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On Subsemigroups of Stone - Čech Compactification for Certain Semigroups

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It is known that $(\beta S, \cdot)$ the Stone - Čech compactification of any discrete semigroup S, can be given a semigroup structure where " $\cdot$ " extends the semigroup operation on S. This paper is concerned with whether or not $\beta S \setminus S$ is a subsemigroup of βS for some well known semigroups S . If S is a rectangular band such that $\beta S \setminus S$ is a subsemigroup of βS we show that then $\beta S \setminus S$ is a rectangular band too. Also we show that , if S and T are infinite semigroups and $\beta S \setminus S$ is a subsemigroup of βS and if there exists $\Phi : S \longmapsto T$ an onto homomorphism such that $\Phi^{-1}(t)$ is finite for any $t \in T$ then $\beta T \setminus T$ is a subsemigroup of βT. Moreover, we conclude that $\beta T \setminus T$ can be subsemigroup of βT even if $\Phi^{-1}(t)$ is infinite for every $t \in T$.

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1. Introduction

Throughout this paper $(S, \cdot)$ will denote an infinite discrete semigroup. Let βS be the Stone-Čech compactification of S. Then βS is compact and Hausdorff such that S is a dense subset of βS and if $f : S \longrightarrow S \subset \beta S$ is a map, then f has a unique extension $\hat{f} : \beta S \longrightarrow \beta S$ since S is discrete. We know that the operation " $\cdot$ " on S extends uniquely to βS in the following way:

S is called right (respectively left) topological semigroup if the mapping $\rho_t : S \longrightarrow S$ (respectively λ_t) $\rho_t(s) = s \cdot t$ (respectively $\lambda_t(s) = t \cdot s$) is continuous for each t in S. Given $y \in S$, $\lambda_y : S \longrightarrow S \subset \beta S$ is a continuous mapping since S is discrete. Then this map has a unique continuous extension $\hat{\lambda}_y : \beta S \longrightarrow \beta S$. For $p \in \beta S \setminus S$, define $y \cdot p = \hat{\lambda}_y(p)$. Hence the operation on $\cdot : S \times S \longrightarrow S$ is extended to $S \times \beta S \longrightarrow \beta S$ such that $(y, p) \longrightarrow y \cdot p$. Now for $p \in \beta S$, $\rho_p : S \longrightarrow \beta S$ is a continuous mapping, so ρ_p has a unique continuos extension $\hat{\rho}_p : \beta S \longrightarrow \beta S$. Similary for $q \in \beta S \setminus S$, define $q \cdot p = \hat{\rho}_p(q)$. So the operation on $S \times \beta S$ is extended to $\beta S \times \beta S$. Then $(\beta S, \cdot)$ is a right topological semigroup and $\hat{\lambda}_s$ is continuous for each $s \in S$.

Let A be an alphabet. We denote by A^+ the free semigroup on A consisting of all non-empty words over A. A semigroup presentation is an ordered pair $< A \mid R >$, where $R \subseteq A^+ \times A^+$. A semigroup S is said to be defined by the semigroup presentation $< A \mid R >$ if S is isomorphic to A^+/ρ, where ρ is the congruence on A^+ generated by R. Let u and v be two words in A^+. We write $u \equiv v$ are identical words and write $u = v$ if $(u, v) \in \rho$, that is v is obtained from u by applying relation from R.(see[4, proposition 1.5.9])

Given any set $A \subseteq S$, $[A]^{<w}$ will denote the set of finite subsets of A. For a set A, $\mid A \mid$ denotes the cardinality of A

We usually write S and st instead of $(S, \cdot)$ and $s \cdot t$ respectively when there is no danger of ambiguity.

This paper is concerned with whether or not $\beta S \setminus S$ is a subsemigroup of βS for some well known semigroups S such as left (or right) zero semigroup, semilattice, chain, rectangular band and direct product of semigroups. For details of these semigroups see [4]. If S is a rectangular band such that $\beta S \setminus S$ is a subsemigroup of βS we show that then $\beta S \setminus S$ is a rectangular band too. Also we show that , if S and T are semigroups , $\beta S \setminus S$ is a subsemigroup of βS and there exists $\Phi : S \longmapsto T$ an onto homomorphism such that $\Phi^{-1}(t)$ is finite for any $t \in T$ then $\beta T \setminus T$ is a subsemigroup of βT. Moreover, we conclude that $\beta T \setminus T$ can be a subsemigroup of βT even if $\Phi^{-1}(t)$ is infinite for every $t \in T$.

