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# Optimal Approximation of Discrete Csiszar $f$-Divergence

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Here are established a number of sharp and nearly optimal discrete probabilistic inequalities giving various types of order of approximation of Discrete Csiszar $f$-divergence between two discrete probability measures. This distance is the most essential and general tool for the comparison of these measures. We give also various types of representation of this discrete Csiszar distance and then we estimate tight their remainders, that is leading to very close discrete probabilistic inequalities involving various norms. We give also plenty of interesting applications.

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*Key Words*: Discrete Csiszar $f$-divergence or discrimination, representation, rate of approximation, probabilistic inequality.

## 1. Basics

The basic general background here has as follows.

Let $f$ by a convex function from $(0, +\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1) = 0$.

Let $(X, A, \lambda)$ be a measure space, where $\lambda$ is a finite or a $\sigma$-finite measure on $(X, A)$. Let $\mu_1$, $\mu_2$ be two probability measures on $(X, A)$ such that $\mu_1 \ll \lambda$, $\mu_2 \ll \lambda$ (absolutely continuous), e.g. $\lambda = \mu_1 + \mu_2$.

Denote by $p = \frac{d\mu_1}{d\lambda}$, $q = \frac{d\mu_2}{d\lambda}$ the (densities) Radon–Nikodym derivatives of $\mu_1$, $\mu_2$ with respect to $\lambda$ Typically we assume that

$$0 < a \leq \frac{p}{q} \leq b, \quad \text{a.e. on } X \text{ and } a \leq 1 \leq b.$$

The quantity

$$\Gamma_f(\mu_1, \mu_2) = \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda(x), \tag{1}$$

was introduced by I. Csiszar in 1967, see [11], and is called *$f$-divergence* of the probability measures $\mu_1$ and $\mu_2$.

By Lemma 1.1 of [11], the integral (1) is well-defined and $\Gamma_f(\mu_1, \mu_2) \geq 0$ with equality only when $\mu_1 = \mu_2$. Furthermore by [11], [6] $\Gamma_f(\mu_1, \mu_2)$ does not depend on the choice of $\lambda$. The concept of $f$-divergence was introduced first in [10] as a generalization of Kullback's "*information for discrimination*" or *I-divergence* (*generalized entropy*) [15], [16], and of Rényi's "*information gain*" (*I-divergence*) *of order* $\alpha$ [17]. In fact the *I-divergence of order* 1 equals

$$\Gamma_{u \log_2 u}(\mu_1, \mu_2).$$

The choice $f(u) = (u-1)^2$ produces again a known measure of difference of distributions that is called *$\chi^2$-divergence*. Of course the *total variation* distance

$$|\mu_1 - \mu_2| = \int_X |p(x) - q(x)|\, d\lambda(x)$$

is equal to $\Gamma_{|u-1|}(\mu_1, \mu_2)$. Here by assuming $f(1) = 0$ we can consider $\Gamma_f(\mu_1, \mu_2)$ the $f$-divergence as a measure of the difference between the probability measures $\mu_1$, $\mu_2$.

This is an expository, survey and applications article where we apply to the discrete case a great variety of interesting results of the above continuous case coming from the author's earlier but recent papers [5], [6], [7], [8].

Throughout this article we use the following. Let again $f$ be a convex function from $(0, +\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1) = 0$. Let $X = \{x_1, x_2, \ldots\}$ be a countable arbitrary set and $A$ be the set of all subsets of $X$. Then any probability distribution $\mu$ on $(X, A)$ is uniquely determined by the sequence $P = \{p_1, p_2, \ldots\}$, $p_i = \mu(\{x_i\})$, $i = 1, \ldots$. We call $M$ the set of all distributions on $(X, A)$ such that all $p_i$, $i = 1, \ldots$, are positive.

Denote here by $\mu_1, \mu_2 \in M$ the distributions with corresponding sequences $P = \{p_1, \ldots\}$, $Q = \{q_1, q_2, \ldots\}$ ($p_i > 0$, $q_i > 0$, $i = 1, 2, \ldots$).

We consider here only the set $M^* \subset M$ of distributions $\mu_1$, $\mu_2$ such that

$$0 < a \leq \frac{p_i}{q_i} \leq b, \quad i = 1, \ldots, \quad \text{on } X \text{ and } a \leq 1 \leq b, \tag{2}$$

where $a, b$ are fixed.

In our discrete case we choose $\lambda$ such that $\lambda(\{x_i\}) = 1$, all $i = 1, 2, \ldots$. Clearly here $\mu_1 \ll \lambda$, $\mu_2 \ll \lambda$. Therefore by (1) we obtain the special case of

$$\Gamma_f(\mu_1, \mu_2) = \sum_{i=1}^{\infty} q_i f\left(\frac{p_i}{q_i}\right). \tag{3}$$

So (3) is the discrete *f-divergence* or *Csiszar distance* between two discrete distributions from $M^*$, see also [10], [12]. The quantity $\Gamma_f$ as in (3) is also called *generalized measure of information.*

The above discrete general concept of $\Gamma_f(\mu_1, \mu_2)$ given by (3) incorporates a large variety of discrete specific $\Gamma_f$ distances such of Kullback–Leibler so called information divergence, Hellinger, Rényi $\alpha$-order entropy, $\chi^2$, Variational, Triangular discrimination, Bhattacharyya, Jeffreys distance. For all these and a whole of very interesting general theory please see S. Dragomir's important book [13], especially his particular article [14]. In there S. Dragomir deals with finite sums similar to (3), while we deal with finite or infinite ones.

The problem of finding and estimating the proper distance (or difference or discrimination) of two discrete probability distributions is one of the major ones in Probability Theory. The above discrete $f$-divergence measure (3) for various specific $f$'s, of finite or infinite sums have been applied alot to Anthropology, Genetics, Finance, Economics, Political Science, Biology, Approximation of Probability distributions, Signal Processing and Pattern Recognition.

In the next the proofs of all formulated and presented results are based alot on this Section 1, especially on the transfer from the continuous to the discrete case.

