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# Function of "Repeating Values" of the Solutions to the Differential Equation $\mathbf{y^{IV} + a(x)y = 0}$

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In this paper we develop the idea of quasi-periodic solutions to differential equations and give some necessary conditions for existence of quasi-periodic solutions with a constant and linear quasi-period to the differential equation of the fourth order (5).

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## 1. Introduction

We introduce the following definition.

**Definition 1.** We say that the function $y = f(x)$, $x \in D_f \subseteq \mathbb{R}$ is a quasi-periodic one if there are functions $\omega = \omega(x)$ and $\lambda = \lambda(\omega)$ such that:

$$\text{if } \ x \in D_f \text{ then } x + \omega \in D_f, \ \text{ and} \qquad (1)$$
$$f(x + \omega) = \lambda f(x)$$

The function $\omega(x)$ is called a **quasi-period (QP)** and $\lambda$ is called a **quasi-periodic coefficient (QPC)** of the function $y = f(x)$.

Example 1. The function $f(x) = e^x \sin x$ is a quasi-periodic one with QP $\omega(x) = 2\pi$ and QPC $\lambda = e^{2\pi}$ since

$$\forall x \in \mathbb{R}, \ \ f(x + 2\pi) = e^{x+2\pi} \sin(x + 2\pi) = e^{2\pi} e^x \sin x = e^{2\pi} f(x)$$

Remark 1. In general $\lambda = \lambda(x, \omega(x))$ and in this case the existence of the relation (1) is a very complex problem that implies, for example, the following essential problems to the differential equations:

– the problem of analytic solutions,

– the problem of oscillating solutions,

– the problem of zeros of the solutions,

– the problem of extremum of the solutions.

If $\lambda = 1$ then $\omega(x)$ is a function of "repeating values" of $y = f(x)$.

If $\omega = const$ and $\lambda = 1$, then (1) is a definition of a periodic function in a classical sense.

If $\lambda = 0$, then the part of repeating the zeros of the function is solved.

In this paper we consider the problem of quasi-periodicity for $\lambda = 1$.

## 2. Problem formulation

Suppose that the function $y = y(x)$ is given implicitly by the differential equation

$$F(x, y, y', \ldots, y^{(n)}, a(x), b(x) \ldots, c(x)) = 0 \tag{2}$$

We want to find a rule $\omega = \omega(x)$ for repeating the values of the solutions to the equation (2), i.e. to find a function $\omega = \omega(x)$ which satisfies both the equation(2) and the relations

$$\begin{cases} y(x+\omega) = y(x) \\ F(t, y(t), y'(t), \ldots, y^{(n)}(t), a(t), b(t), \ldots, c(t))_{/t=x+\omega} = 0 \end{cases} \tag{3}$$

Equations (2) and (3) make a system which defines the functions $y(x)$ and $\omega(x)$.

Since the nature of the solutions to the equation (2) depends on its coefficients, we should determine some conditions for the coefficients that allow repeating the values of the solutions (as a special case of the quasi-periodicity).

According to the above arguments, the problem is equivalent to the system

$$\begin{cases} F(x, y, y', y'' \ldots, y^{(n)}, a(x), b(x), \ldots, c(x)) = 0 \\ F(t, y(t), y'(t), \ldots, y^{(n)}(t), a(t), \ldots, c(t))_{/t=x+\omega} = 0 \\ y(x+\omega) = y(x) \\ \dfrac{d^m}{dx^m} y(x+\omega) = y^{(m)}(x), \quad (m = 1, \ldots, n) \end{cases} \tag{4}$$

which generally leads to a nonlinear differential equation. Thus, solving such a system is not a simple problem, especially for the differential equation of higher order.

If the equation (2) is linear, the system (4) is a simpler one and it is linear in $y(x)$ and its derivatives, but it is nonlinear in $\omega(x)$.

Here we consider this problem in the case of a linear equation of the fourth order.

Let (2) be the equation

$$y^{IV} + a(x)y = 0 \tag{5}$$

where $a(x)$ is a continuous function for $x \geq x_0$. It is known ([1]) that if

$$\begin{cases} a(x) > 0 \\ \int\limits_{x_0}^{\infty} x^2 a(x)dx = \infty \end{cases} \tag{6}$$

then the solutions of (5) are all oscillating functions, which means that they are also quasi-periodic functions and they generate an elliptic trigonometry of IV order ([2]).

