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A Study of the Constant in the Strengthened Cauchy Inequality for 3D Elasticity Problems

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Presented by Bl. Sendov

The constant γ in the strengthened Cauchy-Bunyakowskii-Schwarz inequality is a basic tool for constructing of two-level and multilevel preconditioning matrices. Therefore many authors consider estimates or computations of this quantity. In this paper the bilinear form arising from 3D linear elasticity problem is considered on a polyhedron. The cosine of the abstract angle between multilevel finite element subspaces is computed by a spectral analysis of a general eigenvalue problem. Octasection and bisection approaches are used for refining of the triangulations. Tetrahedron, pentahedron and hexahedron meshes are considered. The dependence of the constant γ on the Poisson ratio is presented graphically.

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Key Words: strengthened Cauchy inequality, linear elasticity problem, finite element method

1. Introduction

Many problems in engineering and natural sciences can be described by boundary value problems, e.g., the heat transfer, the deformation of bodies under given loads, electrical and magnetic fields. Multilevel methods are often used practical tools for solving of the above problems. There are different approaches for constructing multilevel methods and different techniques for the convergence analysis of these methods. For convergence proofs without regularity assumptions on the solution the strengthened Cauchy-Buniakowski-Schwarz (C.B.S.) inequality is the main ingredient (see, e.g., [?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?, ?]). In these cases one gets estimates of the convergence factor in dependence on the constant in the C.B.S. inequality. For that reason it is of practical interest to have estimates of this constant. In the literature the C.B.S. inequality related to different boundary value problems is considered (see, e.g.,

[?, ?]). In our paper we want to concentrate on elasticity problems in three-dimensional domains discretized by means of the finite element method with tetrahedral, pentahedral, and hexahedral elements. For elasticity problems in two-dimensional domains there are several results published. Jung considers in [?] and [?] discretizations of linear elasticity problems by means of hierarchical piecewise linear ansatz functions on right isosceles triangles with mesh refinement by a bisection and a non-standard division of each triangle into four congruent subtriangles, respectively. In that papers the dependence of the constant in the C.B.S. inequality on the Poissons ratio is given by formulas. Margenov proves in [?] that $\sqrt{0.75}$ is an upper bound of the C.B.S. constant for triangulations with right isosceles triangles and standard division into four subtriangles. Additionally, he shows by numerical computations that $\sqrt{0.75}$ is also an upper bound for arbitrary right triangles. Achchab and Maitre consider in [?] discretizations with arbitrary triangular elements and hierarchical piecewise linear ansatz functions and prove that $\sqrt{0.75}$ is an upper bound of the C.B.S. constant in this general case. In [?], Jung and Maitre prove that between the C.B.S. constant $(\gamma^{\ell})^2$ in the case of hierarchical piecewise linear ansatz functions and the constant $(\gamma^{\mathrm{q}})^2$ in the case of piecewise linear/piecewise quadratic ansatz functions the relation $(\gamma^{\ell})^2 = 0.75(\gamma^{\mathrm{q}})^2$ holds for all bilinear forms which are a linear combination of terms of the type $\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx$, $i = 1, 2$. Additionally, they derive the dependence of the C.B.S. constant on the Poisson ratio for discretizations with right isosceles triangles and standard division of each triangle into four subtriangles in the mesh refinement. Furthermore, they consider the reference tetrahedron to get a first result for three-dimensional elasticity problems and give the numerically determined estimate $\sqrt{0.9}$ for the C.B.S. constant. Achchab, Axelsson, Laayouni, and Souissi present in [?, ?] an analytical proof that $\sqrt{0.9}$ is an upper bound of the C.B.S. constant in the case of triangulations with arbitrary tetrahedral elements. This result is generalized in [?] to bilinear forms which are linear combination of terms of the type $\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx$, $i = 1, 2, 3$.

In the present paper we give numerically determined upper bounds of the C.B.S. constant in dependence on the Poisson ratio. Hereby we consider discretizations with tetrahedra, pentahedra, and hexahedra and different polynomial degree of the finite element ansatz functions. We discuss Octasection and bisection approaches.

The paper is organized as follows: In Section 2 we describe the considered elasticity problem and introduce some notation. In Section 3 some general remarks on the computation of the C.B.S. constant are summarized. In Sections 4,5 and 6 estimates of the C.B.S. constant on tetrahedral, pentahedral, and hexahedral triangulations are given. Finally, we discuss the presented results.

2. Setting of the problem

Let us consider the following linear elasticity problem in three-dimensional domain Ω with the boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, $\mathrm{meas}(\Gamma_0) \neq 0$:

Find the displacement vector $\mathbf{u} = [u_i]_{i=1}^3 \in [C^2(\Omega) \cap C^1(\Omega \cup \Gamma_1) \cap C(\bar{\Omega})]^3$ such that

$$\begin{aligned} \sum_{j=1}^{3} \frac{\partial \sigma_{ij}(\mathbf{u})}{\partial x_j} + f_i &= 0 \quad \text{in } \Omega, \qquad i = 1,2,3, \\ u_i &= 0 \quad \text{on } \Gamma_0, \qquad i = 1,2,3, \\ \sum_{j=1}^{3} \sigma_{ij}(\mathbf{u}) n_j &= g_i \quad \text{on } \Gamma_1, \qquad i = 1,2,3, \end{aligned} \tag{1}$$

where

$$\sigma_{ij}(\mathbf{u}) = \lambda \sum_{k=1}^{3} \varepsilon_{kk}(\mathbf{u}) \delta_{ij} + 2\mu \varepsilon_{ij}(\mathbf{u}).$$

are the components of the stress tensor and

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i,j = 1,2,3,$$

are the components of the strain tensor. The Lamé coefficients λ and μ can be expressed by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)} \tag{2}$$

with Young's elasticity modulus E and the Poisson ratio ν. The vector $\mathbf{n} = \{n_1, n_2, n_3\}$ denotes the outward unit normal to Γ,