## 2. Results

**Part I.** Here we apply results from [5].

We present

**Theorem 1.** *Let $f \in C^1([a,b])$ and $\mu_1, \mu_2 \in M^*$. Then*

$$\Gamma_f(\mu_1, \mu_2) \le \|f'\|_{\infty,[a,b]} \left( \sum_{i=1}^{\infty} |p_i - q_i| \right). \tag{4}$$

*Inequality (4) is sharp. The optimal function is $f^*(y) = |y-1|^\alpha$, $\alpha > 1$ under the condition, either*

*i)* $\max(b-1, 1-a) < 1$ *and*

$$C := \sup(Q) < +\infty, \tag{5}$$

*or*

*ii)*

$$|p_i - q_i| \le q_i \le 1, \quad i = 1, 2, \dots . \tag{6}$$

P r o o f. Based on Theorem 1 of [5]. ■

Next we give the more general

**Theorem 2.** *Let* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$, *such that* $f^{(k)}(1) = 0$, $k = 0, 2, \ldots, n$, *and* $\mu_1, \mu_2 \in M^*$. *Then*

$$\Gamma_f(\mu_1, \mu_2) \le \frac{\|f^{(n+1)}\|_{\infty,[a,b]}}{(n+1)!} \left( \sum_{i=1}^{\infty} q_i^{-n} |p_i - q_i|^{n+1} \right). \tag{7}$$

*Inequality (7) is sharp. The optimal function is* $\tilde{f}(y) := |y-1|^{n+\alpha}$, $\alpha > 1$ *under the condition (i) or (ii) of Theorem 1.*

P r o o f. Based on Theorem 2 of [5]. ■

As a special case we have

**Corollary 1** *(to Theorem 2). Let* $f \in C^2([a,b])$ *and* $\mu_1, \mu_2 \in M^*$. *Then*

$$\Gamma_f(\mu_1, \mu_2) \le \frac{\|f^{(2)}\|_{\infty,[a,b]}}{2} \left( \sum_{i=1}^{\infty} q_i^{-1} (p_i - q_i)^2 \right). \tag{8}$$

*Inequality (8) is sharp. The optimal function is* $\tilde{f}(y) := |y-1|^{1+\alpha}$, $\alpha > 1$ *under the condition (i) or (ii) of Theorem 1.*

P r o o f. By Corollary 1 of [5]. ■

Next we connect discrete Csiszar $f$-divergence to the usual first modulus of continuity $\omega_1$.

**Theorem 3.** *Assume that*

$$0 < h := \sum_{i=1}^{\infty} |p_i - q_i| \le \min(1-a, b-1). \tag{9}$$

*Let* $\mu_1, \mu_2 \in M^*$. *Then*

$$\Gamma_f(\mu_1, \mu_2) \le \omega_1(f, h). \tag{10}$$

*Inequality (10) is sharp, namely it is attained by* $f^*(x) := |x-1|$.

P r o o f. By Theorem 3 of [5]. ■

**Part II.** Here we apply results from [6]. But first we need some basic concepts from [9], [2].

Let $\nu > 0$, $n := [\nu]$ and $\alpha = \nu - n$ $(0 < \alpha < 1)$. Let $x, x_0 \in [a,b] \subseteq \mathbb{R}$ such that $x \geq x_0$, where $x_0$ is fixed. Let $f \in C([a,b])$ and define

$$(11) \qquad (J_\nu^{x_0} f)(x) := \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x-s)^{\nu-1} f(s)\,ds, \quad x_0 \leq x \leq b,$$

the *generalized Riemann–Liouville integral*, where $\Gamma$ stands for the gamma function. We define the subspace

$$(12) \qquad C_{x_0}^{\nu}([a,b]) := \left\{ f \in C^n([a,b])\colon J_{1-\alpha}^{x_0} D^n f \in C^1([x_0,b]) \right\}.$$

For $f \in C_{x_0}^{\nu}([a,b])$ we define the *generalized $\nu$-fractional derivative of $f$ over* $[x_0, b]$ as

$$(13) \qquad D_{x_0}^{\nu} f := D J_{1-\alpha}^{x_0} f^{(n)} \quad \left( D := \frac{d}{dx} \right).$$

We present the following.

**Theorem 4.** *Let $a < b$, $1 \leq \nu < 2$, $f \in C_a^{\nu}([a,b])$, $\mu_1, \mu_2 \in M^*$. Then*

$$(14) \qquad \Gamma_f(\mu_1,\mu_2) \leq \frac{\|D_a^{\nu} f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} \left( \sum_{i=1}^{\infty} q_i^{1-\nu} (p_i - a q_i)^{\nu} \right).$$

Proof. By Theorem 2 of [6]. ∎

The counterpart of the previous result follows.

**Theorem 5.** *Let $a < b$, $\nu \geq 2$, $n := [\nu]$, $f \in C_a^{\nu}([a,b])$, $f^{(i)}(a) = 0$, $i = 0,1,\ldots,n-1$, $\mu_1, \mu_2 \in M^*$. Then*

$$(15) \qquad \Gamma_f(\mu_1,\mu_2) \leq \frac{\|D_a^{\nu} f\|_{\infty,[a,b]}}{\Gamma(\nu+1)} \left( \sum_{i=1}^{\infty} q_i^{1-\nu} (p_i - a q_i)^{\nu} \right).$$

Proof. By Theorem 3 of [6]. ∎

Next we give an $L_{\tilde{\alpha}}$ estimate.

**Theorem 6.** *Let $a < b$, $\nu \geq 1$, $n := [\nu]$, $f \in C_a^{\nu}([a,b])$, $f^{(i)}(a) = 0$, $i = 0,1,\ldots,n-1$, $\mu_1, \mu_2 \in M^*$. Let $\tilde{\alpha}, \beta > 1$: $\frac{1}{\tilde{\alpha}} + \frac{1}{\beta} = 1$. Then*

$$(16) \qquad \Gamma_f(\mu_1,\mu_2) \leq \frac{\|D_a^{\nu} f\|_{\tilde{\alpha},[a,b]}}{\Gamma(\nu)(\beta(\nu-1)+1)^{1/\beta}} \left( \sum_{i=1}^{\infty} q_i^{2-\nu-\frac{1}{\beta}} (p_i - a q_i)^{\nu-1+\frac{1}{\beta}} \right).$$

P r o o f. By Theorem 4 of [6]. ■

It follows an $L_\infty$ estimate.

