 u^{(r,m)} - F_h[u]^{(r,m)} \leq \delta_0 v^{(r,m)} - F_h[v]^{(r,m)},\ \ (t^{(r)}, x^{(m)}) \in E_h', \tag{3.2}$$

and the initial boundary estimate

$$u^{(r,m)} \leq v^{(r,m)} \ \text{ on } \ E_{h.0} \cup \partial_0 E_h \tag{3.3}$$

are satisfied.

Then

$$u^{(r,m)} \leq v^{(r,m)} \ \text{ on } \ E_h^*. \tag{3.4}$$

Proof. We prove (3.4) by induction on r. It follows from (3.3) that assertion (3.4) is satisfied for $r \le 0$ and $(t^{(r)}, x^{(m)}) \in E_h$. Suppose now that $u^{(i,m)} \le v^{(i,m)}$ for $(t^{(i)}, x^{(m)}) \in E_{h.r}$ where $0 \le r < K$. It follows easily that

$$(u-v)^{(r+1,m)} \le (u-v)^{(r,m)}$$
$$+h_0\Big[f_h(t^{(r)}, x^{(m)}, u^{(r,m)}, u_{[r,m]}, \delta u^{(r+1,m)}) - f(t^{(r)}, x^{(m)}, v^{(r,m)}, v_{[r,m]}, \delta v^{(r+1,m)})\Big]$$
$$\le (u-v)^{(r,m)}\Big[1 + h_0 \partial_p f(Q)\Big]$$
$$+h_0\Big[f_h(t^{(r)}, x^{(m)}, v^{(r,m)}, u_{[r,m]}, \delta u^{(r+1,m)}) - f_h(t^{(r)}, x^{(m)}, v^{(r,m)}, v_{[r,m]}, \delta u^{(r+1,m)})\Big]$$
$$+h_0 \sum_{j=1}^{n} \partial_{q_j} f(Q) \delta_j (u-v)^{(r+1,m)}$$

where $x^{(m)} \in B_h$ and Q is an intermediate point. We thus get

$$(u-v)^{(r+1,m)} \le h_0 \sum_{j=1}^{n} \partial_{q_j} f(Q) \delta_j (u-v)^{(r+1,m)},$$

where $x^{(m)} \in B_h$. It follows from (3.3) and from Theorem (3.1) that $(u - v)^{(r+1,m)} \le 0$ for $x^{(m)} \in \partial_0 B_h$. Then we obtain (3.4) by induction and the theorem follows. ■

4. Implicit difference methods for mixed problems

We formulate sufficient conditions for the convergence of the method (2.3), (2.4). We first prove that there exists exactly one solution $u_h : E_h^* \to R$ of problem (2.3), (2.4). For each $x^{(m)} \in B_h$ we put

$$\Delta^{(m)} = \{x^{(m+e_j)} : 1 \le j \le \kappa\} \cup \{x^{(m-e_j)} : \kappa + 1 \le j \le n\}.$$

Lemma 4.1. *If Assumption $H[f_h]$ is satisfied and $\varphi_h : E_{h.0} \cup \partial_0 E_h \to R$ then there exists exactly one solution $u_h : E_h^* \to R$ of (2.3), (2.4).*

Proof. Is follows from (2.4) that u_h is defined on $E_{h.0}$. Suppose now that $0 \le r < K$ is fixed and that u_h is defined on $E_{h.r}$. Consider the problem

$$z^{(r+1,m)} = u_h^{(r,m)} + h_0 f_h(t^{(r)}, x^{(m)}, u_h^{(r,m)}, (u_h)_{[r,m]}, \delta z^{(r+1,m)}) \tag{4.1}$$

$$u_h^{(r+1,m)} = \varphi_h^{(r+1,m)} \text{ for } x^{(m)} \in \partial_0 B_h. \tag{4.2}$$

Suppose now that the numbers $u_h(t^{(r+1)}, y)$ where $y \in \Delta^{(m)}$ are known. Write

$$\psi(\tau) = u_h^{(r,m)} + h_0 f_h(t^{(r)}, x^{(m)}, u_h^{(r,m)}, (u_h)_{[r,m]}, Q^{(r+1,m)}(\tau)),$$

where

$$Q^{(r+1,m)}(\tau) = \Big(\frac{1}{h_1}\Big(u_h^{(r+1,m+e_1)} - \tau\Big), \dots, \frac{1}{h_\kappa}\Big(u_h^{(r+1,m+e_\kappa)} - \tau\Big),$$
$$\frac{1}{h_{\kappa+1}}\Big(\tau - u_h^{(r+1,m-e_{\kappa+1})}\Big) \dots, \frac{1}{h_n}\Big(\tau - u_h^{(r+1,m-e_n)}\Big)\Big).$$