We will use the following theorem proved by Hindman [3, Theorem 2.5]. Further details of this subject are given in [5,6,7,8].

Theorem 1. *$\beta S \setminus S$ is a subsemigroup of βS if and only if for any $A \in [S]^{<w}$ and any infinite subset B of S, there exists a $F \in [B]^{<w}$ such that $\bigcap_{x \in F}(x^{-1} \cdot A)$ is finite, where $x^{-1} \cdot A = \{y \in S : \ x \cdot y \in A\}$.*

2. Main Results

Theorem 2. *Let S and T be infinite semigroups and let $\beta S \setminus S$ be a subsemigroup of βS. If there exists $\Phi : S \longmapsto T$ an onto homomorphism such that $\Phi^{-1}(t)$ is finite for any $t \in T$ then $\beta T \setminus T$ is a subsemigroup of βT.*

P r o o f. Let $A \in [T]^{<w}$and let B be any infinite subset of T. Since Φ is onto and $\Phi^{-1}(t)$ is finite for any $t \in T$, $\Phi^{-1}(B)$ is an infinite subset of S and $\phi \neq \Phi^{-1}(A) \in [S]^{<w}$. Then since $\beta S \setminus S$ is a subsemigroup of βS, there exists a $F \in [\Phi^{-1}(B)]^{<w}$ such that $\bigcap_{x \in F}(x^{-1} \cdot \Phi^{-1}(A))$ is finite. Let $\bigcap_{x \in F}(x^{-1} \cdot \Phi^{-1}(A)) = \{s_1, s_2, ..., s_n\}$ and $F = \{x_1, x_2, ..., x_m\}$ for $n, m \in \mathbb{N}$.

Then for any $i = 1, 2, .., n$ and $j = 1, 2, ..., m$, $s_i \in {x_j}^{-1} \cdot \Phi^{-1}(A)$. Therefore $x_j \cdot s_i \in \Phi^{-1}(A)$ for any i, j. Then $\Phi(x_j \cdot s_i) = \Phi(x_j) \cdot \Phi(s_i) \in \Phi(\Phi^{-1}(A)) = A$ for any i, j. Thus $\Phi(s_i) \in \Phi(t_j)^{-1} \cdot A$ for any i, j. Assume that there exists a $t \in T$ such that $t \in \Phi(t_j)^{-1} \cdot A$ and $\Phi(s_i) \neq t$ for any i, j. Then there exists a $s \in S$ such that $\Phi(s) = t$ and $s_i \neq s$ for any i and $\Phi(x_j) \cdot \Phi(s) = \Phi(x_j \cdot s) \in A$ for any j. So $x_j \cdot s \in \Phi^{-1}(A)$ for any j. Then $s \in \bigcap_{x \in F}(x^{-1} \cdot \Phi^{-1}(A))$ and $s = s_i$ for any i. This is a contradiction. Therefore

$$\bigcap_{\Phi(x) \in \Phi(F)} \left(\Phi(x)^{-1} \cdot A\right) = \{\Phi(s_1), \ldots, \Phi(s_n)\}$$

Since $\Phi(F) \in [B]^{<w}$, $\beta T \setminus T$ is a subsemigroup of βT by theorem 1. ■

$\beta T \setminus T$ can be subsemigroup of βT even if $\Phi^{-1}(t)$ is infinite for every $t \in T$. To see this case, let

$$T = \langle A \mid R \rangle = < a, b \mid a^2 = a, b^2 = b > \text{ and } S = A^+.$$

Then

$$T = A^+/\rho = \left\{a\rho : a \in A^+\right\} = \{a, b, ab, ba, aba, bab, abab, baba, ...\}.$$

It is easy to show that $\beta S \setminus S$ and $\beta T \setminus T$ is a subsemigroup of βS and βT respectively. If $\Phi : S \longmapsto T$ is canonic homomorphism, that is $\Phi(a) = a\rho$ for $a \in S = A^+$, then Φ is onto homomorphism and $\Phi^{-1}(t)$ is infinite for every $t \in T$.