**Theorem 7.** *Let $a < b$, $\nu \geq 1$, $n := [\nu]$, $f \in C_a^\nu([a,b])$, $f^{(i)}(a) = 0$, $i = 0, 1, \ldots, n-1$, $\mu_1, \mu_2 \in M^*$. Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{\|D_a^\nu f\|_{1,[a,b]}}{\Gamma(\nu)} \left( \sum_{i=1}^{\infty} q_i^{2-\nu} (p_i - a q_i)^{\nu-1} \right). \tag{17}$$

P r o o f. Based on Theorem 5 of [6]. ■

We continue with

**Theorem 8.** *Let $f \in C^1([a,b])$, $a \neq b$, $\mu_1, \mu_2 \in M^*$. Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{1}{b-a} \int_a^b f(x)dx - \sum_{i=1}^{\infty} f'\left(\frac{p_i}{q_i}\right) \left( \frac{q_i(a+b)}{2} - p_i \right). \tag{18}$$

P r o o f. Based on Theorem 6 of [6]. ■

Furthermore we get

**Theorem 9.** *Let $n$ odd and $f \in C^{n+1}([a,b])$, such that $f^{(n+1)} \geq 0$ ($\leq 0$), $\mu_1, \mu_2 \in M^*$. Then*

$$\begin{aligned} \Gamma_f(\mu_1, \mu_2) \;\; \leq \;\; (\geq) \frac{1}{b-a} \int_a^b f(x)dx - \sum_{i=1}^{n} \frac{1}{(i+1)!} \left( \sum_{k=0}^{i} \left( \sum_{j=1}^{\infty} f^{(i)}\left(\frac{p_j}{q_j}\right) q_j^{1-i} \right.\right. \\ \left.\left. \cdot \, (bq_j - p_j)^{(i-k)} (aq_j - p_j)^k \right) \right). \end{aligned} \tag{19}$$

P r o o f. Based on Theorem 7 of [6]. ■

**Part III.** Here we apply results from [7]. We begin with

**Theorem 10.** *Let $f, g \in C^1([a,b])$ where $f$ as in this article, $g' \neq 0$ over $[a,b]$, $\mu_1, \mu_2 \in M^*$. Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \left\| \frac{f'}{g'} \right\|_{\infty,[a,b]} \left( \sum_{i=1}^{\infty} q_i \left| g\left(\frac{p_i}{q_i}\right) - g(1) \right| \right). \tag{20}$$

Proof. Based on Theorem 1 of [7]. ∎

Example 1. (to Theorem 10).

1) Let $g(x) = \frac{1}{x}$. Then

$$\Gamma_f(\mu_1, \mu_2) \le \|x^2 f'\|_{\infty,[a,b]} \left( \sum_{i=1}^{\infty} \frac{q_i}{p_i} |p_i - q_i| \right). \tag{21}$$

2) Let $g(x) = e^x$. Then

$$\Gamma_f(\mu_1, \mu_2) \le \|e^{-x} f'\|_{\infty,[a,b]} \left( \sum_{i=1}^{\infty} q_i \left| e^{\frac{p_i}{q_i}} - e \right| \right). \tag{22}$$

3) Let $g(x) = e^{-x}$. Then

$$\Gamma_f(\mu_1, \mu_2) \le \|e^{x} f'\|_{\infty,[a,b]} \left( \sum_{i=1}^{\infty} q_i \left| e^{-\frac{p_i}{q_i}} - e^{-1} \right| \right). \tag{23}$$

4) Let $g(x) = \ln x$. Then

$$\Gamma_f(\mu_1, \mu_2) \le \|x f'\|_{\infty,[a,b]} \left( \sum_{i=1}^{\infty} q_i \left| \ln\left( \frac{p_i}{q_i} \right) \right| \right). \tag{24}$$

5) Let $g(x) = x \ln x$, with $a > e^{-1}$. Then

$$\Gamma_f(\mu_1, \mu_2) \le \left\| \frac{f'}{1 + \ln x} \right\|_{\infty,[a,b]} \left( \sum_{i=1}^{\infty} p_i \left| \ln\left( \frac{p_i}{q_i} \right) \right| \right). \tag{25}$$

6) Let $g(x) = \sqrt{x}$. Then

$$\Gamma_f(\mu_1, \mu_2) \le 2\|\sqrt{x} f'\|_{\infty,[a,b]} \left( \sum_{i=1}^{\infty} \sqrt{q_i} \, |\sqrt{p_i} - \sqrt{q_i}| \right). \tag{26}$$

7) Let $g(x) = x^\alpha$, $\alpha > 1$. Then

$$\Gamma_f(\mu_1, \mu_2) \le \frac{1}{\alpha} \left\| \frac{f'}{x^{\alpha-1}} \right\|_{\infty,[a,b]} \left( \sum_{i=1}^{\infty} q_i^{1-\alpha} |p_i^\alpha - q_i^\alpha| \right). \tag{27}$$

Next we give

**Theorem 11.** *Let $f \in C^1([a,b])$, $a \neq b$, $\mu_1, \mu_2 \in M^*$. Then*

$$\left|\Gamma_f(\mu_1,\mu_2) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{\|f'\|_{\infty,[a,b]}}{(b-a)}\left[\left(\frac{a^2+b^2}{2}\right) - (a+b) + \sum_{i=1}^{\infty} q_i^{-1}p_i^2\right]. \tag{28}$$

P r o o f. Based on Theorem 2 of [7]. ∎

It follows

**Theorem 12.** *Let $f \in C^{(2)}([a,b])$, $a \neq b$, $\mu_1, \mu_2 \in M^*$. Then*

$$\left|\Gamma_f(\mu_1,\mu_2) - \frac{1}{b-a}\int_a^b f(x)dx - \left(\frac{f(b)-f(a)}{b-a}\right)\left(1 - \frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{2}\|f''\|_{\infty,[a,b]}\left\{\sum_{i=1}^{\infty} q_i\left[\frac{\left(\frac{p_i}{q_i} - \left(\frac{a+b}{2}\right)\right)^2}{(b-a)^2} + \frac{1}{4}\right]^2 + \frac{1}{12}\right\}. \tag{29}$$

P r o o f. Based on (33) of [7]. ∎

Working more generally we have

**Theorem 13.** *Let $f \in C^1([a,b])$ and $g \in C([a,b])$ of bounded variation, $g(a) \neq g(b)$, $\mu_1, \mu_2 \in M^*$. Then*

$$\left|\Gamma_f(\mu_1,\mu_2) - \frac{1}{g(b)-g(a)}\int_a^b f(x)dg(x)\right| \leq \|f''\|_{\infty,[a,b]}\left\{\sum_{i=1}^{\infty} q_i\left(\int_a^b\left|P\left(g\left(\frac{p_i}{q_i}\right), q(t)\right)\right|dt\right)\right\}, \tag{30}$$

