Mathematica
Balkanica

New Series Vol. 20, 2006, Fasc. 2

Fixed Point Theorems for Set-Valued Non-Self Mappings on Convex Metric Spaces

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In this paper some fixed-point theorems for multi-valued non-self mappings in metrically convex metric spaces are proved. Our theorems generalize and extend Theorem of Rhoades [6], Theorem 1 and Theorem 2 of Itoh [5], Theorem of Assad [1] and the main Theorem of Assad and Kirk [2].

AMS Subj. Classification: 47H10, 54H25
Key Words: multi-valued non-self mapping, metrically convex metric space

1. Introduction

Let (X, d) be a metric space and K a closed subset of X. Let $\partial\, K$ denote the boundary of K, $CB(X)$ the family of all non'empty closed and bounded subsets of X, H the Hausdorff metric induced by d on $CB(X)$ and for any subset A of X, $D(x, A)$ denotes the distance from x to A. Many applications of the contraction mapping theorems occur in a convex setting and the mapping involved is not necessarily self-mapping. Assad and Kirk [2] first gave sufficient conditions for a multi-valued nonself mapping from K into $CB(X)$ to have a fixed point. They proved the following result.

Theorem 1.1. *([2], Theorem 1).* Let X be a complete and convex metric space, K a non'empty closed subset of X, and T a mapping from K into $CB(X)$ such that

$$H(Tx, Ty) \leq \alpha d(x, y),$$

where $\alpha < 1$. If $Tx \subset K$ for each $x \in \partial\, K$ then there exists $x_0 \in K$ such that $x_0 \in Tx_0$ (i.e. T has a fixed point in K).

The main Theorem 1 of Assad and Kirk [2] is further generalized by Itoh [5].

Recently Rhoades [6] proved the following theorem.

Theorem 1.2. *(Rhoades [6]).* Let (X, d) be a complete metrically convex metric space, K a non'empty closed subset of X. Let $T : K \to CB(X)$ satisfy the following contractive condition:

$$\begin{aligned} H(Tx,Ty) &\leq \alpha d(x,y) + \beta \max\{D(x,Tx), D(y,Ty)\} \\ &\quad + \gamma[D(x,Ty) + D(y,Tx)] \end{aligned} \tag{1}$$

for all x, y in K, where $\alpha, \beta, \gamma \geq 0$ and such that

$$\lambda = [(1+\alpha+\gamma)/(1-\beta-\gamma)] \cdot [(\alpha+\beta+\gamma)/(1-\gamma)] < 1. \tag{2}$$

If $Tx \subset K$ for each $x \in \partial\ K$, then there exists a $z \in K$ such that $z \in Tz$.

In this paper we shall consider a wider class of multi-valued non-self mappings than those considered in [6] and in [5]. We proved two fixed point theorems for mappings from that class. These theorems generalize the corresponding Theorem of Rhoades [6], Theorem 1 and Theorem 2 of Itoh [5], Theorem of Assad [1] and the main theorem of Assad and Kirk [2]. Also we prove a simple lemma in the form which enables to simplify proofs in almost all theorems, related to contractive multi-valued mappings.

2. Results

First we prove a simple lemma which we shall use in the proof of our theorems in a convex metric space, but it is applicable in any metric space (c.f.[4]).

Lemma 1. If $A, B \in CB(X)$ and $a \in A$, then for any positive number $q < 1$ there exists $b = b(a)$ in B such that

$$qd(a,b) \leq H(A,B). \tag{3}$$

P r o o f. If $H(A, B) = 0$, then $A = B$ and (3) trivially holds for $b(a) = a$.