The weak formulation of problem (1) reads as:

Find $\mathbf{u} \in V = \{\mathbf{v} \in [H^1(\Omega)]^3 \mid \mathbf{v} = 0 \text{ on } \Gamma_0\}$ such that

$$a(\mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}), \quad \forall \mathbf{v} \in V.$$

with

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \lambda \int_\Omega \mathrm{div}(\mathbf{u}) \mathrm{div}(\mathbf{v})\, dx + 2\mu \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v})\, dx, \\ \ell(\mathbf{v}) &= \int_\Omega (\mathbf{f}, \mathbf{v})\, dx + \int_{\Gamma_1} (\mathbf{g}, \mathbf{v})\, d\sigma\,. \end{aligned}$$

We suppose that

$$\mathbf{f} = [f_i]_{i=1}^3 \in [L^2(\Omega)]^3 \quad \text{and} \quad \mathbf{g} = [g_i]_{i=1}^3 \in [L^2(\Gamma_1)]^3 .$$

Using the representation (2) of the Lamé coefficients we can express the bilinear form $a(.,.)$ by

$$(3) \quad a(\mathbf{u},\mathbf{v}) = 2\mu \left(\frac{\nu}{1-2\nu} \int_\Omega \mathrm{div}(\mathbf{u})\mathrm{div}(\mathbf{v})\, dx + \sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(\mathbf{u})\varepsilon_{ij}(\mathbf{v})\, dx \right).$$

3. General remarks on the computation of the C.B.S. constant

In this Section, we summarize known techniques for the computation of the constant in the strengthened C.B.S. inequality, see, e.g., [**?**].

Let $\hat{T}$ be a finite element of reference. The element $\hat{T}$ will be concretized further concerning different triangulations. We fix an initial triangulation

$$\tau_0 = \{T \in \tau_0 \mid T = F_T(\hat{T}), \;\; F_T \text{ is an invertible affine transformation}\}$$

of the domain Ω and generate a sequence of triangulations $\{\tau_k\}, \;\; k = 0,1,2,...$ that form successive refinements of τ_0. Let V_{k-1} be a finite element space associated with the triangulation τ_{k-1} and $\widetilde{V}_k$ be the correspondent hierarchical space such that V_k is the direct sum of V_{k-1} and $\widetilde{V}_k$, $V_{k-1} \cap \widetilde{V}_k = \{0\}$, [**?**].

Our aim is to get upper estimates of the constant γ in the C.B.S. inequality.

$$(4) \qquad |a(\mathbf{u},\mathbf{v})| \le \gamma\, (a(\mathbf{u},\mathbf{u}))^{\frac{1}{2}}\, (a(\mathbf{v},\mathbf{v}))^{\frac{1}{2}}, \;\; \forall u \in V_{k-1}, \;\; \forall v \in \widetilde{V}_k.$$

Estimates of the constant γ can be obtained locally, i.e. elementwise, see e.g., [**?**, **?**]. We consider on each element of the triangulation τ_{k-1} the strengthened C.B.S. inequality

$$(5) \; |a_T(\mathbf{u},\mathbf{v})| \le \gamma_T\, (a_T(\mathbf{u},\mathbf{u}))^{\frac{1}{2}}\, (a_T(\mathbf{v},\mathbf{v}))^{\frac{1}{2}}, \; \forall u \in V_{k-1}, \; \forall v \in \widetilde{V}_k, \; T \in \tau_{k-1},$$

where the bilinear form $a_T(\mathbf{u},\mathbf{v})$ is the restriction of the bilinear form $a(\mathbf{u},\mathbf{v})$ on the finite element T. Then for the constant γ in (4) holds

$$\gamma = \max_{T \in \tau_{k-1}} \gamma_T, \;\; k = 1,2,\ldots$$

(see [?, ?, ?, ?]). To compute the constant γ_T in (5) we construct for each element $T \in \tau_{k-1}$ the so-called two-level element stiffness matrix on level k:

$$A_T^{(k)} = \left(\begin{array}{cc} A_{T,11} & A_{T,12} \\ A_{T,21} & A_{T,22} \end{array} \right), \quad T \in \tau_{k-1}\,,$$

where the indices "1" and "2" correspond to the new nodes in the triangulation τ_k and to the nodes in the triangulation τ_{k-1}, respectively, see [?]. Then, $A_{T,22} = A_T^{(k-1)}$ is valid. We note that the element stiffness matrix $A_T^{(k-1)}$ of level $k-1$ and the corresponding Schur complement

$$S_T = A_T^{(k-1)} - A_{T,21} A_{T,11}^{-1} A_{T,12}$$

have one and the same null space i.e. $\mathbb{N} = \mathcal{N}(S_T) = \mathcal{N}(A_T^{(k-1)})$. In the case of three-dimensional linear elasticity problems the space $\mathbb{N}$ has the dimension 6.