*where,*

$$P(g(z),g(t)) := \begin{cases} \dfrac{g(t)-g(a)}{g(b)-g(a)}, & a \leq t \leq z, \\[2ex] \dfrac{g(t)-g(b)}{g(b)-g(a)}, & z < t \leq b. \end{cases} \tag{31}$$

Proof. By Theorem 5 of [7]. ∎

**Example 2** (to Theorem 13). Let $f \in C^1([a,b])$, $a \neq b$, and $g(x) = e^x$, $\ln x$, $\sqrt{x}$, $x^\alpha$, $\alpha > 1$; $x > 0$. Then by (71)–(74) of [7] we obtain

1)

$$\left|\Gamma_f(\mu_1,\mu_2) - \frac{1}{e^b - e^a}\int_a^b f(x) - e^x dx\right| \le \frac{\|f'\|_{\infty,[a,b]}}{(e^b - e^a)}\left\{2\sum_{i=1}^{\infty} q_i e^{\frac{p_i}{q_i}} - e^a(2-a) + e^b(b-2)\right\}, \tag{32}$$

2)

$$\left|\Gamma_f(\mu_1,\mu_2) - \left(\ln\left(\frac{b}{a}\right)\right)^{-1}\int_a^b \frac{f(x)}{x}dx\right| \le \|f'\|_{\infty,[a,b]}\left(\ln\left(\frac{b}{a}\right)\right)^{-1}\left\{2\sum_{i=1}^{\infty} p_i \ln\frac{p_i}{q_i} - \ln(ab) + (a+b-2)\right\}, \tag{33}$$

3)

$$\left|\Gamma_f(\mu_1,\mu_2) - \frac{1}{2(\sqrt{b}-\sqrt{a})}\int_a^b \frac{f(x)}{\sqrt{x}}dx\right| \le \frac{\|f'\|_{\infty,[a,b]}}{(\sqrt{b}-\sqrt{a})}\left\{\frac{4}{3}\sum_{i=1}^{\infty} q_i^{-1/2} p_i^{3/2} + \frac{(a^{3/2}+b^{3/2})}{3} - (\sqrt{a}+\sqrt{b})\right\}, \tag{34}$$

4) ($\alpha > 1$)

$$\left|\Gamma_f(\mu_1,\mu_2) - \frac{\alpha}{(b^\alpha - a^\alpha)}\int_a^b f(x)x^{\alpha-1}dx\right| \le \frac{\|f'\|_{\infty,[a,b]}}{(b^\alpha - a^\alpha)} \times \left\{\frac{2}{(\alpha+1)}\left(\sum_{i=1}^{\infty} q_i^{-\alpha} p_i^{\alpha+1}\right) + \left(\frac{\alpha}{\alpha+1}(a^{\alpha+1} + b^{\alpha+1}\right) - (a^\alpha + b^\alpha)\right\}. \tag{35}$$

We continue with

**Theorem 14.** *Let $f \in C^{(2)}([a,b])$, $g \in C([a,b])$ and of bounded variation,*

$g(a) \neq g(b)$, $\mu_1, \mu_2 \in M^*$. *Then*

$$\Bigg|\Gamma_f(\mu_1,\mu_2) - \frac{1}{(g(b)-g(a))}\int_a^b f(x)dg(x) - \frac{1}{(g(b)-g(a))}\left(\int_a^b f'(x)dg(x)\right)\left[\sum_{i=1}^{\infty}\left(q_i\left(\int_a^b P\left(g\left(\frac{p_i}{q_i}\right),g(x)\right)dx\right)\right)\right]\Bigg|$$

$$\leq \|f''\|_{\infty,[a,b]}\left(\sum_{i=1}^{\infty}\left(q_i\left(\int_a^b\int_a^b \left|P\left(g\left(\frac{p_i}{q_i}\right),g(x)\right)\right| |P(g(x),g(x_1))|dx_1dx\right)\right)\right). \tag{36}$$

P r o o f. Apply (79) of [7]. ■

**Remark 1.** Next we define

$$P^*(z,s) := \begin{cases} s-a, & s \in [a,z] \\ s-b, & s \in (z,b]. \end{cases} \tag{37}$$

Let $f \in C^1([a,b])$, $a \neq b$, $\mu_1, \mu_2 \in M^*$. Then by (25) of [7] we get the representation

$$\Gamma_f(\mu_1,\mu_2) = \frac{1}{b-a}\left(\int_a^b f(x)dx + \mathcal{R}_1\right), \tag{38}$$

where by (26) of [7] we have

$$\mathcal{R}_1 := \sum_{i=1}^{\infty} q_i\left(\int_a^b P^*\left(\frac{p_i}{q_i},x\right)f'(x)dx\right). \tag{39}$$

Let again $f \in C^1([a,b])$, $g \in C([a,b])$ and of bounded variation, $g(a) \neq g(b)$, $\mu_1, \mu_2 \in M^*$.

Then by (47) of [7] we get the representation formula

$$\Gamma_f(\mu_1,\mu_2) = \frac{1}{g(b)-g(a)}\int_a^b f(x)dg(x) + \mathcal{R}_4, \tag{40}$$

where by (48) of [7] we have

$$\mathcal{R}_4 := \sum_{i=1}^{\infty} q_i\left(\int_a^b P\left(g\left(\frac{p_i}{q_i}\right),g(x)\right)f'(x)dx\right). \tag{41}$$

Let $\tilde{\alpha}, \beta > 1$: $\frac{1}{\tilde{\alpha}} + \frac{1}{\beta} = 1$, then by (89) of [7] we have

$$|\mathcal{R}_1| \leq \|f'\|_{\tilde{\alpha},[a,b]}\left(\sum_{i=1}^{\infty} q_i\left\|P^*\left(\frac{p_i}{q_i},\cdot\right)\right\|_{\beta,[a,b]}\right) \tag{42}$$

and by (90) of [7] we get

$$|\mathcal{R}_1| \le \|f'\|_{1,[a,b]} \left( \sum_{i=1}^{\infty} q_i \left\| P^* \left( \frac{p_i}{q_i}, \cdot \right) \right\|_{\infty,[a,b]} \right). \tag{43}$$