Then $\psi : R \to R$ is of class C^1 and

$$\psi'(\tau) = -h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\partial_{q_j} f_h(t^{(r)}, x^{(m)}, u_h^{(r,m)}, (u_h)_{[r,m]}, Q^{(r+1,m)}(\tau))| \le 0$$

for $\tau \in R$. Then equation $\tau = \psi(\tau)$ has exactly one solution and consequently the number $u_h^{(r+1,m)}$ can be calculated.

Since $u_h^{(r+1,m)}$ is given for $x^{(m)} \in \partial_0 B_h$ it follows that there exists exactly one solution $u_h^{(r+1,m)}$ of (4.1), (4.2) for $x^{(m)} \in B_h$. Then u_h is defined on $E_{h.r+1}$. Then by induction the solution exists and it is unique on E_h^*. ■

Suppose that the functions $v_h : E_h^* \to R$ and $\alpha_0, \gamma : H \to R_+$ are such that

$$|\delta_0 v_h^{(r,m)} - F_h[v_h]^{(r,m)}| \le \gamma(h) \text{ on } E_h', \tag{4.3}$$

$$|\varphi_h^{(r,m)} - v_h^{(r,m)}| \le \alpha_0(h) \text{ on } E_{h.0} \cup \partial_0 E_h \tag{4.4}$$

and

$$\lim_{h\to 0} \gamma(h) = 0, \quad \lim_{h\to 0} \alpha_0(h) = 0. \tag{4.5}$$

The function v_h satisfying the above relations can be considered as an approximate solution to problem (2.3), (2.4).

We give a theorem on the estimate of the difference between the exact and approximate solutions to problem (2.3), (2.4).

Assumption $H[f_h, \sigma_h]$. Suppose that Assumption $H[f_h]$ is satisfied and

1) there is $L \in R_+$ such that $|\partial_p f_h(P)| \le L$ for $P = (t, x, p, w, q) \in \Omega$,

2) there is $\sigma_h : I_h' \times R_+ \to R_+$ such that

(i) for each $t \in I'_h$ the function $\sigma_h(t, \cdot) : R_+ \to R_+$ is continuous and it is nondecreasing,

(ii) $\sigma_h(t, 0) = 0$ for $t \in I'_h$ and the difference problem,

$$\eta^{(r+1)} = \eta^{(r)} + h_0\sigma_h(t^{(r)}, \eta^{(r)}) + h_0 L\eta^{(r)}, \quad 0 \le r \le K-1, \tag{4.6}$$

$$\eta^{(0)} = 0 \tag{4.7}$$

is stable in the following sense: if $\gamma, \alpha_0 : H \to R_+$ are such functions that

$$\lim_{h\to 0} \gamma(h) = 0, \quad \lim_{h\to 0} \alpha_0(h) = 0$$

and $\eta_h : I_h \to R_+$ is a solution of the problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0\sigma_h(t^{(r)}, \eta^{(r)}) + h_0 L\eta^{(r)} + h_0\gamma(h), \tag{4.8}$$

$$\eta^{(0)} = \alpha_0(h), \tag{4.9}$$

where $0 \le r \le K-1$, then there is $\alpha : H \to R_+$ such that

$$\eta_h^{(r)} \le \alpha(h) \tag{4.10}$$

for $0 \le r \le K$ and $\lim_{h\to 0} \alpha(h) = 0$,

(iii) if $w, \overline{w} \in F(D_h, R)$ and $w \le \overline{w}$, then

$$f_h(t^{(r)}, x^{(m)}, p, w, q) - f_h(t^{(r)}, x^{(m)}, p, \overline{w}, q) \le \sigma_h(t^{(r)}, \| w - \overline{w} \|_{D_h}).$$

Now we formulate the main result in this section.

Theorem 4.2. *Suppose that Assumption $H[f_h, \sigma_h]$ is satisfied and*

1) $\varphi_h : E_{h.0} \cup \partial_0 E_h$ is a given function and $u_h : E_h^* \to R$ is a solution of problem (2.3), (2.4),

2) the functions $v_h : E_h^* \to R$ and $\gamma, \alpha_0 : H \to R_+$ are such that relations (4.3)-(4.5) hold.