Suppose now that $H(A, B) > 0$. By definition of $D(a, B)$ and $H(A, B)$, for any positive number ε there exists a $b \in B$ such that

$$d(a,b) \leq D(a,B) + \varepsilon \leq H(A,B) + \varepsilon. \tag{4}$$

Let $0 < q < 1$. Then $q^{-1} - 1 > 0$. Since $H(A, B) > 0$,

$$\varepsilon = (q^{-1} - 1)H(A,B) > 0.$$

By inserting this ε in (4) we get (3). ■

Recall that X is said to be a convex metric space in the sense of Menger if X has the property that for each x, y in X with $x \neq y$ there exists z in X, $x \neq z$, $y \neq z$, such that

$$d(x,z) + d(z,y) = d(x,y).$$

Further (see [2, 3]), if K is a closed subset of X and if $x \in K$ and $y \notin K$, then there exists a point z in ∂K, the boundary of K, such that

$$d(x,z) + d(z,y) = d(x,y).$$

Theorem 2.1. Let (X,d) be a complete metrically convex metric space, K a non'empty closed subset of X. Let T be a mapping of K into $CB(X)$ such that

$$\begin{aligned} &H(Tx,Ty) \leq \alpha d(x,y) + \beta \max\{D(x,Tx), D(y,Ty)\} \\ &+\gamma[D(x,Ty) + D(y,Tx)] + \delta[D(x,Tx) + D(y,Ty)], \end{aligned} \tag{5}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ are such that

$$\lambda = \alpha + 2\beta + (3+\alpha)(\gamma+\delta) < 1. \tag{6}$$

If $Tx \subseteq K$ for each $x \in \partial K$, then there exists $u \in K$ such that $u \in Tu$.

Proof. We prove Theorem for the case when $T(\partial K) \subset K$, but not necessarily $T(K) \subset K$. The case $T(K) \subset K$ is much more simpler. In this case, from the proof which follows, it is easy to see that the condition (6) can be weakened to $\lambda = \alpha + \beta + 2\gamma + 2\delta < 1$ and the hypothesis of convexity of X can be omitted.

We select two sequences $\{x_n\}$ and $\{y_n\}$ in K and X, respectively, in the following way. Let x_0 in $\rightarrow K$ and $x_1 = y_1 \in Tx_0$ be arbitrary. Let a be any fixed number such that $0 < a < \frac{1}{2}$. Put

$$q = \lambda^a.$$

Then from (6), $q < 1$. By Lemma 1 we can choose $y_2 \in Tx_1$ such that

$$qd(y_1,y_2) \leq H(Tx_0,Tx_1).$$

If $y_2 \in K$, put $x_2 = y_2$. If $y_2 \notin K$, then, as X is convex, we can choose $x_2 \in \partial K$ such that

$$d(x_1,x_2) + d(x_2,y_2) = d(x_1,y_2).$$

Let $y_3 \in Tx_2$ be such that

$$qd(y_2, y_3) \leq H(Tx_1, Tx_2).$$

By induction we may obtain sequences $\{x_n\}$ and $\{y_n\}$ such that for $n = 1, 2, \ldots$

(i) $y_n \in Tx_{n-1}$,

(ii) $qd(y_n, y_{n+1}) \leq H(Tx_{n-1}, Tx_n)$,

(iii) $x_{n+1} = y_{n+1}$, if $y_{n+1} \in K$, or

(iv) $x_{n+1} \in \partial K$ and $d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1})$, if $y_{n+1} \notin K$ for all n.

Define

$$P = \{x_i \in \{x_n\} : x_i = y_i\}, \quad Q = \{x_i \in \{x_n\} : x_i \neq y_i\}.$$

Observe that if $x_n \in Q$ for some n, then x_{n-1} and x_{n+1} belong to P, as two consecutive terms of $\{x_n\}$ cannot be in Q.

We wish to estimate $d(x_n, x_{n+1})$. Three cases need to be considered.

