Mathematica
Balkanica

New Series Vol. 20, 2006, Fasc. 2

Test Starlike Functions for Approximation by Subordinate Polynomials

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We construct a function $f(z)$, univalent and starlike in the unit disc, with some interesting geometric and analytic properties. It turns out that its convolutions $f * \lambda_n$ with a specific class of univalent polynomials $\lambda_n(z)$ are not subordinate to the function itself. This provides a counterexample to a result of Greiner and Ruscheweyh.

AMS Subj. Classification: Primary 30C45, 30C10

Key Words: Univalent function, starlike functions, subordinate polynomial, convolution

1. Introduction

Let $D = \{z \ : \ |z| < 1\}$ be the unit disc in the complex plane and $\mathcal{A}(D)$ be the set of analytic functions in D. A function $f \in \mathcal{A}(D)$ is univalent in D if $f(z_1) \neq f(z_2)$ whenever $z_1, z_2 \in D$, $z_1 \neq z_2$. Andriecskii and Ruscheweyh [1] proved the following interesting result about polynomial approximation to conformal maps of D.

Theorem A. *There exists a constant $c > 0$ such that, for each $f(z)$ univalent in D, there exists a sequence of polynomials $p_n(z)$, all univalent in D with $p_n(0) = f(0)$, such that*

$$f(\rho_n D) \subset p_n(D) \subset f(D), \quad \rho_n = 1 - c/n,$$

for every $n > 2c$.

[1]Research supported by the Brazilian foudations CNPq under Grant 300645/95-3 and FAPESP under Grant 03/01874-2.

If $f, g \in \mathcal{A}(D)$, the function g is called subordinate to f when there exists $\varphi \in \mathcal{A}(D)$, such that $|\varphi(z)| \leq |z|$ for any $z \in D$ and $f = g \circ \varphi$. If g is subordinate to f we write $g \prec f$. When f is univalent, the subordination $g \prec f$ is equivalent to the fact that $g(0) = f(0)$ and $g(D) \subset f(D)$ hold simultaneously. Thus the conclusion of Theorem 1 is that $f_{\rho_n} \prec p_n \prec f$, where $f_\rho(z) = f(\rho z)$.

The natural questions if the order of convergence is $1/n$ and about upper and lower bounds for the constant c in Theorem 1 arose and were already partially answered. Greiner [4] proved that $c \leq 73$ and Greiner and Ruscheweyh [5] provided an example which shows that $c \geq \pi$. Most probably the idea of the latter authors was that the Koebe function $k(z) = z/(1-z)^2$ is extremal for a variety of problems for univalent functions and a sequence of polynomials constructed by Suffridge [6] provides a "good" approximation to $k(z)$ in the sense of Theorem 1.

Suffridge defined and studied the classes of univalent polynomials

$$S_n(j; z) = \sum_{k=1}^{n} \frac{n-k+1}{n} \frac{\sin kj\pi/(n+1)}{\sin j\pi/(n+1)} z^k, \quad j \in \mathbb{N},$$

establishing various extremal properties. It is interesting that $S_n(1; z)$ is the desired approximation to $k(z)$. Among the other facts in support of this observation, they obey an "asymptotic Koebe 1/4-theorem". More precisely,

$$\inf_{z \in \partial D} |S_n(1; z)| \longrightarrow 1/4 \quad \text{as} \quad n \to \infty,$$

and, for each $n \in \mathbb{N}$, the above infimum is attained for $z = -1$.