For getting an estimate of γ_T we consider the generalized eigenvalue problem

$$S_T \mathbf{v}_T = \lambda A_T^{(k-1)} \mathbf{v}_T. \tag{6}$$

The bilinear form (3) is coercive, symmetric and continuous on Ω. Since we have hierarchical refinement for obtaining of any fine triangulation, the two-level matrices A_T, $T \in \tau_{k-1}$ are symmetric and positive semidefinite with one invertible block in the main diagonal. Then we can apply the pure algebraic approach described by Eijkhout and Vassilevski in [?] for obtaining

$$\lambda_{T,\,\min} = 1 - \gamma_T^2. \tag{7}$$

where $\lambda_{T,\,\min}$ is the smallest non-zero eigenvalue of problem (6). Let n be the dimension of the space V_{k-1}. We define a matrix $B_T = [\mathbf{w}_1\ \mathbf{w}_2\ \ldots\ \mathbf{w}_p]_{n\times p}$, where $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p \notin \mathcal{N}(A_T^{(k-1)})$ and $R^n = \mathcal{N}(A_T^{(k-1)}) + \mathrm{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\}$. Then, the smallest non-zero eigenvalue of (6) is equal to the smallest eigenvalue of the following general eigenvalue problem [?, Lemma 2]

$$\mathcal{S}_T \mathbf{v}_T = \lambda \mathcal{A}_T^{(k-1)} \mathbf{v}_T, \tag{8}$$

with $\mathcal{S}_T = B_T^\top S_T B_T$ and $\mathcal{A}_T^{(k-1)} = B_T^\top A_T^{(k-1)} B_T$ (see also [?, ?, ?]). Then, we shall use (8) for computing the element γ_T-constant.

Using the fact that

$$\begin{aligned} \ker\{a_T\} &= \left\{ \mathbf{v} \in V_{k|T} \,:\, a_T(\mathbf{v}, \mathbf{z}) = 0, \quad \forall \mathbf{z} \in V_{k|T} \right\} \subset V_{k-1|T} \\ &= \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -x_3 \\ x_2 \end{bmatrix}, \begin{bmatrix} -x_3 \\ 0 \\ x_1 \end{bmatrix} \right\} \end{aligned}$$

we can determine vectors spanning the null space of the matrix $A_T^{(k-1)}$ and find then appropriate vectors for defining the matrices B_T.

We note that μ is a factor in the bilinear form (3). Therefore having in mind (6), (7), and (8) we conclude that the constant γ_T is independent of μ.

4. Splitting of the finite element spaces on the tetrahedron meshes

For our further considerations we make the following assumptions on the domain Ω:

(H1) Ω is a polyhedron with all faces parallel to the coordinate planes;

(H2) all edges of Ω have rational lengths.

We define a class of similarity $[T]$ in an arbitrary triangulation τ of the domain Ω by

$$[T] = \{L \in [T] \ | \ L \sim T, \ L \in \tau\},$$

i.e. one finite element L belongs to the class $[T]$ when L is geometrically similar to the finite element $T \in \tau$.

We introduce two special elements: the finite element of reference $\hat{T}$ and a regular pyramid K. Let $\hat{T}$ be the canonical 3D simplex with vertices $\hat{a}_1(1,0,0)$, $\hat{a}_2(0,1,0)$, $\hat{a}_3(0,0,1)$, $\hat{a}_4(0,0,0)$ (see Figure 1). The regular tetrahedron K is defined by the vertices $b_1(1,1,-1)$, $b_2(-1,1,1)$, $b_3(1,-1,1)$, and $b_4(-1,-1,-1)$.

a_3
a_{10}
a_6 a_8
a_4
a_7 a_9
a_1 a_5 a_2

$$\begin{aligned}
T\{1\} &= \Delta a_1 a_5 a_6 a_7,\\
T\{2\} &= \Delta a_5 a_2 a_8 a_9,\\
T\{3\} &= \Delta a_6 a_8 a_3 a_{10},\\
T\{4\} &= \Delta a_7 a_9 a_{10} a_4,\\
T\{5\} &= \Delta a_5 a_6 a_7 a_9,\\
T\{6\} &= \Delta a_5 a_6 a_8 a_9,\\
T\{7\} &= \Delta a_6 a_8 a_9 a_{10},\\
T\{8\} &= \Delta a_6 a_7 a_9 a_{10}.
\end{aligned}$$

Figure 1: The decomposition of the tetrahedron $T = \Delta a_1 a_2 a_3 a_4$ into eight subtetrahedra

We consider sequences of triangulations $\{\tau_k\}$, $k = 0, 1, 2, \ldots,$ of the domain Ω satisfying the following conditions:

(i) The initial triangulation τ_0 contains only tetrahedra;

(ii) The triangulation τ_1 is obtained from τ_0 by dividing each element of τ_0 into eight elements as it is shown in Figure 1. The nodes are numbered using the strategy of Bey [**?**];

(iii) The triangulation τ_k is obtained from τ_{k-1} by the same way as τ_1 is obtained from τ_0 in (ii).

This approach for obtaining the triangulation τ_k is called regular refinement strategy.

Theorem 1. *Let the domain* Ω *fulfils the hypotheses (H1) and (H2). Then using regular refinements we can obtain a sequence of triangulations* $\{\tau_k\},\ \ k = 0, 1, 2, \dots$ *, such that there are only five classes of similarity for all elements in all triangulations* τ_k.

Proof. Since the hypotheses (H1) and (H2) hold, the domain Ω can be partitioned by cubes. There exists a partition of any cube into five tetrahedra, four of them from the class $[\hat{T}]$ and one from the class $[K]$ [**?**, p. 56]. To obtain an initial triangulation of the domain Ω we perform two steps:

(a) Decompose Ω by cubes;

(b) Decompose each cube from (a) into five tetrahedra of classes $[\hat{T}]$ and $[K]$.