Case 1. $x_n \in P$ and $x_{n+1} \in P$. Then from (ii) and (5) we have

$$\begin{aligned} & qd(x_n, x_{n+1}) = qd(y_n, y_{n+1}) \leq H(Tx_{n-1}, Tx_n) \\ \leq\ & \alpha d(x_{n-1}, x_n) + \beta \max\{D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n)\} \\ & +\gamma[D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})] + \delta[D(x_{n-1}, Tx_{n-1}) + D(x_n, Tx_n)] \\ \leq\ & \alpha d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\ & +\gamma d(x_{n-1}, x_{n+1}) + \delta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \end{aligned}$$

Hence, using the triangle inequality for $d(x_{n-1}, x_{n+1})$,

$$\begin{aligned} qd(x_n, x_{n+1}) \leq\ & (\alpha + \gamma + \delta) d(x_{n-1}, x_n) \\ & +\beta \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + (\gamma + \delta) d(x_n, x_{n+1}). \end{aligned} \tag{7}$$

If we suppose that $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$, then from (7) and by (6) we get

$$\begin{aligned} qd(x_n, x_{n+1}) &\leq (\alpha + \gamma + \delta) d(x_{n-1}, x_n) + (\beta + \gamma + \delta) d(x_n, x_{n+1}) \\ &\leq (\alpha + \beta + 2\gamma + 2\delta) d(x_n, x_{n+1}) \leq \lambda d(x_n, x_{n+1}) < \lambda^a d(x_n, x_{n+1}) \\ &= qd(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction. Therefore, from (7) we obtain

$$qd(x_n, x_{n+1}) \leq (\alpha + \beta + \gamma + \delta)d(x_{n-1}, x_n) + (\gamma + \delta)d(x_n, x_{n+1})$$

and hence

$$d(x_n, x_{n+1}) \leq [(\alpha + \beta + \gamma + \delta)/(q - \gamma - \delta)]d(x_{n-1}, x_n). \tag{8}$$

Case 2. $x_n \in P$ and $x_{n+1} \in Q$. Then by (iv)

$$d(x_n, x_{n+1}) = d(x_n, y_{n+1}) - d(x_{n+1}, y_{n+1}) \leq d(x_n, y_{n+1}) = d(y_n, y_{n+1}).$$

By the same method as in Case 1, we have

$$d(x_n, x_{n+1}) \leq [(\alpha + \beta + \gamma + \delta)/(q - \gamma - \delta)]d(x_{n-1}, x_n). \tag{9}$$

Case 3. $x_n \in Q$ and $x_{n+1} \in P$. By the triangle inequality we have

$$d(x_n, x_{n+1}) \leq d(x_n, y_n) + d(y_n, x_{n+1}) = d(x_n, y_n) + d(y_n, y_{n+1}).$$

Then from (ii) and (5) we have

$$\begin{aligned} qd(x_n, x_{n+1}) &\leq qd(x_n, y_n) + qd(y_n, y_{n+1}) \leq qd(x_n, y_n) + H(Tx_{n-1}, Tx_n) \\ &\leq qd(x_n, y_n) + \alpha d(x_{n-1}, x_n) + \beta \max\{D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n)\} \\ &\quad +\gamma[D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})] + \delta[D(x_{n-1}, Tx_{n-1}) + D(x_n, Tx_n)] \\ &\leq qd(x_n, y_n) + \alpha d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, y_n), d(x_n, y_{n+1})\} \\ &\quad +\gamma[d(x_{n-1}, x_{n+1}) + d(x_n, y_n)] + \delta[(d(x_{n-1}, y_n) + d(x_n, y_{n+1})]. \end{aligned} \tag{10}$$

Since two consecutive terms of $\{x_n\}$ cannot be in Q, $x_{n-1} \in P$. Then by (iv) $d(x_n, y_n) + d(x_{n-1}, x_n) = d(x_{n-1}, y_n)$ and hence, as $\alpha < \lambda^a = q$, we have

$$qd(x_n, y_n) + \alpha d(x_{n-1}, x_n) \leq qd(x_{n-1}, y_n).$$

Also, by the triangle inequality,

$$\begin{aligned} \gamma[d(x_{n-1}, x_{n+1}) + d(x_n, y_n)] &\leq \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, y_n)] \\ &= \gamma d(x_{n-1}, y_n) + \gamma d(x_n, x_{n+1}). \end{aligned}$$