Furthermore by (95) and (96) of [7] we obtain

$$|\mathcal{R}_4| \le \|f'\|_{\tilde{\alpha},[a,b]} \left( \sum_{i=1}^{\infty} q_i \left\| P \left( g\left( \frac{p_i}{q_i} \right), g(\cdot) \right) \right\|_{\beta,[a,b]} \right)$$

and

$$|\mathcal{R}_4| \le \|f'\|_{1,[a,b]} \left( \sum_{i=1}^{\infty} q_i \left\| P \left( g\left( \frac{p_i}{q_i} \right), g(\cdot) \right) \right\|_{\infty,[a,b]} \right). \tag{44}$$

**Remark 2.** Let $a < b$.

(i) By (105) of [7] we get

$$|\mathcal{R}_1| \le \|f'\|_{1,[a,b]} \left( \sum_{i=1}^{\infty} q_i \max \left( \frac{p_i}{q_i} - a, b - \frac{p_i}{q_i} \right) \right), \quad f \in C^1([a,b]). \tag{45}$$

(ii) Let $g$ be strictly increasing and continuous over $[a,b]$, e.g. $g(x) = e^x$, $\ln x$, $\sqrt{x}$, $x^{\alpha}$ with $\alpha > 1$, and $x > 0$ whenever is needed. Also let $f \in C^1([a,b])$ and as in this article always. Then by (106) of [7] we find

$$|\mathcal{R}_4| \le \frac{\|f'\|_{1,[a,b]}}{(g(b) - g(a))} \left( \sum_{i=1}^{\infty} q_i \max \left( g\left( \frac{p_i}{q_i} \right) - g(a), g(b) - g\left( \frac{p_i}{q_i} \right) \right) \right). \tag{46}$$

In particular via (107)–(110) of [7] we obtain

1)

$$|\mathcal{R}_4| \le \frac{\|f'\|_{1,[a,b]}}{(e^b - e^a)} \left( \sum_{i=1}^{\infty} q_i \max \left( e^{\frac{p_i}{q_i}} - e^a, e^b - e^{\frac{p_i}{q_i}} \right) \right), \tag{47}$$

2)

$$|\mathcal{R}_4| \le \frac{\|f'\|_{1,[a,b]}}{\ln b - \ln a} \left( \sum_{i=1}^{\infty} q_i \max \left( \ln\left( \frac{p_i}{q_i} \right) - \ln a, \ln b - \ln\left( \frac{p_i}{q_i} \right) \right) \right), \tag{48}$$

3)

$$|\mathcal{R}_4| \le \frac{\|f'\|_{1,[a,b]}}{\sqrt{b}-\sqrt{a}} \left( \sum_{i=1}^{\infty} q_i \max\left( \sqrt{\frac{p_i}{q_i}} - \sqrt{a}, \sqrt{b} - \sqrt{\frac{p_i}{q_i}} \right) \right), \tag{49}$$

and finally for $\alpha > 1$ we obtain

4)

$$|\mathcal{R}_4| \le \frac{\|f'\|_{1,[a,b]}}{b^\alpha - a^\alpha} \left( \sum_{i=1}^{\infty} q_i \max\left( \frac{p_i^\alpha}{q_i^\alpha} - a^\alpha, b^\alpha - \frac{p_i^\alpha}{q_i^\alpha} \right) \right). \tag{50}$$

iii) At last let $\tilde{\alpha}, \beta > 1$ such that $\frac{1}{\tilde{\alpha}} + \frac{1}{\beta} = 1$, then by (115) of [7] we have

$$|\mathcal{R}_1| \le \|f'\|_{\tilde{\alpha},[a,b]} \left( \sum_{i=1}^{\infty} \left( \sqrt[\beta]{\frac{(p_i - aq_i)^{\beta+1} + (bq_i - p_i)^{\beta+1}}{(\beta+1)q_i}} \right) \right). \tag{51}$$

**Part IV.** Here we apply results from [8].

We start with

**Theorem 15.** *Let $0 < a < 1 < b$, $f$ as in this paper and $f \in C^n([a,b])$, $n \ge 1$ with $|f^{(n)}(t) - f^{(n)}(1)|$ be a convex function in $t$. Let $0 < h < \min(1-a, b-1)$ be fixed, $\mu_1, \mu_2 \in M^*$. Then*

$$\begin{aligned} \Gamma_f(\mu_1,\mu_2) \ \le \ & \sum_{k=2}^{n} \frac{|f^{(k)}(1)|}{k!} \left| \sum_{i=1}^{\infty} q_i^{1-k}(p_i - q_i)^k \right| \\ & + \frac{\omega_1(f^{(n)},h)}{(n+1)!h} \left( \sum_{i=1}^{\infty} q_i^{-n} |p_i - q_i|^{n+1} \right). \end{aligned} \tag{52}$$

*Here $\omega_1$ is the usual first modulus of continuity. If $f^{(k)}(1) = 0$, $k = 2, \dots, n$, then*

$$\Gamma_f(\mu_1,\mu_2) \le \frac{\omega_1(f^{(n)},h)}{(n+1)!h} \left( \sum_{i=1}^{\infty} q_i^{-n} |p_i - q_i|^{n+1} \right). \tag{53}$$

*Inequalities (52) and (53) when $n$ is even are attained by*

$$\tilde{f}(t) := \frac{|t-1|^{n+1}}{(n+1)!}, \quad a \le t \le b. \tag{54}$$

P r o o f. By Theorem 1 of [8]. ■

We continue with the general

**Theorem 16.** *Let* $f \in C^n([a,b])$, $n \geq 1$, $\mu_1, \mu_2 \in M^*$. *Assume that* $\omega_1(f^{(n)}, \delta) \leq w$, *where* $0 < \delta \leq b-a$, $w > 0$. *Let* $x \in \mathbb{R}$ *and denote by*

$$\phi_n(x) := \int_0^{|x|} \left\lceil \frac{t}{\delta} \right\rceil \frac{(|x|-t)^{n-1}}{(n-1)!} dt, \tag{55}$$

*where* $\lceil \cdot \rceil$ *is the ceiling of the number. Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \sum_{k=2}^{n} \frac{|f^{(k)}(1)|}{k!} \left| \sum_{i=1}^{\infty} q_i^{1-k}(p_i - q_i)^k \right| + w \left( \sum_{i=1}^{\infty} q_i \phi_n\left( \frac{p_i - q_i}{q_i} \right) \right). \tag{56}$$