An arbitrary element $T \in \tau_0$ can be obtained by $T = F_T(\hat{T})$ or by $T = F_T(K)$, with the invertible affine linear transformation F_T. Then it is sufficient to decompose the elements $\hat{T}$ and K for obtaining any triangulation τ_k, $k = 1, 2, 3, \dots$.

Following the numeration strategy of Bey [**?**] we decompose the finite element of reference $\hat{T}$ into eight subtetrahedra $\hat{T}\{i_1\},\ 1 \le i_1 \le 8$ (see Figure 1). To obtain the second level we divide each element $\hat{T}\{i_1\},\ 1 \le i_1 \le 8$, again into eight subtetrahedra $\hat{T}\{i_1, i_2\},\ 1 \le i_2 \le 8$. Thus we get a family of tetrahedra:

$$\{\hat{T},\ \hat{T}\{i_1\},\ \hat{T}\{i_1,i_2\},\ \hat{T}\{i_1,i_2,i_3\},\dots,\hat{T}\{i_1,i_2,i_3,\dots,i_k\},\dots \mid \quad 1 \le i_n \le 8,\ k \in \mathbf{N}\}. \tag{9}$$

Considering all tetrahedra in (9) we could obtain at most six classes of similarity [**?**, p. 59]. But having in mind that we decompose the reference tetrahedron we can prove that we get only three classes. We have the following:

$$[\hat{T}],\ \ [\hat{T}\{5\}],\ \ [\hat{T}\{6\}] \tag{10}$$

in the first level. Further we decompose the tetrahedra of the first level and obtain in the second level:

$$\begin{array}{llll} \hat{T}\{5,i_2\} \in [\hat{T}\{5\}]\,, & 1 \le i_2 \le 4, & \hat{T}\{5,i_2\} \in [\hat{T}\{6\}]\,, & i_2 = 5,7, \\ \hat{T}\{5,i_2\} \in [\hat{T}]\,, & i_2 = 6,8, & & \\ \hat{T}\{6,i_2\} \in [\hat{T}\{6\}]\,, & 1 \le i_2 \le 4, & \hat{T}\{6,i_2\} \in [\hat{T}\{5\}]\,, & i_2 = 5,7, \\ \hat{T}\{6,i_2\} \in [\hat{T}]\,, & i_2 = 6,8. & & \end{array}$$

From this we can conclude that all tetrahedra in the next levels belong to one of the three classes given in (9).

Let us now consider the regular tetrahedron K. In the first level we have eight subtetrahedra as follows:

$$K\{i_1\} \in [K]\,,\ \ 1 \le i_1 \le 4 \quad \text{and} \quad K\{i_1\} \in [K\{5\}],\ \ 5 \le i_1 \le 8.$$

In the second level we have to decompose the finite element $K\{5\}$. We obtain

$$K\{5,i_2\} \in [K]\,,\ \ i_2 = 6,8 \quad \text{and} \quad K\{5,i_2\} \in [K\{5\}]\,,\ \ i_2 = 1,2,...,5,7\,.$$

Therefore, there are only two classes of similarity decomposing the regular pyramid K. Consequently, we have the following classes:

$$[\hat{T}],\ \ [\hat{T}\{5\}],\ \ [\hat{T}\{6\}],\ \ [K],\ \ [K\{5\}]$$

for all level of triangulations of the domain Ω ∎

Corollary 1. *Let the conditions of Theorem 1 be satisfied and the initial triangulation of the domain Ω is obtained by steps (a) and (b). If the sequence of triangulations $\{\tau_k\}$, $k = 0,1,2,\dots$ is obtained by a regular refinement strategy then the constant γ in (4) can be computed by*

$$\gamma = \max\left\{\gamma(\hat{T}),\ \ \gamma(\hat{T}\{5\}),\ \ \gamma(\hat{T}\{6\}),\ \ \gamma(K),\ \ \gamma(K\{5\})\right\}$$

where $\gamma(T)$ is the local γ-constant obtained on the finite element T.

P r o o f. The corollary follows directly from Theorem 1 and [?]. ∎

Further we present estimates of the constant γ_T for the five classes of similarity. Different possibilities for the definition of the finite element spaces V_{k-1} and $\widetilde{V}_k$ are considered. We show in Figures 2 – 6 how the constant γ_T depends on the Poisson's ratio ν. The results are obtained by solving the generalized eigenvalue problem (8) numerically and computing γ_T according to (7). To distinguish the constants γ_T for different finite element discretizations we denote it by an additional upper index, i.e. in the case of piecewise polynomial finite element ansatz functions of degree m we use the notation $\gamma_T^{(m)}$. In the figures, we mark the plots for $\gamma(\hat{T})$, $\gamma(\hat{T}\{6\})$, $\gamma(\hat{T}\{5\})$, $\gamma(K)$, and $\gamma(K\{5\})$ by **1**, **2**, **3**, **4**, and **5**, respectively.