Suppose that $d(x_{n-1}, y_n) < d(x_n, y_{n+1}) = d(x_n, x_{n+1})$. Then from (10) we get

$$qd(x_n, x_{n+1}) \leq qd(x_{n-1}, y_n) + \beta d(x_n, x_{n+1}) + (\gamma+\delta)d(x_{n-1}, y_n) + (\gamma+\delta)d(x_n, x_{n+1})$$

and hence

$$d(x_n, x_{n+1}) \le [(q+\gamma+\delta)/(q-\beta-\gamma-\delta)]d(x_{n-1}, y_n). \tag{11}$$

If $d(x_{n-1}, y_n) \ge d(x_n, x_{n+1})$, then, as $1 \le (q+\gamma+\delta)/(q-\beta-\gamma-\delta)$, we have again

$$d(x_n, x_{n+1}) \le [(q+\gamma+\delta)/(q-\beta-\gamma-\delta)]d(x_{n-1}, y_n).$$

Therefore, (11) holds for all n.

Since $x_{n-1} = y_{n-1}$, it follows $d(x_{n-1}, y_n) = d(y_{n-1}, y_n)$. Then, as in Case 1, we have

$$d(x_{n-1}, y_n) \le [(\alpha+\beta+\gamma+\delta)/(q-\gamma-\delta)]d(x_{n-2}, x_{n-1}). \tag{12}$$

By (11) and (12) we obtain

$$d(x_n, x_{n+1}) \le \left[\frac{\alpha+\beta+\gamma+\delta}{q-\gamma-\delta} \cdot \frac{q+\gamma+\delta}{q-\beta-\gamma-\delta}\right] d(x_{n-2}, x_{n-1}). \tag{13}$$

Since $q = \lambda^a < 1$, we have

$$\begin{aligned}
h &= \left[\frac{(\alpha+\beta+\gamma+\delta)}{(q-\beta-\gamma-\delta)}\right] \cdot \left[\frac{q+\gamma+\delta}{q-\gamma-\delta}\right] \\
&= 1 + \frac{(\alpha+\beta+\gamma+\delta)(q+\gamma+\delta) - (\gamma+\delta)(\beta+\gamma+\delta) + q(\beta+2\gamma+2\delta) - q^2}{q^2 - q(\beta+2\gamma+2\delta) + (\gamma+\delta)(\beta+\gamma+\delta)} \\
(14) \quad &\le 1 + \frac{(\alpha+\beta+\gamma+\delta)(1+\gamma+\delta) - (\gamma+\delta)(\beta+\gamma+\delta) + \beta+2\gamma+2\delta - q^2}{q^2 - q(\beta+2\gamma+2\delta) + (\gamma+\delta)(\beta+\gamma+\delta)} \\
&= 1 - \frac{q^2 - [\alpha+2\beta+(3+\alpha)(\gamma+\delta)]}{(q-\beta-\gamma-\delta)(q-\gamma-\delta)} \\
&\le 1 - \frac{\lambda^{2a} - \lambda}{(\lambda^a-\beta-\gamma-\delta)(\lambda^a-\gamma-\delta)}.
\end{aligned}$$

Since $\lambda^{2a} > \lambda$, we conclude that $h < 1$.

By (8), (9) and (13) we conclude that in all cases

$$d(x_n, x_{n+1}) \le h \max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\} \tag{15}$$

for all $n \ge 2$, where h is given by (14).

Now it is easily shown by induction that from (2.15) we have

$$d(x_n, x_{n+1}) \le h^{(n-1)/2} \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

For $m > n > N$,

$$d(x_n, x_m) \leq \sum_{i=N}^{\infty} d(x_i, x_{i+1}) \leq \left[\frac{h^{N/2}}{(h^{1/2} - h)}\right] \max\{d(x_0, x_1), d(x_1, x_2)\}.$$