*Inequality (56) is sharp, namely it is attained by the function*

$$\tilde{f}_n(t) := w\phi_n(t-1), \quad a \leq t \leq b, \tag{57}$$

*when* $n$ *is even.*

P r o o f. By Theorem 2 of [8]. ■

It follows

**Corollary 2.** *(to Theorem 16) Let* $\mu_1, \mu_2 \in M^*$. *It holds* $(0 < \delta \leq b-a)$

$$\begin{aligned} \Gamma_f(\mu_1,\mu_2) &\leq \sum_{k=2}^{n} \frac{|f^{(k)}(1)|}{k!} \left( \sum_{i=1}^{\infty} q_i^{1-k}(p_i - q_i)^k \right) \\ &+\omega_1(f^{(n)},\delta) \left\{ \frac{1}{(n+1)!\delta} \left( \sum_{i=1}^{\infty} q_i^{-n}|p_i - q_i|^{n+1} \right) + \frac{1}{2n!} \left( \sum_{i=1}^{\infty} q_i^{1-n}|p_i - q_i|^{n} \right) \right. \\ &\left. + \frac{\delta}{8(n-1)!} \left( \sum_{i=1}^{\infty} q_i^{2-n}|p_i - q_i|^{n-1} \right) \right\}, \end{aligned} \tag{58}$$

*and*

$$\begin{aligned} \Gamma_f(\mu_1,\mu_2) \ &\leq \ \sum_{k=2}^{n} \frac{|f^{(k)}(1)|}{k!} \left| \sum_{i=1}^{\infty} q_i^{1-k}(p_i - q_i)^k \right| \\ &+ \omega_1(f^{(n)},\delta) \left\{ \frac{1}{n!} \left( \sum_{i=1}^{\infty} q_i^{1-n}|p_i - q_i|^{n} \right) \right. \\ &\left. + \frac{1}{n!(n+1)\delta} \left( \sum_{i=1}^{\infty} q_i^{-n}|p_i - q_i|^{n+1} \right) \right\}. \end{aligned} \tag{59}$$

*In particular we have*

$$\Gamma_f(\mu_1,\mu_2) \le \omega_1(f',\delta)\left\{\frac{1}{2\delta}\left(\sum_{i=1}^{\infty} q_i^{-1}(p_i-q_i)^2\right)+\frac{1}{2}\left(\sum_{i=1}^{\infty}|p_i-q_i|\right)+\frac{\delta}{8}\right\}, \tag{60}$$

*and*

$$\Gamma_f(\mu_1,\mu_2) \le \omega_1(f',\delta)\left\{\left(\sum_{i=1}^{\infty}|p_i-q_i|\right)+\frac{1}{2\delta}\left(\sum_{i=1}^{\infty} q_i^{-1}(p_i-q_i)^2\right)\right\}. \tag{61}$$

P r o o f. Based on Corollary 1 of [8]. ■

Also we have

**Corollary 3** *(to Theorem 16). Let $\mu_1,\mu_2 \in M^*$. Assume here that $f^{(n)}$ is of Lipschitz type of order $\alpha$, $0<\alpha\le 1$, i.e.*

$$\omega_1(f^{(n)},\delta) \le K\delta^{\alpha}, \quad K>0, \tag{62}$$

*for any $0<\delta\le b-a$. Then*

$$\Gamma_f(\mu_1,\mu_2) \le \sum_{k=2}^{n}\frac{|f^{(k)}(1)|}{k!}\left|\sum_{i=1}^{\infty} q_i^{1-k}(p_i-q_i)^k\right| + \frac{K}{\prod_{i=1}^{n}(\alpha+i)}\left(\sum_{i=1}^{\infty} q_i^{1-n-\alpha}|p_i-q_i|^{n+\alpha}\right). \tag{63}$$

*When $n$ is even (63) is attained by $f^*(x)=c|x-1|^{n+\alpha}$, where*

$$c := K/\Big(\prod_{i=1}^{n}(\alpha+i)\Big) > 0\,.$$

P r o o f. Based on Corollary 2 of [8]. ■

**Corollary 4** (to Theorem 16). *Let $\mu_1,\mu_2 \in M^*$. Assume that*

$$b-a \ge \sum_{i=1}^{\infty} q_i^{-1}p_i^2 - 1 > 0. \tag{64}$$

*Then*

$$\Gamma_f(\mu_1,\mu_2) \le \omega_1\left(f',\left(\sum_{i=1}^{\infty} q_i^{-1}p_i^2-1\right)\right)\left\{\sum_{i=1}^{\infty}|p_i-q_i|+\frac{1}{2}\right\}. \tag{65}$$

P r o o f. Based on Corollary 3 of [8]. ■

**Corollary 5** (to Theorem 16). *Let* $\mu_1, \mu_2 \in M^*$. *Assume that*

$$\sum_{i=1}^{\infty} |p_i - q_i| > 0. \tag{66}$$

*Then*

$$\begin{aligned} \Gamma_f(\mu_1, \mu_2) &\leq \omega_1\left(f', \left(\sum_{i=1}^{\infty} |p_i - q_i|\right)\right)\left\{\left(\sum_{i=1}^{\infty} |p_i - q_i|\right) + \frac{b-a}{2}\right\} \\ &\leq \frac{3}{2}(b-a)\omega_1\left(f', \left(\sum_{i=1}^{\infty} |p_i - q_i|\right)\right). \end{aligned} \tag{67}$$

P r o o f. Based on Corollary 5 of [8]. ■

We give

**Proposition 1.** *Let* $f \in C([a,b])$, $\mu_1, \mu_2 \in M^*$.

i) *Assume that*

$$\sum_{i=1}^{\infty} |p_i - q_i| > 0. \tag{68}$$

*Then*

$$\Gamma_f(\mu_1, \mu_2) \leq 2\omega_1\left(f, \left(\sum_{i=1}^{\infty} |p_i - q_i|\right)\right). \tag{69}$$

ii) *Let* $r > 0$ *and*

$$b - a \geq r\left(\sum_{i=1}^{\infty} |p_i - q_i|\right) > 0. \tag{70}$$

*Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \left(1 + \frac{1}{r}\right)\omega_1\left(f, r\left(\sum_{i=1}^{\infty} |p_i - q_i|\right)\right). \tag{71}$$

P r o o f. Based on Proposition 1 of [8]. ■

Also we give

**Proposition 2.** *Let $f$ as in this setting and $f$ is a Lipschitz function of order $\alpha$, $0 < \alpha \le 1$, i.e. there exists $K > 0$ such that*

$$|f(x) - f(y)| \le K|x - y|^{\alpha}, \quad \textit{all } x, y \in [a, b]. \tag{72}$$

*Let $\mu_1, \mu_2 \in M^*$. Then*

$$\Gamma_f(\mu_1, \mu_2) \le K \left( \sum_{i=1}^{\infty} q_i^{1-\alpha} |p_i - q_i|^{\alpha} \right). \tag{73}$$

Proof. Based on Proposition 2 of [8]. ∎

Next we present some alternative type of results.