Let us start with the case, where the finite element spaces V_{k-1} and $\widetilde{V}_k$ are defined by piecewise polynomials of degree not exceeding m, i.e.

$$V_{k-1}^{(m)} = \{\mathbf{v} = (v_1, v_2, v_3)^\top \in [C^0(\overline{\Omega})]^3 \mid v_{i|T} = \hat{v}_i \circ F_T^{-1}, \\ \hat{v}_i \in P_m(\hat{T}),\ \ i = 1,2,3,\ \ \forall T \in \tau_{k-1}\},\ \ m = 1,2,3\,. \tag{11}$$

First we consider the case with piecewise linear ansatz functions for defining the spaces V_{k-1} and $\widetilde{V}_k$, respectively. Figure 2a) shows the dependence of the C.B.S. constant $\gamma_T^{(1)}$ on the Poisson's ratio ν. The plots show that the C.B.S. constant $(\gamma_T^{(1)})^2$ is bounded by 0.9 as it is proved analytically by Achchab, Axelsson, Laayouni, and Souissi in [?] and correspond to the results given in [?].

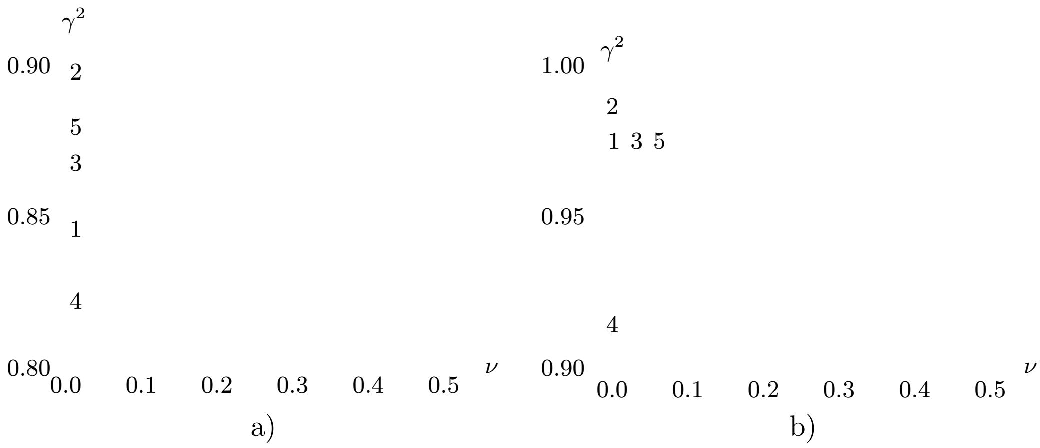


Figure 2: a) Piecewise linear functions b) Piecewise quadratic functions:

If the finite element spaces V_{k-1} and $\widetilde{V}_k$ are defined by means of piecewise quadratic or piecewise cubic functions, we get the constants $\gamma_T^{(2)}$ or $\gamma_T^{(3)}$, respectively, which are illustrated in Figure 2b) and Figure 3.

We have

$$\max\left(\gamma^{(m)}(\hat{T}\{5\}), \gamma^{(m)}(K\{5\})\right) < \gamma^{(m)}(\hat{T}\{6\}),\ \ \forall\nu \in \left(0, \frac{1}{2}\right),\ \ m = 2,3\,,$$

$$0.981 < \gamma^{(2)}(\hat{T}\{6\}) < 1, \quad \forall\nu \in \left(0, \frac{1}{2}\right),\ \ m = 2,3$$

and

$$0.99 < \gamma^{(3)}(\hat{T}\{6\}) < 1,\ \ \forall\nu \in \left(0, \frac{1}{2}\right).$$

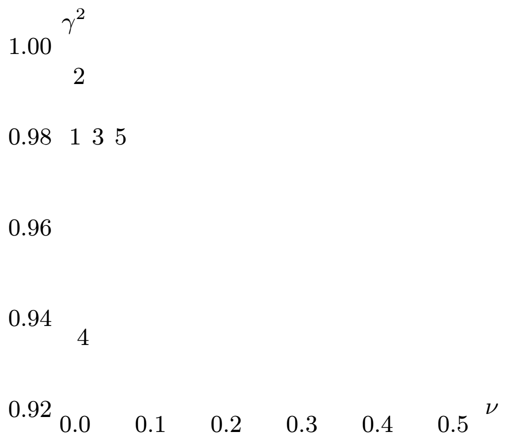


Figure 3: Piecewise cubic functions: $\gamma^{(3)}(\nu)$.

Now, we discuss the case where the space V_{k-1} is defined by piecewise linear functions (case $m = 1$ in (11)) and the space $\widetilde{V}_k$ will be defined by

$$\begin{aligned}\widetilde{V}_k = \mathcal{V}^{(2)} &= \{\mathbf{w} = (w_1, w_2, w_3) \mid w_{i|T} = \hat{w}_i \circ F_T^{-1}, \;\; \hat{w}_i \in P_2(\hat{T}), \\ &\quad \hat{w}_i(\hat{a}_j) = 0, \;\; i = 1, 2, 3, \;\; j = 1, 2, 3, 4, \;\; \forall T \in \tau_{k-1}\}\end{aligned}$$

spanned by quadratic bump functions.

Further we consider a splitting of the finite element spaces over one triangulation (see Figure 4). Let V be a finite element space associated with a triangulation τ of the domain Ω. Let also the space $V = V_1 \oplus V_2$, where V_1 and V_2 are disjoint subspaces of V. Then we denote the cosine of the abstract angle between the subspaces V_1 and V_2 by $\gamma(V_1, V_2)$.

The space $H = V^{(1)} \oplus \mathcal{V}^{(2)}$ is spanned on the piecewise linear functions corresponding to the vertices of the elements added with the basis functions of $V^{(2)}$ corresponding to the edge nodes. The graphic of the constant

$$\widetilde{\gamma}^{(1,2)} = \gamma(V^{(1)}, \mathcal{V}^{(2)})$$

is presented in Figure 5 a).