For each pair of functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

their convolution (or Hadamard product) is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

In [5], the polynomials

$$\lambda_n(z) = 1 + \frac{\cot(\pi/(2n+2))}{2n+2} \sum_{k=1}^{n} \frac{n+1-k}{k} \sin \frac{k\pi}{n+1} z^k$$

were considered. It is easily seen that $(k * \lambda_n)(z) = z\lambda_n'(z)$ and this polynomial is a constant multiple of $S_n(1; z)$, the constant of normalization being chosen in such a way that $(k * \lambda_n)(-1) = -1/4$. Then, it was pointed out in [5], that $k_{e_n} \prec k * \lambda_n \prec k$ for

$$e_n = \frac{1 - \sin(\pi/(2n+2))}{1 + \sin(\pi/(2n+2))}. \tag{1.1}$$

This fact, together with the observation that $k_\varepsilon(D) \not\subset (k * \lambda_n)(D)$ for $\varepsilon > e_n$, implies that $c \geq \pi$ at least when one considers subordination of starlike functions. Recall that $f(z)$ is said to be starlike (with respect to the origin) if f is univalent in D and, together with any of its points w, the image $f(D)$ contains the entire segment $\{tw \ : \ 0 \leq t \leq 1\}$. Motivated by their result, Greiner and Ruscheweyh formulated the following

Conjecture A. *Let f be a univalent mapping from D onto some domain $\Omega \subset \mathbb{C}$, starlike with respect to the origin. Then $f * \lambda_n$ is a univalent polynomial of degree n with*

$$f_{e_n} \prec f * \lambda_n \prec f, \tag{1.2}$$

where e_n is defined by (1.1). *If f is a rotation of the Koebe function, then e_n cannot be replaced by any greater number.*

As a first step towards establishing the truth of the conjecture, they furnished a proof of the right-hand side subordination (1.2), namely $f * \lambda_n \prec f$ for any univalent and starlike function (see Theorem 3 in [5]).

2. Test starlike functions and counterexamples

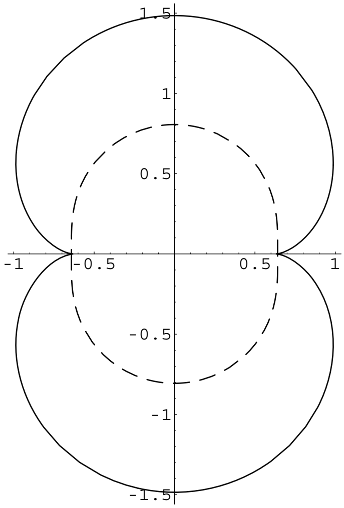
While trying to better understand the properties of starlike functions, we constructed in [2] various examples, looking for functions with specifically distributed zeros. It turns out that some of the examples constructed there provide counterexamples to the statement of Greiner and Ruscheweyh mentioned in the end of the previous section. Formally, we state

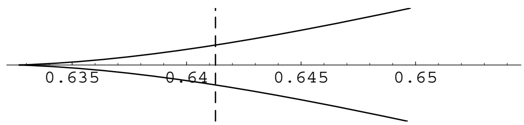
Proposition 1. *The function*

$$f(z) = z \frac{\sqrt{3}}{\sinh \sqrt{3}} \frac{\sin \sqrt{3(z^2-1)}}{\sqrt{3(z^2-1)}}. \tag{2.1}$$

is an entire function, it is univalent and starlike in D and $f * \lambda_4 \not\prec f$. More precisely, $f(1)$ and $(f * \lambda_4)(1)$ are real and $0 < f(1) < (f * \lambda_4)(1)$.

Before we furnish the simple proof, note that our statement is illustrated in the figures below. The first one shows the images of the unit circumference ∂D through $f(z)$ in continuous line and through $(f * \lambda_4)(z)$ in dashed line. The second figure shows the portions of these images for z close to 1.





P r o o f. Let $C = \sqrt{3}/\sinh\sqrt{3}$. By Hadamard's factorization of the sine function we obtain

$$f(z) = Cz\prod_{k=1}^{\infty}\left(1 - \frac{z^2-1}{k^2\pi^2/3}\right). \tag{2.2}$$

Hence f is an entire function.