Hence we conclude that $\{x_n\}$ is a Cauchy sequence, hence convergent. Call the limit u. From the way in which the $\{x_n\}$ were chosen, there exists an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in P$. Then for $n_k = m$, we have

$$\begin{aligned} D(x_{n_k}, Tu) &\leq H(Tx_{m-1}, Tu) \\ &\leq \alpha d(x_{m-1}, u) + \beta \max\{D(x_{m-1}, Tx_{m-1}), D(u, Tu)\} \\ &\quad + \gamma[D(x_{m-1}, Tu) + D(u, Tx_{m-1})] + \delta[D(x_{m-1}, Tx_m) + D(u, Tu)] \\ &\leq \alpha d(x_{m-1}, u) + \beta \max\{d(x_{m-1}, x_m), D(u, Tu)\} \\ &\quad + \gamma[D(x_{m-1}, Tu) + d(u, x_m)] + \delta[d(x_{m-1}, x_m) + D(u, Tu)]. \end{aligned}$$

Taking the limit as $k \to \infty$ yields

$$D(u, Tu) \leq (\beta + \gamma + \delta) D(u, Tu),$$

which implies, as $\beta + \gamma + \delta < 1$, that $D(u, Tu) = 0$. Since Tu is closed, $u \in Tu$. ■

Remark 1. If in Theorem 2.1 $\beta = 0$, $\gamma = 0$ and $\delta = 0$, then we obtain Theorem 1.1 (i.e. Theorem 1 in [2]).

Remark 2. If in Theorem 2.1 $\beta = 0$, then we obtain Theorem 1 of Itoh in [5], since the contractive condition in Theorem 1:

$$(\alpha + \gamma + \delta)(1 + \gamma + \delta)/(1 - \gamma - \delta)^2 < 1 \tag{16}$$

implies (when $\gamma + \delta > 0$) that

$$\begin{aligned} \frac{(\alpha + \gamma + \delta)(1 + \gamma + \delta)}{(1 - \gamma - \delta)^2} &< \frac{(\alpha + \gamma + \delta)(1 + \gamma + \delta) + (\gamma + \delta)(2 - \gamma - \delta)}{(1 - \gamma - \delta)^2 + (\gamma + \delta)(2 - \gamma - \delta)} \\ &= \alpha + (3 + \alpha)(\gamma + \delta). \end{aligned}$$

Remark 3. Theorem 2.1 is a generalization of the above Theorem 1.2 of Rhoades.[6] Since

$$\left(\frac{1 + \alpha + \gamma}{1 - \beta - \gamma}\right)\left(\frac{\alpha + \beta + \gamma}{1 - \gamma}\right) = \frac{\alpha + 2\beta + 3\gamma + \alpha\gamma + \alpha(\alpha + \beta + \gamma) + \gamma(\beta + \gamma) - \beta - 2\gamma}{1 + \gamma(\beta + \gamma) - \beta - 2\gamma},$$

Rhoades condition (1.2) implies that

$$\alpha + 2\beta + 3\gamma + \alpha(\alpha + \beta + \gamma^2) < 1\,.$$

Therefore, Theorem 2.1 is a generalization of Theorem of Rhoades, even if $\delta = 0$. Note that for $\beta = 0$, $\gamma = 0$ and $\delta = 0$ the condition (6) reduces to $\alpha < 1$, and (1.2) to $\alpha < (\sqrt{5} - 1)/2$.

Now we shall give a fixed point theorem for a continuous multi-valued mapping, weaking the condition (6), not requiring that the constant λ be less than 1. We need the following:

Definition. Let K be a non'empty subset of a metric space (X, d). A mapping $T : K \to CB(X)$ is said to be *continuous at* $x_0 \in K$ if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $H(Tx, Tx_0) < \varepsilon$, whenever $d(x, x_0) < \delta$. If T is continuous at each point of K, then T is said to be *continuous on* K.

Theorem 2.3. *Let (X, d) be a complete and metrically convex metric space, K a non'empty compact subset of X. Let T be a continuous mapping of K into $CB(X)$ such that for all $x, y \in K$ with $x \neq y$,*

$$\begin{aligned} H(Tx, Ty) \;<\; & \alpha d(x, y) + \beta \max\{D(x, Tx), D(y, Ty)\} + \gamma[D(x, Ty) + D(y, Tx)] \\ & + \delta[D(x, Tx) + D(y, Ty)], \end{aligned} \tag{17}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and such that

$$\alpha + 2\beta + (3 + \alpha)(\gamma + \delta) \leq 1. \tag{18}$$

If $Tx \subset K$ for each $x \in \partial K$, then there exists an $u \in K$ such that $u \in Tu$.