Then there exists a function $\alpha : H \to R_+$ such that

$$|u_h^{(r,m)} - v_h^{(r,m)}| \le \alpha(h) \ \ \text{on} \ \ E_h \ \ \text{and} \ \ \lim_{h\to 0} \alpha(h) = 0. \tag{4.11}$$

P r o o f. Let the function $C_h : E'_h \to R$ be defined by

$$\delta_0 v_h^{(r,m)} = F_h[v_h]^{(r,m)} + C_h^{(r,m)}. \tag{4.12}$$

It follows from (4.3) that $|C_h^{(r,m)}| \le \gamma(h)$ on E'_h.

Consider the difference problem (4.8), (4.9) and its solution $\omega_h : I_h \to R$. Write

$$\widetilde{v}_h^{(r,m)} = v_h^{(r,m)} + \alpha_0(h) \text{ on } E_{h.0},$$
$$\widetilde{v}_h^{(r,m)} = v_h^{(r,m)} + \omega_h^{(r,m)} \text{ on } E_h \cup \partial_0 E_h.$$

We prove that $\widetilde{v}_h$ satisfies the difference inequality

$$\delta_0 \widetilde{v}_h^{(r,m)} \ge F_h[\widetilde{v}_h]^{(r,m)} \text{ on } E'_h. \tag{4.13}$$

We conclude from Assumption $H[f_h, \sigma_h]$ and (4.12) that

$$\delta_0 \widetilde{v}_h^{(r,m)} = F_h[\widetilde{v}_h]^{(r,m)} + \frac{1}{h_0}\Big(\omega_h^{(r+1)} - \omega_h^{(r)}\Big) + C_h^{(r,m)}$$
$$+\Big[f_h(t^{(r)}, x^{(m)}, v_h^{(r,m)}, (v_h)_{[r,m]}, \delta v_h^{(r+1,m)})$$
$$-f_h(t^{(r)}, x^{(m)}, \widetilde{v}_h^{(r,m)}, (\widetilde{v}_h)_{[r,m]}, \delta v_h^{(r+1,m)})\Big]$$
$$\ge F_h[\widetilde{v}_h]^{(r,m)} + \frac{1}{h_0}\Big(\omega_h^{(r+1,m)} - \omega_h^{(r,m)}\Big) - \sigma_h(t^{(r)}, \omega_h^{(r)}) - L\omega_h^{(r)} - \gamma(h),$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. The above inequality and (4.8) imply (4.13). By the initial boundary estimate

$$u_h^{(r,m)} \le \widetilde{v}_h^{(r,m)} \text{ on } E_{h.0} \cup \partial_0 E_h \tag{4.14}$$

and (4.13) and Theorem 3.2 we obtain

$$u_h^{(r,m)} \le v_h^{(r,m)} + \omega_h^{(r,m)} \text{ on } E_h. \tag{4.15}$$

In a similar manner we can see that

$$v_h^{(r,m)} - \omega_h^{(r,m)} \le u_h^{(r,m)} \text{ on } E_h. \tag{4.16}$$

Now we obtain the assertion of Theorem 4.2 from (4.15), (4.16) and from the stability of problem (4.6), (4.7). ■

We consider a class of difference problems (2.3), (2.4) where f_h is a superposition of f and a suitable interpolating operator.

Assumption $H[T_h]$. Suppose that the operator $T_h : F(D_h, R) \to F(D, R)$ satisfies the conditions

1) if $w \in F(D_h, R)$ then $T_h[w] \in C(D, R)$ and

$$\| T_h[w] \|_D = \| w \|_{D_h},$$

2) if $w, \widetilde{w} \in F(D_h, R)$ then $T_h[w + \widetilde{w}] = T_h[w] + T_h[\widetilde{w}]$, if we assume that $w \leq \widetilde{w}$ then $T_h[w] \leq T_h[\widetilde{w}]$,

3) if $w : D \to R$ is of class C^1 then there is $\gamma : H \to R_+$ such that

$$\| T_h[w_h] - w \|_D \leq \gamma(h) \text{ and } \lim_{h \to 0} \gamma(h) = 0,$$

where w_h is the restriction of w to the set D_h.

R e m a r k 4.3. It is easy to see that the interpolating operator considered in [3] satisfies Assumption $H[T_h]$.