We start with

**Theorem 17.** *All elements involved here as in this article and $f \in C^{n+1}([a, b])$, $n \in \mathbb{N}$. Let $\mu_1, \mu_2 \in M^*$. Then*

$$\Gamma_f(\mu_1, \mu_2) = \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} \left( \sum_{i=1}^{\infty} f^{(k+1)}\left(\frac{p_i}{q_i}\right) q_i^{-k} (p_i - q_i)^{k+1} \right) + \psi_2, \tag{74}$$

*where*

$$\psi_2 := \frac{(-1)^n}{n!} \left( \sum_{i=1}^{\infty} q_i \left( \int_1^{\frac{p_i}{q_i}} (t-1)^n f^{(n+1)}(t) dt \right) \right). \tag{75}$$

Proof. By Theorem 12 of [8]. ∎

Next we estimate $\psi_2$. We give

**Theorem 18.** *All here as in Theorem 17. It holds*

$$|\psi_2| \le \min \ \textit{of} \begin{cases} B_1 := \dfrac{\|f^{(n+1)}\|_{\infty,[a,b]}}{(n+1)!} \left( \displaystyle\sum_{i=1}^{\infty} q_i^{-n} |p_i - q_i|^{n+1} \right), \\[2ex] B_2 := \dfrac{\|f^{(n+1)}\|_{1,[a,b]}}{n!} \left( \displaystyle\sum_{i=1}^{\infty} q_i^{1-n} |p_i - q_i|^{n} \right), \\[2ex] \qquad \textit{and for } \tilde{\alpha}, \beta > 1\textit{: } \dfrac{1}{\tilde{\alpha}} + \dfrac{1}{\beta} = 1, \\[2ex] B_3 := \dfrac{\|f^{(n+1)}\|_{\tilde{\alpha},[a,b]}}{n!(n\beta+1)^{1/\beta}} \left( \displaystyle\sum_{i=1}^{\infty} q_i^{1-n-\frac{1}{\beta}} |p_i - q_i|^{n+\frac{1}{\beta}} \right). \end{cases} \tag{76}$$

*Also it holds*

(77)
$$\Gamma_f(\mu_1,\mu_2) \le \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left| \sum_{i=1}^{\infty} f^{(k+1)}\left(\frac{p_i}{q_i}\right) q_i^{-k}(p_i-q_i)^{k+1} \right| + \min(B_1,B_2,B_3).$$

P r o o f. By Theorem 13 of [8]. ∎

**Corollary 6** (to Theorem 18). *Case of* $n=1$. *It holds*

(78)
$$|\psi_2| \le \min \ \textit{of} \begin{cases} B_{1,1} := \dfrac{\|f''\|_{\infty,[a,b]}}{2}\left(\displaystyle\sum_{i=1}^{\infty} q_i^{-1}(p_i-q_i)^2\right), \\ B_{2,1} := \|f''\|_{1,[a,b]}\left(\displaystyle\sum_{i=1}^{\infty} |p_i-q_i|\right), \\ \qquad \textit{and for } \tilde{\alpha},\beta>1\colon \dfrac{1}{\tilde{\alpha}}+\dfrac{1}{\beta}=1, \\ B_{3,1} := \dfrac{\|f''\|_{\tilde{\alpha},[a,b]}}{(\beta+1)^{1/\beta}}\left(\displaystyle\sum_{i=1}^{\infty} q_i^{-\frac{1}{\beta}}|p_i-q_i|^{1+\frac{1}{\beta}}\right). \end{cases}$$

*Also it holds*

(79)
$$\Gamma_f(\mu_1,\mu_2) \le \left| \sum_{i=1}^{\infty} f'\left(\frac{p_i}{q_i}\right)(p_i-q_i) \right| + \min(B_{1,1},B_{2,1},B_{3,1}).$$

P r o o f. By Corollary 8 of [8]. ∎

We further present

**Theorem 19.** *Let all elements involved as in this article,* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$ *and* $\mu_1,\mu_2 \in M^*$. *Then*

(80)
$$\begin{aligned} \Gamma_f(\mu_1,\mu_2) &= \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!}\left(\sum_{i=1}^{\infty} f^{(k+1)}\left(\frac{p_i}{q_i}\right) q_i^{-k}(p_i-q_i)^{k+1}\right) \\ &\quad + \frac{(-1)^n}{(n+1)!} f^{(n+1)}(1)\left(\sum_{i=1}^{\infty} q_i^{-n}(p_i-q_i)^{n+1}\right) + \psi_4, \end{aligned}$$

*where*

$$\psi_4 := \sum_{i=1}^{\infty} q_i \psi_3 \left( \frac{p_i}{q_i} \right), \tag{81}$$

*with*

$$\psi_3(x) := \frac{(-1)^n}{n!} \int_1^x (s-1)^n (f^{(n+1)}(s) - f^{(n+1)}(1)) ds. \tag{82}$$

P r o o f. Based on Theorem 14 of [8]. ∎

Next we present estimations of $\psi_4$.