For (5) the system (4) has the form:

$$\begin{cases} y^{IV} + a(x)y = 0 \\ y^{IV}(x+\omega) + a(x+\omega)y(x+\omega) = 0 \\ y(x+\omega) = y(x) \\ y'(x+\omega)\cdot(1+\omega') = y'(x) \\ y''(x+\omega)\cdot(1+\omega')^2 + y'(x+\omega)\cdot\omega'' = y''(x) \\ y'''(x+\omega)\cdot(1+\omega')^3 + 3y''(x+\omega)(1+\omega')\omega'' + y'(x+\omega)\cdot\omega''' = y'''(x) \\ y^{IV}(x+\omega)\cdot(1+\omega')^4 + 6y'''(x+\omega)(1+\omega')^2\cdot\omega'' + 3y''(x+\omega)\cdot\omega''^2 + \\ \qquad +4y''(x+\omega)\cdot(1+\omega')\omega''' + y'(x+\omega)\cdot\omega^{IV} = y^{IV}(x) \end{cases} \tag{7}$$

If we eliminate $y^{(m)}(x+\omega)$, $m = 1,2,3,4,$ when $1+\omega' \neq 0$, we obtain the equation

$$
\begin{aligned}
&-a(x+\omega)y(x)(1+\omega')^4+\\
&+\frac{6\omega''}{(1+\omega')}\left[\frac{3y'(x)\omega''^2}{(1+\omega')^2}-\frac{3y''(x)\omega''}{(1+\omega')}-\frac{\omega'''\cdot y'(x)}{(1+\omega')}+y'''(x)\right]+\\
&+\frac{3\omega''^2}{(1+\omega')^2}\left[y''(x)-\frac{\omega''\cdot y'(x)}{(1+\omega')}\right]+\frac{4\omega'''}{(1+\omega')}\left[y''(x)-\omega''\cdot\frac{y'(x)}{(1+\omega')}\right]+\\
&+\frac{y'(x)}{(1+\omega')}\omega^{IV}=-a(x)y(x)
\end{aligned}
\tag{8}
$$

which is a linear differential equation of III order in $y(x)$, but it is a nonlinear one of IV order in $\omega(x)$.

If we rearrange (8) with respect to $y$ and its derivatives, it will take the form

$$
\begin{cases}
y'''\Big[6\omega''\cdot(1+\omega')^3\Big]+y''\Big[-15\omega''^2\cdot(1+\omega')+4\omega'''\cdot(1+\omega')^2\Big]+\\
\qquad +y'\Big[15\omega''^3-10\omega''\cdot\omega'''\cdot(1+\omega')+\omega^{IV}\cdot(1+\omega')^2\Big]+\\
\qquad\qquad +y\Big[a(x)-a(x+\omega)\cdot(1+\omega')^4\Big](1+\omega')^3=0
\end{cases}
\tag{9}
$$

Since (9) is a nonlinear equation of IV order and it still depends on $a(x)$ and $a(x+\omega)$, it follows that finding $\omega(x)$, using (9), is not a trivial problem at all, and in general it may be difficult to solve equation (9).

## 4. Main results

Here we consider some particular cases.

**Lemma 1.** *Let $y(x)$ be a quasi-periodic solution for (5) with QP $\omega(x)=kx+b$, where $k^2+b^2\neq 0$, $k\neq -1$. Then*

$$
y(x)\Big[a((k+1)x+b)\cdot(1+k)^4-a(x)\Big]=0 \tag{10}
$$

P r o o f. For $\omega(x)=kx+b$ the system (7) has the form:

$$
\begin{cases}
y^{IV}(x)+a(x)y(x)=0\\
y^{IV}(t)+a(t)y(t)=0_{/t=(1+k)x+b}\\
y(t)=y(x)\\
y^{IV}(t)\cdot(1+k)^4=y^{IV}(x)
\end{cases}
$$

Observing that $1 + k \neq 0$, by eliminating $y(t)$, $y^{IV}(t)$, $y^{IV}(x)$ we obtain (10). ■

Using Lemma 1, we get the following assertion.

**Proposition 1.** *Let the equation (5) have a quasi-periodic solution with a constant QP $\omega = b$. Then*

$1.1^0$ $y(x) \equiv 0$ *(a trivial solution)*

*or*

$1.2^0$ $a(x)$ *is a quasi-periodic function with the same period* $\omega = b$ *as the solution* $y(x)$

P r o o f. If $\omega(x) = b = konst$, i.e. $k = 0$, $b \neq 0$, then, from (10) we have

$$y(x) \cdot \Big[a(x+b) - a(x)\Big] = 0 \tag{11}$$

which is equivalent with $y(x) \equiv 0$ or $a(x+b) - a(x) \equiv 0$. ■

R e m a r k 2. Condition $1.2^0$ in Proposition 1 is necessary, but it is not sufficient for $\omega(x)$ to repeat the values of the solutions to (5).