Assumption $H[f, \sigma]$. Suppose that the function $f : \Omega \to R$ of the variables (t, x, p, z, q) is such that

1) f is continuous and it is nondecreasing with respect to the functional variable,

2) for each $P = (t, x, p, w, q) \in \Omega$ there exist the partial derivatives

$$\partial_p f(P), \;\; \partial_q f(P) = (\partial_{q_1} f(P), \dots, \partial_{q_n} f(P))$$

and $\partial_p f \in C(\Omega, R)$, $\partial_q f \in C(\Omega, R^n)$,

3) for each $P \in \Omega$ we have

$$\partial_q f(P) \diamond \theta \geq 0 \text{ and } 1 + h_0 \partial_p f(P) \geq 0$$

and there is $L \in R_+$ such that $|\partial_p f(P)| \leq L$ on Ω,

4) there is $\sigma : [0, a] \times R_+ \to R_+$ such that

   (i) σ is continuous and it is nondecreasing with respect to both variables,

   (ii) $\sigma(t, 0) = 0$ for $t \in [0, a]$ and for each $\varepsilon \geq 0$ the maximal solution $\omega(\cdot, \varepsilon)$ of the Cauchy problem

   $$w'(t) = Lw(t) + \sigma(t, w(t)) + \varepsilon, \;\; w(0) = \varepsilon \qquad (4.17)$$

   is defined on $[0, a]$ and $\omega(t, 0) = 0$ for $t \in (0, a)$,

(iii) the estimate

$$f(t,x,p,w,q) - f(t,x,p,\overline{w},q) \le \sigma(t, \| w - \overline{w} \|_D)$$

is satisfied on Ω for $w \ge \overline{w}$.

We will approximate solutions of (1.1), (1.2) with solutions of the difference functional equation

$$\delta_0 z^{(r,m)} = f(t^{(r)}, x^{(m)}, z^{(r,m)}, T_h z_{[r,m]}, \delta z^{(r+1,m)}) \tag{4.18}$$

with the initial boundary condition (2.4).

Theorem 4.4. *Suppose that Assumptions $H[T_h]$ and $H[f,\sigma]$ are satisfied and*

1) *$h \in H$ and the function $u_h : E_h^* \to R$ is a solution of (2.4), (4.18) and there is $\alpha_0 : H \to R_+$ such that*

$$|\varphi^{(r,m)} - \varphi_h^{(r,m)}| \le \alpha_0(h) \ \text{ on } \ E_{h.0} \cup \partial_0 E_h \ \text{ and } \ \lim_{h\to 0} \alpha_0(h) = 0, \tag{4.19}$$

2) *$v : E^* \to R$ is a solution of (1.1), (1.2) and v is of class C^1 on E^*.*

Then there is $\alpha : H \to R_+$ such that

$$|u_h^{(r,m)} - v_h^{(r,m)}| \le \alpha(h) \ \text{ and } \ \lim_{h\to 0} \alpha(h) = 0. \tag{4.20}$$

Proof. Write

$$f_h(t,x,p,z,q) = f(t,x,p,T_h z,q) \ \text{ on } \ \Omega_h$$

Then f_h is nondecreasing with respect to the functional variable. Moreover we have the estimate

$$f_h(t,x,p,w,q) - f_h(t,x,p,\overline{w},q) \le \sigma(t, \| w - \overline{w} \|_{D_h})$$

on Ω_h for $w \ge \overline{w}$. Let the function $C_h : E_h' \to R$ be defined by (4.12). Then there is $\gamma : H \to R_+$ such that $|C_h^{(r,m)}| \le \gamma(h)$ on E_h' and $\lim_{h\to 0} \gamma(h) = 0$. It follows that v_h is an approximate solution to (2.4), (4.18). Now we prove that the difference problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0 \sigma(t^{(r)}, \eta^{(r)}) + h_0 L \eta^{(r)} \ \text{ for } \ 0 \le r \le K-1, \ \eta(0) = 0$$

is stable. Suppose that $\gamma, \alpha_0 : H \to R_+$ are such functions that

$$\lim_{h\to 0} \gamma(h) = 0, \quad \lim_{h\to o} \alpha_0(h) = 0.$$

Consider the solution $\eta_h : I_h \to R_+$ of the problem

$$\eta^{(r+1)} = \eta^{(r)} + h_0\sigma(t^{(r)}, \eta^{(r)}) + h_0 L\eta^{(r)} + h_0\gamma(h), \quad 0 \le r \le K-1,$$