**Theorem 20.** *Let all elements involved here as in this article and* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$, $\mu_1, \mu_2 \in M^*$ *and*

$$0 < \frac{1}{(n+2)} \left( \sum_{i=1}^{\infty} q_i^{-n-1} |p_i - q_i|^{n+2} \right) \leq b - a. \tag{83}$$

*Then (80) is again valid and*

$$\begin{aligned} |\psi_4| \quad &\leq \quad \frac{1}{n!} \omega_1 \left( f^{(n+1)}, \frac{1}{(n+2)} \left( \sum_{i=1}^{\infty} q_i^{-n-1} |p_i - q_i|^{n+2} \right) \right) \\ &\quad \cdot \left( 1 + \frac{1}{(n+1)} \left( \sum_{i=1}^{\infty} q_i^{-n} |p_i - q_i|^{n+1} \right) \right). \end{aligned} \tag{84}$$

P r o o f. Based on Theorem 15 of [8]. ∎

Also we have

**Theorem 21.** *All elements involved here as in this article and* $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$, $\mu_1, \mu_2 \in M^*$. *Furthermore we assume that* $|f^{(n+1)}(t) - f^{(n+1)}(1)|$ *is convex in* $t$ *and*

$$0 < \frac{1}{n!(n+2)} \left( \sum_{i=1}^{\infty} q_i^{-n-1} |p_i - q_i|^{n+2} \right) \leq \min(1-a, b-1). \tag{85}$$

*Then (80) is again valid and*

$$|\psi_4| \leq \omega_1 \left( f^{(n+1)}, \frac{1}{n!(n+2)} \left( \sum_{i=1}^{\infty} q_i^{-n-1} |p_i - q_i|^{n+2} \right) \right). \tag{86}$$

*The last (86) is attained by*

$$\hat{f}(s) := \frac{(s-1)^{n+2}}{(n+2)!}, \quad a \leq s \leq b. \tag{87}$$

*when $n$ is even.*

P r o o f. By Theorem 16 of [8]. ■

When $n = 1$ we obtain

**Corollary 7** (to Theorem 20). *Let $f \in C^2([a,b])$, $\mu_1, \mu_2 \in M^*$ and*

$$0 < \frac{1}{3}\left(\sum_{i=1}^{\infty} q_i^{-2}|p_i - q_i|^3\right) \leq b - a. \tag{88}$$

*Then*

$$\Gamma_f(\mu_1,\mu_2) = \sum_{i=1}^{\infty} f'\left(\frac{p_i}{q_i}\right)(p_i - q_i) - \frac{f''(1)}{2}\left(\sum_{i=1}^{\infty} q_i^{-1}(p_i - q_i)^2\right) + \psi_{4,1}, \tag{89}$$

*where*

$$\psi_{4,1} := \sum_{i=1}^{\infty} q_i \psi_{3,1}\left(\frac{p_i}{q_i}\right), \tag{90}$$

*with*

$$\psi_{3,1}(x) := -\int_1^x (t-1)(f''(t) - f''(1))dt, \quad x \in [a,b]. \tag{91}$$

*Furthermore it holds*

$$|\psi_{4,1}| \leq \omega_1\left(f'', \frac{1}{3}\left(\sum_{i=1}^{\infty} q_i^{-2}|p_i - q_i|^3\right)\right)\frac{1}{2}\left(\left(\sum_{i=1}^{\infty} q_i^{-1}p_i^2\right) + 1\right). \tag{92}$$

P r o o f. Based on Corollary 9 of [8]. ■

Also we have

**Corollary 8** (to Theorem 21). *Let $f \in C^2([a,b])$ and $|f''(t) - f''(1)|$ is convex in $t$. Here $\mu_1, \mu_2 \in M^*$ such that*

$$0 < \frac{1}{3}\left(\sum_{i=1}^{\infty} q_i^{-2}|p_i - q_i|^3\right) \leq \min(1-a, b-1). \tag{93}$$

*Then again (89) is valid and*

$$|\psi_{4,1}| \le \omega_1 \left( f'', \frac{1}{3} \left( \sum_{i=1}^{\infty} q_i^{-2} |p_i - q_i|^3 \right) \right). \tag{94}$$

P r o o f. By Corollary 10 of [8]. ∎

The last main result follows.

**Theorem 22.** *All elements involved here in this article and $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$, $\mu_1, \mu_2 \in M^*$. Additionally we assume that*

$$|f^{(n+1)}(x) - f^{(n+1)}(y)| \le K|x-y|^{\alpha}, \tag{95}$$

*$K > 0$, $0 < \alpha \le 1$, all $x, y \in [a,b]$ with $(x-1)(y-1) \ge 0$. Then*

$$\begin{aligned} \Gamma_f(\mu_1, \mu_2) &= \sum_{k=0}^{n-1} \frac{(-1)^k}{(k+1)!} \left( \sum_{i=1}^{\infty} f^{(k+1)} \left( \frac{p_i}{q_i} \right) q_i^{-k} (p_i - q_i)^{k+1} \right) \\ &\quad + \frac{(-1)^n}{(n+1)!} f^{(n+1)}(1) \left( \sum_{i=1}^{\infty} q_i^{-n} (p_i - q_i)^{n+1} \right) + \psi_4. \end{aligned} \tag{96}$$

*Here it holds*

$$|\psi_4| \le \frac{K}{n!(n+\alpha+1)} \left( \sum_{i=1}^{\infty} q_i^{-n-\alpha} |p_i - q_i|^{n+\alpha+1} \right). \tag{97}$$

*Inequality (97) is attained when $n$ is even by*

$$f^*(x) = \tilde{c}|x-1|^{n+\alpha+1}, \quad x \in [a,b] \tag{98}$$

*where*

$$\tilde{c} := \frac{K}{\prod_{j=0}^{n} (n+\alpha+1-j)}. \tag{99}$$

P r o o f. By Theorem 17 of [8]. ∎

Next we give

**Corollary 9** (to Theorem 22). *Let $f \in C^2([a,b])$, $\mu_1, \mu_2 \in M^*$. Additionally we assume that*

$$|f''(x) - f''(y)| \le K|x-y|^{\alpha}, \tag{100}$$

*$K > 0$, $0 < \alpha \leq 1$, all $x, y \in [a,b]$ with $(x-1)(y-1) \geq 0$. Then*

$$\Gamma_f(\mu_1,\mu_2) = \sum_{i=1}^{\infty} f'\left(\frac{p_i}{q_i}\right)(p_i - q_i) - \frac{f''(1)}{2}\left(\sum_{i=1}^{\infty} q_i^{-1}(p_i - q_i)^2\right) + \psi_{4,1}. \tag{101}$$

*Here*

$$\psi_{4,1} := \sum_{i=1}^{\infty} q_i \psi_{3,1}\left(\frac{p_i}{q_i}\right) \tag{102}$$

*with*

$$\psi_{3,1}(x) := -\int_1^x (t-1)(f''(t) - f''(1))dt, \quad x \in [a,b]. \tag{103}$$

*It holds that*

$$|\psi_{4,1}| \leq \frac{K}{(\alpha+2)}\left(\sum_{i=1}^{\infty} q_i^{-1-\alpha}|p_i - q_i|^{\alpha+2}\right). \tag{104}$$

P r o o f. By Corollary 11 of [8]. ■

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