Proof. Let $f(x) = D(x, Tx)$ for each $x \in K$. Since for each $x, y \in K$

$$D(x, Tx) \leq d(x, y) + D(y, Tx); \quad D(y, Tx) \leq D(y, Ty) + H(Ty, Tx),$$

we have

$$\begin{aligned} |f(x) - f(y)| &\leq |D(x, Tx) - D(y, Tx)| + |D(y, Tx) - D(y, Ty)| \\ &\leq d(x, y) + H(Tx, Ty). \end{aligned}$$

Hence, as T is continuous, $f(x)$ is continuous.

Since K is compact, there exists a $z \in K$ such that

$$f(z) = \min\{f(x) : x \in K\}\,,$$

i.e., such that

$$D(z, Tz) \leq D(x, Tx)\} \tag{19}$$

for each $x \in K$. We shall show that $D(z, Tz) = 0$. Assume the contrary that $f(x) > 0$ for all $x \in K$. Let $\{x_n\}$ be a sequence in Tz such that

$$\lim_{n\to\infty} d(z, x_n) = D(z, Tz). \tag{20}$$

Suppose at first that there exists an infinite subsequence of $\{x_n\}$ which is contained in a compact subset K. Then there exists a subsequence $\{x_{n_i}\}$ which converges to some x_0. Since Tz is closed, $x_0 \in Tz$. Thus $d(z, x_0) = D(z, Tz)$. From (17) we obtain, as $D(z, Tz) > 0$ implies that $z \neq x_0$,

$$\begin{aligned} D(x_0, Tx_0) &\leq H(Tz, Tx_0) \\ &< \alpha d(z, x_0) + \beta \max\{D(z, Tz), D(x_0, Tx_0)\} \\ &+ \gamma D(z, Tx_0) + \delta[D(z, Tz) + D(x_0, Tx_0)] \end{aligned}$$

and hence, as $D(z, Tx_0) \leq d(z, x_0) + D(x_0, Tx_0) = D(z, Tz) + D(x_0, Tx_0)$, we have

$$D(x_0, Tx_0) < \alpha D(z, Tz) + \beta D(x_0, Tx_0) + (\gamma + \delta)[D(z, Tz) + D(x_0, Tx_0)].$$

Hence, using that $D(z, Tz) \leq D(x_0, Tx_0)$ and $\alpha + \beta + 2\gamma + 2\delta \leq 1$, we have

$$D(x_0, Tx_0) < (\alpha + \beta + 2\gamma + 2\delta) D(x_0, Tx_0) \leq D(x_0, Tx_0),$$

a contradiction.

Suppose now that $x_n \notin K$ for all sufficiently large n. Since X is convex and $z \in K$, for each such x_n there exists $y_n \in \partial K$ such that

$$d(z, y_n) + d(y_n, x_n) = d(z, x_n). \tag{21}$$

Since ∂X is compact, we may suppose, for the sake of convenience, that $\{y_n\}$ converges to some $y_0 \in \partial K$. Since f is continuous,

$$\lim_{n\to\infty} D(y_n, Ty_n) = D(y_0, Ty_0). \tag{22}$$

By the triangle inequality, (17) and (21) we have, as $z \neq y_n$,

$$\begin{aligned} & D(y_n, Ty_n) \leq d(y_n, x_n) + D(x_n, Ty_n) \leq d(z, x_n) - d(z, y_n) + H(Tz, Ty_n) \\ < \; & d(z, x_n) - d(z, y_n) + \alpha d(z, y_n) + \beta \max\{D(z, Tz), D(y_n, Ty_n)\} \\ & + \gamma[D(z, Ty_n) + D(y_n, Tz)] + \delta[D(z, Tz) + D(y_n, Ty_n)] \\ \leq \; & d(z, x_n) + \beta \max\{D(z, Tz), D(y_n, Ty_n)\} \\ & + \gamma[d(z, y_n) + D(y_n, Ty_n) + d(y_n, x_n)] + \delta[D(z, Tz) + D(y_n, Ty_n)] \\ = \; & d(z, x_n) + \beta D(y_n, Ty_n) \\ & + \gamma[d(z, x_n) + D(y_n, Ty_n)] + \delta[D(z, Tz) + D(y_n, Ty_n)]. \end{aligned}$$