If $S =< A \mid R >$ and $a\rho$ is finite for ecah $a \in A^+$,where ρ is the congruence on A^+ generated by R, then $\beta S \setminus S$ is a subsemigroup of βS.

Let L be a non-empty set. Then we define a multiplication $s \cdot t = s$ for all $s, t \in L$. With respect to this multiplication L is a semigroup, which is called a left zero semigroup. Similary we define a right zero semigroup.

Let S be a semigroup which is a disjoint union of some of its subsemigroups. Then S is called a union of semigroups. Suppose further that there exists a band(or semilattice) Y and a family $(S_\alpha)_{\alpha \in Y}$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$, for all $\alpha, \beta \in Y$. Then S is called the band(or semilattice) of semigroups $(S_\alpha)_{\alpha \in Y}$.

Theorem 3. *If S is a left (or right) zero semigroup then $\beta S \setminus S$ is a subsemigroup of βS. Moreover $\beta S \setminus S$ is a left(or right) zero semigroup and βS is the band of semigoups $\beta S \setminus S$ and S.*

Proof. Suppose S is a left zero semigroup. If $A \in [S]^{<w}$ and B is any infinite subset of S, then there exits a $F \in [B]^{<w}$such that $A \cap F = \phi$. Then,

for every $x \in F$ and $y \in S$, $xy = x \notin A$. Therefore $\bigcap_{x \in F}(x^{-1} \cdot A) = \phi$. Thus $\beta S \setminus S$ is a subsemigroup of βS , from Theorem 1.

Let $p, q \in \beta S \setminus S$. Then there exists a net $< p_\alpha >_{\alpha \in I} \subset S$ such that $\lim_\alpha p_\alpha = p$. Since $\rho_p : S \longrightarrow \beta S$ is continuous ,

$$p \cdot q = \hat{\rho}_p(q) = \lim_\alpha \rho_q(p_\alpha) = \lim_\alpha (p_\alpha q) = \lim_\alpha p_\alpha = p.$$

Thus $\beta S \setminus S$ is a left zero semigroup.

Let $p \in S$ and $q \in \beta S \setminus S$. Then there exists a net $< q_\alpha >_{\alpha \in I} \subset S$ such that $\lim_\alpha q_\alpha = q$. Since $\hat{\lambda}_p$ is continuous,

$$p \cdot q = \hat{\lambda}_p(q) = \lim_\alpha \hat{\lambda}_p(q_\alpha) = \lim_\alpha \lambda_p(q_\alpha) = \lim_\alpha (p \cdot q_\alpha) = \lim_\alpha p = p.$$

Thus $S \cdot (\beta S \setminus S) \subseteq S$. Similarly $(\beta S \setminus S) \cdot S \subseteq \beta S \setminus S$. Let $S_1 = S$, $S_2 = \beta S \setminus S$ and Y be a band such that $Y = \{1, 2\}$ and $1^2 = 1$, $2^2 = 2$, $1 \cdot 2 = 1$, $2 \cdot 1 = 2$. Then βS is the band of semigoups S_1 and S_2. ∎

Let S be a well ordered set with the relation " $\leq$ " . We define a binary operation " $\cdot$ " on S as $x \cdot y = y \cdot x = y$ for all $x, y \in S$ and $x \leq y$. Then $(S, \cdot)$ is a semigroup called chain. Further details for chain is given in [1].

Theorem 4. *Let S be a chain. Then $\beta S \setminus S$ is a subsemigroup of βS. Moreover $\beta S \setminus S$ is right zero semigroup and βS is the semilattice of semigoups $\beta S \setminus S$ and S.*

Proof. Let $A \in [S]^{<w}$ and let B be any infinite subset of S. Since B is infinite, there exists $x \in B$ such that $x \notin A$. If $x > a$ for every $a \in A$ then $x^{-1} \cdot A = \phi$. If $x \leq a$ for every $a \in A$ then $x^{-1} \cdot A \subseteq A$. Therefore we can take $F = \{x\}$.