It is well known (see [3]) that a function $f \in \mathcal{A}(D)$ is starlike in D if and only if

$$\Re\left(z\frac{f'(z)}{f(z)}\right) > 0 \quad \text{for every} \quad z \in D. \tag{2.3}$$

We shall prove that the function $f(z)$, defined by (2.1), or equivalently by (2.2), obeys this property. It follows immediately from the representation (2.2) that

$$z\frac{f'(z)}{f(z)} = 1 + \sum_{k=1}^{\infty}\frac{6z^2}{3z^2 - (k^2\pi^2+3)}. \tag{2.4}$$

It is easy to see that

$$\Re\,\frac{6z^2}{3z^2-(k^2\pi^2+3)} \geq -\frac{6}{k^2\pi^2} \quad \text{for every} \quad z \in \overline{D}, \tag{2.5}$$

where $\overline{D}$ denotes the closed unit disc. Moreover, equality in (2.5) is attained if and only if $z = \pm 1$. Therefore, by (2.4) and (2.5) we obtain

$$\Re\left(z\frac{f'(z)}{f(z)}\right) \geq 1 - \frac{6}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2} = 0 \quad \text{for every} \quad z \in \overline{D},$$

with equality only at $z = \pm 1$. Thus $f(z)$ does satisfy (2.3) and it is starlike.

Straightforward calculations show that the Maclaurin expansion of $f(z)$ is

$$f(z) = z + \frac{1 - \sqrt{3}\coth\sqrt{3}}{2}z^3 + \frac{3(2 - \sqrt{3}\coth\sqrt{3})}{8}z^5 + \cdots.$$

Then $(f * \lambda_4)$ reduces to a cubic polynomial,

$$(f * \lambda_4)(z) = \frac{2}{5 - \sqrt{5}}z + \frac{(5 + \sqrt{5})(1 - \sqrt{3}\coth\sqrt{3})}{60(\sqrt{5} - 1)}z^3.$$

Hence $f(1) = \sqrt{3}/\sinh\sqrt{3} \approx 0.632677$ and

$$(f * \lambda_4)(1) = \frac{25\sqrt{5} + 5 - \sqrt{3}(5 + \sqrt{5})\coth\sqrt{3}}{60(\sqrt{5} - 1)} \approx 0.641261.$$

Thus $f(1) < (f * \lambda_4)(1)$. This inequality can be proved formally, without using the numerical approximation of the two sides. We omit the technical details. ■

It is worth mentioning that the even Maclaurin coefficients of $f(z)$ are equal to zero. Also, the quantity (2.3) vanishes on the boundary of D for $f(z)$. These might be the reasons that make it a good test starlike function when one investigates approximation by subordinate polynomials. Functions with similar properties are

$$f_m(z) = z\frac{\sqrt{6/m}}{\sinh\sqrt{6/m}}\frac{\sin\sqrt{6(z^m - 1)/m}}{\sqrt{6(z^m - 1)/m}}, \quad m \in \mathbb{N}.$$

Observe that $f_2(z)$ is exactly the function which appears in Proposition 1. The only nonzero Maclaurin coefficients of $f_m(z)$ are those of z^{mk+1}, $k = 0, 1, \ldots$, so that $f_m(z) = zg_m(z^m)$ for some entire function $g_m(z)$. The fact that the sequences of the Maclaurin coefficients of $f_m(z)$ possesses gaps, allow us to construct many examples similar to the one given in Proposition 1. Exhaustive experiments show that $f_m * \lambda_{km+\nu} \not\prec f_m$ for various values of the integers m, k, ν, $m \geq 2$, $2 \leq k \leq m$.

In the author's opinion, the reason for these counterexamples is most probably the fact that the function defined in Lemma 6 in [5] is not close-to-convex. However, we believe that it is still possible to fix the above conjecture choosing the normalization constant in $\lambda_n(z)$ a bit smaller than $\cot(\pi/(2n+2))/(2n+2)$.

Acknowledgments

I thank Professor Ruscheweyh who read carefully this short note and then proposed a simpler counterexample. He kindly informed me that, for the function $z/(1-z^2)$, which is starlike in D, and omits the rays it and $-it$, for $t \geq 1/2$, the corresponding polynomial

$$\lambda_3 * \frac{z}{1-z^2} = \frac{\cot(\pi/8)}{24\sqrt{2}}\left(9z+z^3\right)$$

is convex univalent in D and takes the approximate value $0.569036i$ at $z=i$.

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Received 30.11.2004