$$\eta(0) = \alpha_0(h).$$

Let us denote by $\omega_h : [0, a) \to R_+$ the solution of problem (4.17) with $\varepsilon = \max\{\gamma(h), \alpha_0(h)\}$. It is easily seen that $\eta_h^{(i)} \le \omega_h^{(i)}$ for $0 \le i \le K$ and

$$\lim_{h\to 0} \omega_h(t) = 0 \text{ uniformly on } [0, a].$$

Then we have $\eta_h^{(i)} \le \omega_h(a)$ for $0 \le i \le K$ and problem (4.17) with $\varepsilon = 0$ is stable. Then all the conditions of Theorem 4.2 are satisfied and the assertion (4.20) follows. ∎

Now we give an error estimate for method (2.4), (4.18).

Lemma 4.5. *Suppose that*

1) *the solution $v : E^* \to R$ of differential problem (1.1), (1.2) is of class C^2 and the assumptions of Theorem 4.4 are satisfied with $\sigma(t, p) = Lp$, $L > 0$,*
2) *there exist $\widetilde{M} = (\widetilde{M}_1, \ldots, \widetilde{M}_n) \in R_+^n$, such that*

$$|\partial_{q_j} f(t, x, p, z, q)| \le \widetilde{M}_j$$

and $\widetilde{C} \in R_+$ is such a constant that

$$|\partial_{tt} v(t, x)| \le \widetilde{C}, \quad |\partial_{x_j x_j} v(t, x)| \le \widetilde{C}$$

where $j = 1, \ldots, n$.

Then

$$|u_h^{(r,m)} - v_h^{(r,m)}| \le \widetilde{\eta}_h^{(r)}, \tag{4.21}$$

where

$$\widetilde{\eta}_h^{(r)} = \alpha_0(h)(1 + 2h_0L)^r + \widetilde{\gamma}(h)\frac{(1 + 2h_0L)^r - 1}{2L}$$

and

$$\widetilde{\gamma}(h) = \frac{1}{2}\widetilde{C}h_0 + L\gamma(h) + \frac{\widetilde{C}}{2}\sum_{j=1}^{n} h_j\widetilde{M}_j.$$

P r o o f. From the assumptions of Lemma we conclude that the operators δ_0, δ satisfy the following conditions

$$|\delta_0 v^{(r,m)} - \partial_t v^{(r,m)}| \leq \frac{1}{2}\widetilde{C}h_0,$$

$$|\delta_j v^{(r,m)} - \partial_{x_j} v^{(r,m)}| \leq \frac{1}{2}\widetilde{C}h_j, \quad j = 1, \ldots, n.$$

It follows from above estimates and from Assumption $H[T_h]$ that

$$|C_h^{(r,m)}| = |\delta_0 v^{(r,m)} - \partial_t v^{(r,m)} +$$

$$f(t^{(r)}, x^{(m)}, v^{(r,m)}, v_{[r,m]}, \partial_x v^{(r,m)}) - f(t^{(r)}, x^{(m)}, v^{(r,m)}, T_h v_{[r,m]}, \delta v^{(r,m)})| \leq \widetilde{\gamma}(h).$$

The function $\widetilde{\eta}_h$ is a solution of the problem

$$\eta^{(r+1)} = \eta^{(r)}(1 + 2h_0 L) + h_0 \widetilde{\gamma}(h).$$

Then from Theorem 3.2 we get the assertion (4.21). ■

R e m a r k 4.6. The stability of difference equations generated by hyperbolic equations or systems is strictly connected with the so-called Courant-Friedrichs-Levy (CFL) condition (see [2], Chapter 3). Let us consider the explicit Euler method

$$\delta_0 z^{(r,m)} = f(t^{(r)}, x^{(m)}, z^{(r,m)}, T_h z_{[r,m]}, \delta z^{(r,m)}) \tag{4.22}$$

with initial boundary condition (2.4).

The (CFL) condition for (4.22) has the form ([3], Chapter 5)

$$1 - h_0 \sum_{j=1}^{n} \frac{1}{h_j} |\partial_{q_j} f(P)| \geq 0, \quad P = (t, x, p, w, q) \in \Omega_h. \tag{4.23}$$

Note that we have omitted assumption (4.23) for implicit Euler method.