Let $q \in \beta S \setminus S$ and $s \in S$. Then there exists a net $< q_\alpha >_{\alpha \in I} \subset S$ such that $\lim_\alpha q_\alpha = q$. Since S is a chain, there exists a $\alpha_0 \in I$ such that $s \cdot q_\alpha = q_\alpha$ for every $\alpha > \alpha_0$. Since $\hat{\lambda}_s$ is continuous,

$$s \cdot q = \hat{\lambda}_s(q) = \lim_\alpha \hat{\lambda}_s(q_\alpha) = \lim_\alpha \lambda_s(q_\alpha) = \lim_\alpha (s \cdot q_\alpha) = \lim_\alpha q_\alpha = q.$$

Thus $S \cdot (\beta S \setminus S) \subseteq \beta S \setminus S$. Similary $(\beta S \setminus S) \cdot S \subseteq \beta S \setminus S$. Let $S_1 = S$, $S_2 = \beta S \setminus S$ and Y be a semilattice such that $Y = \{1, 2\}$ and $1^2 = 1$, $2^2 = 2$, $1 \cdot 2 = 2 \cdot 1 = 2$. Then βS is the semilattice of semigoups S_1 and S_2.

Let $p, q \in \beta S \setminus S$. Then there exists a net $< p_\alpha >_{\alpha \in I} \subset S$ such that $\lim_\alpha p_\alpha = p$. Since $p_\alpha \in S$ and $q \in \beta S \setminus S$, $p_\alpha \cdot q = q$ for every $\alpha \in I$. Therefore since $\hat{\rho}_q$ is continuous,

$$p \cdot q = \hat{\rho}_q(p) = \lim_\alpha \hat{\rho}_q(p_\alpha) = \lim_\alpha \rho_q(p_\alpha) = \lim_\alpha (p_\alpha \cdot q) = \lim_\alpha q = q.$$

Thus $\beta S \setminus S$ is a right zero semigroup. ∎

Let X be a set and let $P(X)$ denote the set of all subsets of X. Then $(P(X) \setminus \{\phi\}, \cup)$, where $\cup$ denotes the union of sets, is a semigroup. This semigroup is called semilattice.

Theorem 5. *If S is a semilattice then $\beta S \setminus S$ is not a subsemigroup of βS.*

Proof. Let $S = (P(X) \setminus \{\phi\}, \cup)$, $A = \{X\}$ and $B = \{X \setminus \{a\} : a \in X\}$. If $x \in B$ then $x^{-1} \cdot A = \{\{a\} \cup K : K \in P(X)\}$ where $x = X \setminus \{a\}$ for a $a \in X$. Thus, if $F = \{x_1, x_2, ..., x_n\}$ is a finite subset of B such that $x_i = X \setminus \{a_i\}$ and $a_i \in X$ for $i = 1, 2, .., n$ it can be seen easily that

$$\bigcap_{x \in F} (x^{-1}.A) = \{\{a_1, a_2, ..., a_n\} \cup K : K \in P(X)\}$$

is infinite. So $\beta S \setminus S$ is not a subsemigroup of βS by theorem 1. ∎

Let I and Λ be two sets. The set $I \times \Lambda = \{(i, \lambda) : i \in I \text{ and } \lambda \in \Lambda\}$ with a binary operation " $\cdot$ "

$$(i, \lambda) \cdot (j, \mu) = (i, \mu) \text{ (for all } (i, \lambda), (j, \mu) \in I \times \Lambda)$$

is a semigroup which is called a rectangular band.

Theorem 6. *Let $S = I \times \Lambda$ be a rectangular band. Then $\beta S \setminus S$ is a subsemigroup of βS if and only if one of I and Λ is finite.*

Proof. Let $\beta S \setminus S$ is a subsemigroup of βS and suppose that I and Λ are infinite. For a fixed $i_0 \in I$ and $\lambda_0 \in \Lambda$, let

$$A = \{(i_0, \lambda_0)\} \text{ and } B = \{(i_0, \lambda) : \lambda \in \Lambda\}.$$

Then $x^{-1} \cdot A = \{(j, \lambda_0) : j \in I\}$ for each $x \in B$.Therefore, for every $F \in [B]^{<w}$, $\bigcap_{x \in F}(x^{-1}.A) = \{(j, \lambda_0) : j \in I\}$ is infinite. Then by theorem 1.1 $\beta S \setminus S$ is not subsemigroup of βS. This is a contradiction. Therefore one of I and Λ is finite.