Introduce finite element space $\mathcal{V}^{(\mathrm{b})}$ spanned on the bulb functions by

$$\mathcal{V}^{(\mathrm{b})} =$$

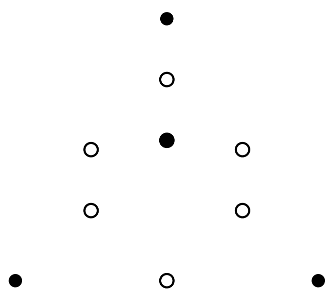
$$\mathrm{span}\{x_i x_j x_k, \; x_1 x_2 x_3 x_4 \mid i < j < k, \; i, j, k \in \{1, 2, 3, 4\}, x_4 = 1 - x_1 - x_2 - x_3\}.$$

The space $\mathcal{V}^{(b)}$ consists of the functions which vanishes on the edges of the element $T \in \tau$. Consider a two-level splitting $H^{(b)} = H \oplus \mathcal{V}^{(b)}$ (see Figure 4) and the corresponding

$$\gamma^{(b)} = \gamma(H, \mathcal{V}^{(b)}),$$

(see Figure 5 b)). This case deserve special attention since $\gamma^{(b)}(\nu) \leq \sqrt{0.97}$, $\forall \nu \in (0, \frac{1}{2})$.

At the end of this section we consider another splitting of the finite element space $H^{(b)} = V^{(1)} \oplus (\mathcal{V}^{(2)} \oplus \mathcal{V}^{(b)})$ with corresponding $\widetilde{\gamma}^{(b)} = \gamma(V^{(1)}, \mathcal{V}^{(2)} \oplus \mathcal{V}^{(b)})$, (see Figure 6).



$\bullet$ – the nodes in the vertices
$\circ$ – the nodes in the middles of the edges
$\square$ – the nodes in the barycenters of the faces
$\times$ – the node in the barycenter of the element

Figure 4: The splitting of the finite element space $H^{(b)}$.

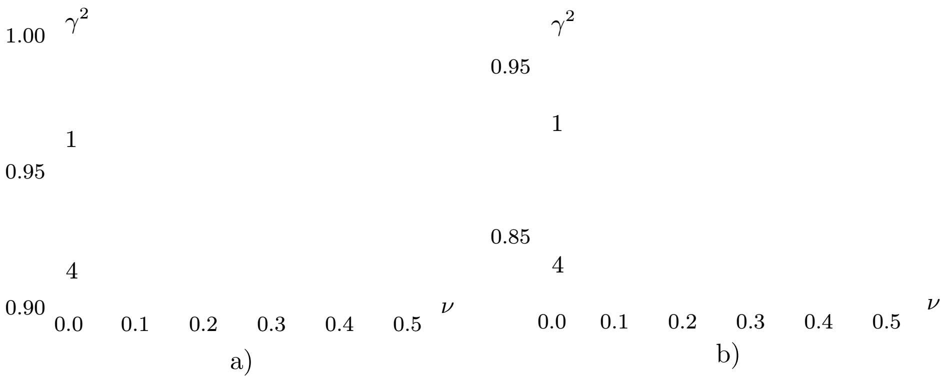


Figure 5: a) $\widetilde{\gamma}_T^{(1,2)}(\nu)$, b) $\gamma^{(b)}(\nu)$.

5. Pentahedron meshes

Again, we assume that the hypotheses (H1) and (H2) are valid. To obtain an initial triangulation τ_0 of the domain Ω we make the following steps:

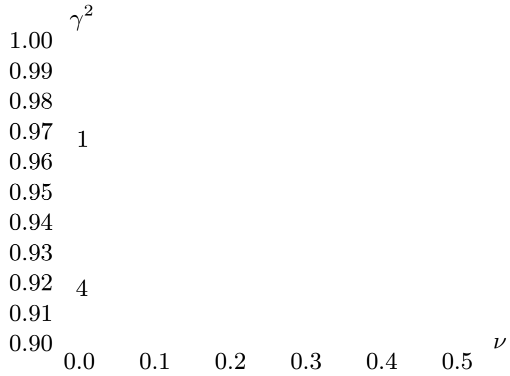


Figure 6: The dependence of the constant $\widetilde{\gamma}^{(\mathrm{b})}$ on the Poisson ratio.

(c) Decompose Ω by cubes;

(d) Decompose each cube from (c) into two pentahedra as it is shown in Figure 7a).

We construct a triangulation τ_1 by dividing each pentahedron from τ_0 into eight pentahedra as it is done in Figure 7b). We generate any refined triangulation τ_k from τ_{k-1} by the same way. Then for the constant γ in (4) we have $\gamma{=}\gamma(\hat{E})$, where $\hat{E}$ is the pentahedron with the vertices

$$\hat{E} = \{(1,0,0),\ (0,1,0),\ (0,0,0),\ (1,0,1),\ (0,1,1),\ (0,0,1)\}. \tag{12}$$