E x a m p l e 2 Let $a(x) = n^4$. Then the equation

$$y^{IV} + n^4 y = 0 \tag{12}$$

has a fundamental system of solutions ([2], [3])

$$\begin{cases} y_1 = e^{\frac{n}{\sqrt{2}}x} \cdot \cos \dfrac{n}{\sqrt{2}}x, & y_2 = e^{\frac{n}{\sqrt{2}}x} \cdot \sin \dfrac{n}{\sqrt{2}}x \\ y_3 = e^{-\frac{n}{\sqrt{2}}x} \cdot \cos \dfrac{n}{\sqrt{2}}x, & y_4 = e^{-\frac{n}{\sqrt{2}}x} \cdot \sin \dfrac{n}{\sqrt{2}}x \end{cases} \tag{13}$$

which are oscillating functions. They are not periodic functions in a classical sense, but they are quasi-periodic ones with the same QP

$$\omega = \frac{2\sqrt{2}\pi}{n},$$

and QPC $\lambda = e^{2\pi} \neq 1$ for $y_1, y_2$ and $\lambda_1 = e^{-2\pi} \neq 1$ for $y_3, y_4$. Hence, it follows that $\omega = \dfrac{2\sqrt{2}\pi}{n}$ does not repeat all of the values of functions (13) but only the zeros of the functions.

Using Lemma 1, we can also prove the following proposition.

**Proposition 2.** *Let the equation* (5) *have a quasi-periodic solution with a linear QP* $\omega(x) = kx + b, k \neq 0, k \neq -1,$ *i.e.*

$$y(x + (kx + b)) = y(x). \tag{14}$$

*Then*

$2.1^o$ $y(x) \equiv 0$ *( a trivial solution);*

*or*

$2.2^o$ $a(x)$ *is a quasi-periodic function with QP* $\omega(x) = kx + b$ $k \neq 0, k \neq -1$ *and QPC* $\lambda_1 = \dfrac{1}{(1+k)^4}$, *i.e. it holds*

$$a((1 + k)x + b) = \frac{1}{(1+k)^4} a(x). \tag{15}$$

Example 3. The coefficient $a(x) = (x - 1)^2$ of the equation

$$y^{IV} + (x - 1)^2 y = 0 \tag{16}$$

satisfies (15) for $k = -2$, $b = 2$ and it also satisfies (6). So, equation (16) has oscillating solutions. One of its fundamental system of solutions is ([2]):

$$\left\{\begin{aligned}
y_1 &= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (6k+1)(6k+2)}{(6n)!} (x-1)^{6n} \\
y_2 &= (x-1) + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (6k+2)(6k+3)}{(6n+1)!} (x-1)^{6n+1} \\
y_3 &= \frac{(x-1)^2}{2!} + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (6k+3)(6k+4)}{(6n+2)!} (x-1)^{6n+2} \\
y_4 &= \frac{(x-1)^3}{3!} + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (6k+4)(6k+5)}{(6n+3)!} (x-1)^{6n+3}
\end{aligned}\right. \tag{17}$$

Functions (17) are quasi-periodic, but they do not repeat all their values. Only zeros of the functions are repeated by $\omega = -2x + 2$.

### 3. Condition of "repeating" extremum

Let $y(x) \in C^2$ be a quasi-periodic function with QP $\omega(x) \in C^2$ and QPC $\lambda = 1$. Let us assume that the conditions

$$y'(x_0) = 0, \;\; y''(x_0) \neq 0$$

are satisfied at the point $x = x_0$. Then, it holds

$$y'(x_0 + \omega(x_0))(1 + \omega'(x_0)) = 0$$

from where we obtain that if $1 + \omega'(x_0) \neq 0$, then

$$y''(x_0 + \omega(x_0))(1 + \omega'(x_0))^2 = y''(x_0).$$

Thus, we have the relation

$$y''(x_0) \cdot y''(x_0 + \omega(x_0)) > 0$$

From the above arguments, we have the following lemma.

**Lemma 2.** *Let $y(x) \in C^2$ be a quasi-periodic function with QP $\omega(x) \in C^2$, $\omega(x) \neq -x + c$. If $y(x)$ has an extremum at the point $x = x_0$, then it also has an extremum of the same kind at the point $x_0 + \omega(x_0)$.*

Using relations (7), (9) and Lemma 2, we obtain the following proposition.

**Proposition 3.** *Let $y(x) \in C^4$ be a quasi-periodic solution to the equation (5) with QP $\omega(x) \in C^4$, and let $y(x)$ have an extremum at the point $x = x_0$. At the point $x_0 + \omega(x_0)$ the solution $y(x)$ has an extremum of the same kind as at the point $x = x_0$, if it holds*

$$\begin{aligned}
&y'''(x_0)\Big[6\omega''(x_0) \cdot (1 + \omega'(x_0))^3\Big] \\
&\quad + y''(x_0)\Big[-15\omega''^2(x_0) + 4\omega'''(x_0) \cdot (1 + \omega'(x_0))\Big](1 + \omega'(x_0)) \\
&\quad + y(x_0)\Big[a(x_0) - a(x_0 + \omega(x_0)) \cdot (1 + \omega'(x_0))^4\Big](1 + \omega'(x_0))^3 = 0
\end{aligned}$$

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