Conversely, assume that I is finite. Then Λ is infinite since S is infinite. Let $A \in [S]^{<w}$and B is any infinite subset of S. Let

$$A = \{(i_k, \lambda_k) : i_k \in I, \lambda_k \in \Lambda \text{ for } k = 1, 2, ..., n \text{ and } n \in \mathbb{N}\}$$

and $A_\Lambda = \{\lambda_k : k = 1, 2, ..., n\}$. Let $x = (j, \lambda) \in B$.

i) If $j = i_k$ for a $k = 1, 2, ..., n$ then $x^{-1} \cdot A = \{(i, \lambda_k) : i \in I \text{ and } \lambda_k \in A_\Lambda\}$ is finite since I is finite.

ii) If $j \neq i_k$ for every $k = 1, 2, ..., n$ then $x^{-1} \cdot A = \phi$.

If Λ is finite and I is infinite then there exits a $F \in [B]^{<w}$ such that $x = (i, \lambda) \in F$ and $i \neq i_k$ for every $k = 1, 2, ..., n$. So $\bigcap_{x \in F}(x^{-1} \cdot A) = \phi$ since $x^{-1} \cdot A = \phi$ for any $x \in F$. Therefore $\beta S \setminus S$ is a subsemigroup of βS by Theorem 1. ■

Let X and Y be semigoups. Then the cartesian product

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$$

becomes a semigroup with a binary operation

$$(x_1,\ y_1) \cdot (x_2,\ y_2) = (x_1 x_2,\ y_1 y_2) \text{ for every } (x_1,\ y_1), (x_2,\ y_2) \in X \times Y.$$

This semigroup is called the direct product of X and Y.

Corollary 7. *Let S be rectangular band. If $\beta S \setminus S$ is a subsemigroup of βS then $\beta S \setminus S$ is a rectangular band too.*

Proof. Let $S = I \times \Lambda$ be a rectangular band. Consider $L = I$ with mutiplication $i \cdot j = i$ and $R = \Lambda$ with multiplication $\lambda \cdot \mu = \mu$. It is immediate that L and R are left and right zero semigroups respectively, and moreover S and $L \times R$ are izomorphic. If $\beta S \setminus S$ is a subsemigroup of βS then L or R is finite by Theorem 6. If R is finite then

$$\beta S = \beta(L \times R) = \beta L \times R$$

by [9, Prop 8.2.]. Thus

$$\beta S \setminus S = (\beta L \times R) \setminus (L \times R) = (\beta L \setminus L) \times R$$

. Since L is a left zero semigroup, then $\beta L \setminus L$ is a left zero semigroup by Theorem 3. Therefore $(\beta L \setminus L) \times R$ is a rectangular band by [4, Theorem 1.1.3]. Similary, if L is finite then

$$\beta S \setminus S = (L \times \beta R) \setminus (L \times R) = L \times (\beta R \setminus R)$$

is a rectangular band since $\beta R \setminus R$ is a right zero semigroup. ■

Let L is an infinite left zero semigroup and R is an infinite right zero semigroup. Then the direct product $S = L \times R$ is a rectangular band (see[4, Theorem 1.1.3]). By Theorem 3, $\beta L \setminus L$ and $\beta R \setminus R$ are subsemigroups of βL and βR, respectively. But, by Theorem 6 $\beta S \setminus S$ is not a subsemigroup of βS since L and R are infinite. Therefore, even if X and Y be infinite semigroups and $\beta X \setminus X$ and $\beta Y \setminus Y$ are subsemigroup of βX and βY, respectively then $\beta(X \times Y) \setminus (X \times Y)$ is not necessary to be a subsemigroup of $\beta(X \times Y)$.

Corollary 8. *Let X and Y be two semigoups. If X (or Y) is finite and $\beta Y \setminus Y$ (or $\beta X \setminus X$) is a subsemigroup of βY (or βX) then $\beta(X \times Y) \setminus (X \times Y)$ is a subsemigroup of $\beta(X \times Y)$.*

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