Taking the limit when n tends to infinity and considering (20) and (22) we get

$$D(y_0, Ty_0) \leq D(z, Tz) + \beta D(y_0, Ty_0)$$
$$+\gamma[D(z, Tz) + D(y_0, Ty_0)] + \delta[D(z, Tz) + D(y_0, Ty_0)].$$

Hence

(23) $$D(y_0, Ty_0) \leq [(1+\gamma+\delta)/(1-\beta-\gamma-\delta)]D(z, Tz).$$

Since $y_0 \in\to K$, $Ty_0 \subset K$. Thus Ty_0 is compact and so there exists $u \in Ty_0$ such that $d(y_0, u) = D(y_0, Ty_0)$.

From (17), as $f(y_0) > 0$ implies that $u \neq y_0$, we have

$$\begin{aligned} D(u, Tu) &\leq H(Ty_0, Tu) \\ &< \alpha d(u, y_0) + \beta \max\{D(u, Tu), D(y_0, Ty_0)\} + \gamma D(y_0, Tu) \\ &\quad + \delta[D(y_0, Ty_0) + D(u, Tu)]. \end{aligned}$$

Since $D(y_0, Tu) \leq d(y_0, u) + D(u, Tu) = D(y_0, Ty_0) + D(u, Tu)$, we have

$$\begin{aligned} D(u, Tu) &< \alpha D(y_0, Ty_0) + \beta \max\{D(u, Tu), D(y_0, Ty_0)\} \\ &\quad + (\gamma+\delta)[D(u, Tu) + D(y_0, Ty_0)] \end{aligned}$$

and hence

$$\begin{aligned} D(u, Tu) &< \max\left[\frac{\alpha+\beta+\gamma+\delta}{\alpha-\gamma-\delta}\right] \cdot \left[\frac{1+\gamma+\delta}{1-\beta-\gamma-\delta}\right] D(y_0, Ty_0) \\ &= [(\alpha+\beta+\gamma+\delta)/(1-\gamma-\delta)]D(y_0, Ty_0). \end{aligned}$$

So by (23) we have

(24) $$D(u, Tu) < \left[\frac{\alpha+\beta+\gamma+\delta}{1-\gamma-\delta}\right] \cdot \left[\frac{1+\gamma+\delta}{1-\beta-\gamma-\delta}\right] D(z, Tz).$$

Since

$$\frac{(\alpha+\beta+\gamma+\delta)(1+\gamma+\delta)}{(1-\gamma-\delta)(1-\beta-\gamma-\delta)} = \frac{\alpha+2\beta+(3+\alpha)(\gamma+\delta)-\beta-(\gamma+\delta)(2-\beta-\gamma-\delta)}{1-\beta-(\gamma+\delta)(2-\beta-\gamma-\delta)},$$

taking in consideration (18) we get

$$(\alpha+\beta+\gamma+\delta)(1+\gamma+\delta)/[(1-\gamma-\delta)(1-\beta-\gamma-\delta)] \leq 1\,.$$

Thus by (24) we have

$$D(u, Tu) < D(z, Tz),$$

a contradiction with (19). Therefore, $D(z, Tz) = 0$. Hence, as Tz is closed, $z \in Tz$. ■

R e m a r k 4. Theorem 2.3 is a generalization of Theorem of Assad [1] and Theorem 2 of Itoh in [5]. The presented method of proof gives a simplification of the corresponding proof of Theorem 2 given by Itoh in [5].

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Received 14.10.2004