First we consider the case, where the spaces V_{k-1} and $\widetilde{V}_k$ are defined by the finite element ansatz functions corresponding to the 6-node pentahedron, called the *Pen 1* to *Pen 1* case. The corresponding constant in the C.B.S. inequality will be denoted by $\gamma^{(\mathrm{pen},1)}$. This constant is bounded by $\sqrt{0.95}$, see Figure 8. If the spaces V_{k-1} and $\widetilde{V}_k$ are defined by the ansatz functions corresponding to the 18-node pentahedron, i.e. in the *Pen 2* to *Pen 2* case, we get the behaviour of the constant $\gamma^{(\mathrm{pen},2)}$ illustrated in Figure 8. Additionally, we consider the case where V_{k-1} is defined by the ansatz functions of the 6-node pentahedron and $\widetilde{V}_k$ is spanned by the nodal basis functions of the 18-node pentahedron in the midpoints of the edges. The dependence of the corresponding constant $\gamma^{(\mathrm{pen},1,2)}$ is shown in Figure 8. Finally, we discuss the serendipity case (15-node pentahedron elements [**?**, p. 462]). By analogy to the constants $\gamma^{(\mathrm{pen},2)}$ and $\gamma^{(\mathrm{pen},1,2)}$ we analyse constants $\gamma_s^{(\mathrm{pen},2)}$ and $\gamma_s^{(\mathrm{pen},1,2)}$. We obtain

$$0.97 < \gamma_s^{(1,2)} < \gamma^{(1,2)} < 1, \quad \forall \nu \in \left(0, \frac{1}{2}\right).$$

$a)$ $b)$

Figure 7: **a)** Dividing of a hexahedron to two pentahedra; **b)** A decomposition of a pentahedron to eight subpentahedra.

γ^2
1.00
0.98 3
0.96 2
0.94
0.92
0.90
0.88 1
0.86
0.0 0.1 0.2 0.3 0.4 0.5 ν

Figure 8: Pentahedron meshes: **1** - $\gamma^{(\text{pen},1)}(\nu)$, **2** - $\gamma^{(\text{pen},2)}(\nu)$, **3** - $\gamma_s^{(\text{pen},2)}(\nu)$.

Figure 9: A partition of a hexahedron to eight elements.

6. Hexahedron meshes

We suppose that the hypotheses (H1) and (H2) concerning the domain Ω hold. We obtain an initial triangulation τ_0 by dividing Ω into cubes. The triangulation τ_1 is obtained from τ_0 by partitioning of each hexahedron into eight hexahedra as it is shown in Figure 9. An arbitrary triangulation τ_k of the domain Ω is obtained from τ_{k-1} by the same way.

Let the finite element spaces $V_{k-1}^{(m)}$ ($V_k^{(m)}$) be spanned by functions which are continuous and are polynomials of degree m in each direction x_i, $i = 1, 2, 3$, on each element $T \in \tau_{k-1}$ ($T \in \tau_k$). Again, we consider the following three cases:

(a) $V_{k-1} = V_{k-1}^{(1)}$ and $\widetilde{V}_k$ is spanned by the ansatz function from $V_k^{(1)}$ which correspond to the new nodes in τ_k.

(b) We use 27-node hexahedra, $V_{k-1} = V_{k-1}^{(2)}$ and $\widetilde{V}_k$ is spanned by the ansatz functions from $V_k^{(2)}$ such that $V_k^{(2)} = V_{k-1}^{(2)} \oplus \widetilde{V}_k$ and $V_{k-1}^{(2)} \cap \widetilde{V}_k = \{0\}$.

(c) $V_{k-1} = V_{k-1}^{(1)}$ and $\widetilde{V}_k$ is spanned by the ansatz function from $V_k^{(2)}$ which correspond to the node in the midpoints of the edges, the centers of the faces, and the center of the hexahedra $T \in \tau_{k-1}$.

The corresponding constants in the C.B.S. inequality are denoted by $\gamma^{(\text{hex},1)}$, $\gamma^{(\text{hex},2)}$, and $\gamma^{(\text{hex},1,2)}$, respectively. The dependence of these constants on the Poisson ratio ν is shown in Figure **??**.

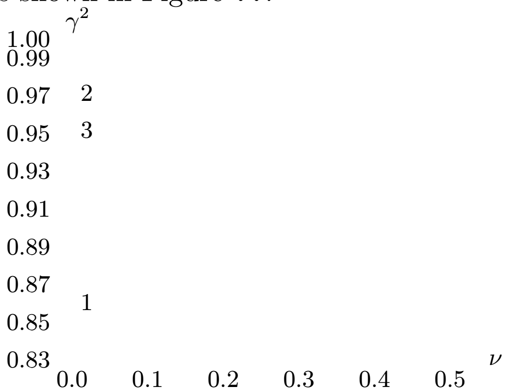


Figure 10: Hexahedron meshes, the octasection method: **1** – $\gamma^{(\text{hex},1)}$, **2** – $\gamma^{(\text{hex},2)}$, **3** – $\gamma^{(\text{hex},1,2)}$.

Finally, we consider the bisection method for hexahedron meshes. Let $T \in \tau_0$ be an arbitrary cube in the initial triangulation. We obtain a refined triangulation τ_1 by dividing of each cube $T \in \tau_0$ into two hexahedra $T\{i_1\}$, $i_1 = 1, 2$. In the first step the refinement procedure is in x_1-direction. Further we make a refinement in x_2-direction dividing the hexahedron $T\{i_1\}$ into two hexahedra $T\{i_1, i_2\}$, $i_1, i_2 \in \{1, 2\}$. In the third step we obtain a triangulation τ_3 refining the triangulation τ_2 in x_3-direction. Thus we obtain $T\{i_1, i_2, i_3\}$, $i_j \in \{1, 2\}$, $j = 1, 2, 3$. We repeat these three steps in the same order for obtaining of arbitrary triangulation τ_k, $k \geq 4$. Then

$$\gamma = \max_{i=0,1,2} \gamma_{b,i},$$

where

$$\gamma_{b,0} = \gamma(T), \quad \gamma_{b,1} = \gamma(T\{i_1\}), \quad \gamma_{b,2} = \gamma(T\{i_1, i_2\}).$$

The dependence of these constants on the Poisson ratio is given in Figure ??.