5. Numerical example

For $n = 2$ we put

$$E = [0, 0.5] \times [-1, 1) \times (-1, 1], \quad E_0 = \{0\} \times [-1, 1] \times [-1, 1],$$

$$\partial_0 E = [0, 0.5] \times \{1\} \times \{-1\}.$$

Consider the quasilinear differential equation with deviated variables

$$\partial_t z(t,x,y) = \Big[1 + y\sin\big(z(t,x,y)\big)\Big]\partial_x z(t,x,y) \tag{5.1}$$
$$-\Big[1 - x\sin\big(z(t,x,y)\big)\Big]\partial_y z(t,x,y)$$
$$+z\Big(t, 0.5(x+y), 0.5(x-y)\Big) + \Big(x^2 - y^2 - 2yt - 2xt\Big)z(t,x,y) - e^{txy},$$

with the initial boundary condition

$$\begin{aligned} z(0,x,y) &= 1, && (x,y)\in[-1,1]\times[-1,1],\\ z(t,1,y) &= e^{t(1-y^2)}, && t\in[0,1],\ \ y\in[-1,1],\\ z(t,x,-1) &= e^{t(x^2-1)}, && t\in[0,1],\ \ x\in[-1,1]. \end{aligned} \tag{5.2}$$

The solution of the above problem is given by

$$\widetilde{z}(t,x) = e^{t(x^2-y^2)}.$$

Let us denote by u_h the solution of the implicit difference problem

$$\delta_0 z^{(r,m,k)} = \Big[1 + y^{(k)}\sin\big(z^{(r,m,k)}\big)\Big]\delta_1 z^{(r+1,m,k)}$$
$$-\Big[1 - x^{(m)}\sin\big(z^{(r,m,k)}\big)\Big]\delta_2 z^{(r+1,m,k)} + T_h z\Big(t^{(r)}, 0.5(x^{(m)}+y^{(k)}), 0.5(x^{(m)}-y^{(k)})\Big)$$
$$+\Big((x^{(m)})^2 - (y^{(k)})^2 - 2y^{(k)}t^{(r)} - 2x^{(m)}t^{(r)}\Big)z^{(r,m,k)} - \exp\big(t^{(r)}x^{(m)}y^{(k)}\big)$$

Let $\widetilde{u}_h$ denote the solution of a classical difference problem corresponding to (5.1), (5.2). We give the following information on errors of the methods. Write

$$\varepsilon_h^{(r)} = \frac{1}{4(N-1)^2}\sum_{m=-N}^{N-1}\sum_{k=-N+1}^{N}|u_h^{(r,m,k)} - \widetilde{z}^{(r,m,k)}|,$$

$$\widetilde{\varepsilon}_h^{(r)} = \frac{1}{4(N-1)^2}\sum_{m=-N}^{N-1}\sum_{k=-N+1}^{N}|\widetilde{u}_h^{(r,m,k)} - \widetilde{z}^{(r,m,k)}|.$$

The numbers $\varepsilon_h^{(r)}$ and $\widetilde{\varepsilon}_h^{(r)}$ are the arithmetical mean of the errors with fixed $t^{(r)}$. The values of the functions ε_h and $\widetilde{\varepsilon}_h$ are listed in the table. We write " $\times$ " for $\widetilde{\varepsilon}_h^{(r)} > 100$.

Table of errors ($\widetilde{\varepsilon}_h$, ε_h)

	$h_0 = 0.01,\ h_1 = h_2 = 0.01$		$h_0 = 0.005,\ h_1 = h_2 = 0.01$	
	$\widetilde{\varepsilon}_h$	ε_h	$\widetilde{\varepsilon}_h$	ε_h
$t = 0.1$	0.018958	0.001302	0.010255	0.000667
$t = 0.2$	$\times$	0.002599	0.379365	0.001360
$t = 0.3$	$\times$	0.003908	0.568249	0.002088
$t = 0.4$	$\times$	0.005250	0.616789	0.002863
$t = 0.5$	$\times$	0.006642	0.629860	0.003697

The results shown in the table are consistent with our mathematical analysis.

Our experiments have the following property. The classical Euler method for problem (5.1), (5.2) is stable for $h_0 = 0.005$, $h_1 = h_2 = 0.01$ and it is not stable if $h_0 = 0.01$, $h_1 = h_2 = 0.01$. The implicit Euler method is stable in the both cases.

The difference methods described in Section 4 have the potential for applications in the numerical solving of differential functional equations with initial boundary conditions.

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