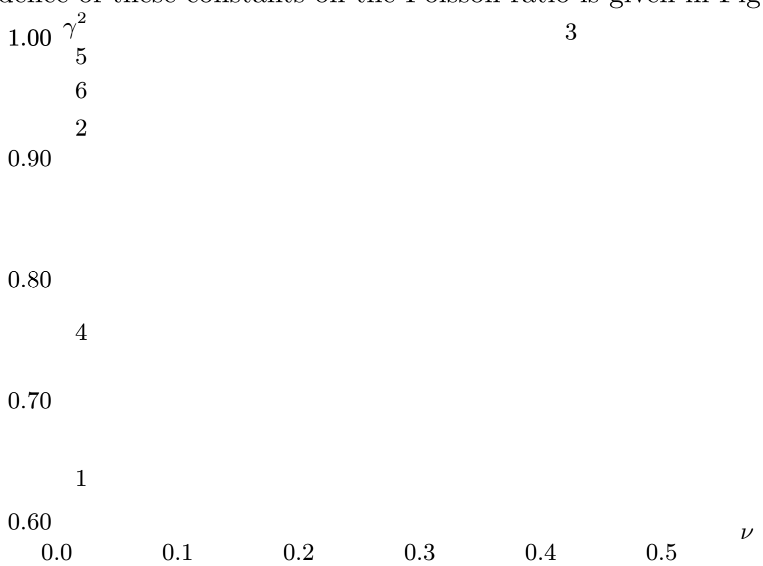


Figure 11: Hexahedron meshes, the bisection method: **1** $- \gamma_{b,0}^{(1)}$, **2** $- \gamma_{b,1}^{(1)}$, **3** $- \gamma_{b,2}^{(1)}$, **4** $- \gamma_{b,0}^{(2)}$, **5** $- \gamma_{b,1}^{(2)}$, **6** $- \gamma_{b,2}^{(2)}$.

7. Discussion

In this section we make a comparison of the results obtained by different discretizations. We present upper and lower bounds for the constant $\gamma(\nu)$, $\underline{\gamma} \leq \gamma(\nu) \leq \overline{\gamma}$, $\nu \in \left(0, \frac{1}{2}\right)$ in Table ??.

We have not a rigorous proof but the experiments show us that $\gamma = \gamma\left(\hat{T}\{6\}\right)$ for the tetrahedron meshes considered in Section 4. We obtain the best result for the regular pyramid. The tetrahedron K has the following phenomenal properties

$$\nabla\varphi_i(x) \parallel \mathbf{b}_i, \quad i = 1, 2, 3, 4,$$

where $\varphi_i(x)$ is an arbitrary linear nodal basis function associated with the vertex node b_i and $\mathbf{b}_i$ is the radius vector of the node b_i.

This properties reflect on the γ constant as follows: $\gamma^{(1)}(K)$ is independent of ν if $\nu \in \left(0, \frac{1}{3}\right]$ and grows linearly when $\frac{1}{3} < \nu < \frac{1}{2}$. The latter denotes that $\gamma^{(1)}(K)$ is independent of ν when the coefficient of

$$\int_\Omega div(\mathbf{u}) div(\mathbf{v})\, dx$$

is smaller than the coefficient of

$$\sum_{i,j=1}^{3} \int_\Omega \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v})\, dx$$

in the representation of the bilinear form (3).

The serendipity pentahedra give worse results than 18-node pentahedron elements for *Pen 2* to *Pen 2* case. But the latter is not true for *Pen 1* to *Pen 2* case.

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Mesh type	γ	$\underline{\gamma}$	$\overline{\gamma}$
tetrahedra	$\gamma^{(1)}$	0.8940742701185053	0.9
tetrahedra	$\gamma^{(1,2)}$	0.9843653209940322	1
tetrahedra	$\gamma^{(2)}$	0.9814419781277343	1
tetrahedra	$\gamma^{(3)}$	0.9903363929671173	1
tetrahedra	$\widetilde{\gamma}^{(1,2)}$	0.9466905004377048	1
tetrahedra	$\gamma^{(b)}$	0.9084540555884962	0.97
tetrahedra	$\widetilde{\gamma}^{(b)}$	0.9554792931954466	1
pentahedra	$\gamma^{(\mathrm{pen},1)}$	0.8714946277928661	0.95454
pentahedra	$\gamma^{(\mathrm{pen},1,2)}$	0.977358542005979	1
pentahedra	$\gamma_s^{(\mathrm{pen},1,2)}$	0.9739370808771585	1
pentahedra	$\gamma^{(\mathrm{pen},2)}$	0.9532267833061236	1
pentahedra	$\gamma_s^{(\mathrm{pen},2)}$	0.98026581438292	1
hexahedra	$\gamma^{(\mathrm{hex},1)}$	0.8324675324675327	0.973
hexahedra	$\gamma^{(\mathrm{hex},1,2)}$	0.9519177881326361	1
hexahedra	$\gamma^{(\mathrm{hex},2)}$	0.9327994977034302	1
hexahedra	$\gamma_{b,2}^{(1)}$	0.940559585231717	1
hexahedra	$\gamma_{b,2}^{(2)}$	0.9683415044643198	1

Table 1: Upper and lower bounds for the constant $\gamma(\nu)$, $\nu \in \left(0, \frac{1}{2}\right)$ with respect to different triangulations.

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