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Boundedness of Some Pseudodifferential Operators on Bessel-Sobolev Space [1]

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We study the continuity of generalized pseudodifferential operator $B_{\alpha,\sigma}$ on Sobolev-Bessel space, with $\alpha > -1/2$ and σ in the class of symbols. Also, we give the analogous result related to the commutator $[B_{\alpha,\sigma}\ ,\ I_\varphi\ ]$ where $I_\varphi = \mathcal{F}_{\mathcal{B}}^{-1}(\varphi \mathcal{F}_{\mathcal{B}}(\cdot))$ and φ is being a suitable function.

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1. Introduction

The continuity of the pseudodifferential operators on Sobolev space has been introduced in literature by using the classical Fourier transform on $\mathbb{R}^n$. Throughout this paper we fix $\alpha > -1/2$, the weight function

$$v_\alpha(x) = \tfrac{1}{2^\alpha \Gamma(\alpha+1)} x^{2\alpha+1}$$

and we define the generalized pseudodifferential operator $B_{\alpha,\sigma}$, on $[0, +\infty[$, by the formula

$$B_{\alpha,\sigma} f(x) = \int_0^\infty j_\alpha(x\xi)\, \sigma(x,\xi)\, \mathcal{F}_{\mathcal{B}} f(\xi)\, v_\alpha(\xi)\, d\xi,$$

for all $f \in \mathcal{S}_*(\mathbb{R})$, where:

- $\mathcal{S}_*(\mathbb{R})$ is the Schwartz's subspace of even functions.
- j_α are the normalized Bessel functions of first kind and order α given by

$$j_\alpha(\lambda) = \tfrac{2\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)} \int_0^1 \left(1-t^2\right)^{\alpha-1/2} \cos(\lambda t) dt.$$

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(See for example [4]).

- σ belongs to $S_*^{1,m}$; the class of even symbols with respect to the second variable and satisfying, for all $\alpha,\beta,\gamma\in\mathbb{N}$,

$$N_{\alpha,\beta,\gamma}^{1,m}(\sigma)=\sup_{x,\xi\geq 0}\frac{(1+x^2)^{\alpha}}{(1+\xi^2)^{m-\gamma}}\left|\left(\frac{1}{x}\frac{\partial}{\partial x}\right)^{\beta}\left(\frac{1}{\xi}\frac{\partial}{\partial \xi}\right)^{\gamma}\sigma(x,\xi)\right|<\infty.$$

- $\mathcal{F}_{\mathcal{B}}$ is the Fourier-Bessel transform given by

$$\mathcal{F}_{\mathcal{B}}f(\lambda)=\int_0^{\infty}j_{\alpha}(\lambda x)f(x)v_{\alpha}(x)\,dx,\qquad (f\in\mathcal{S}_*(\mathbb{R}),\lambda\in\mathbb{R}).$$

It has been proved in [5] that $\mathcal{F}_{\mathcal{B}}$ is an isomorphism from $\mathcal{S}_*(\mathbb{R})$ into itself and its inverse is $\mathcal{F}_{\mathcal{B}}^{-1}=\mathcal{F}_{\mathcal{B}}$.
In this work we shall be interested in the continuity of $B_{\alpha,\sigma}$ and of the commutator $[B_{\alpha,\sigma},\,I_{\varphi}]$ on the Bessel-Sobolev space $E_{\alpha}^{s,p}$, where $I_{\varphi}=\mathcal{F}_{\mathcal{B}}^{-1}(\varphi\mathcal{F}_{\mathcal{B}}(\cdot))$ with φ is a differentiable even function adequately chosen. We recall here that $E_{\alpha}^{s,p}$ is the set of even distributions f on $\mathbb{R}$ satisfying

$$\|f\|_{E_{\alpha}^{s,p}}=\left(\int_0^{\infty}\left|\left(1+\xi^2\right)^{s}\mathcal{F}_{\mathcal{B}}f(\xi)\right|^{p}v_{\alpha}(\xi)\,d\xi\right)^{1/p}<\infty.$$

For more details, we refer to the works of M. Assal and M. Nessibi [1]. See also Pathak and Pandey ([2], [3]. Our results are the following:

Theorem 1. *Let $s,r\in\mathbb{R}$, $1\leq p,q<\infty$ and $\sigma\in S_*^{1,m}$. If one of the following assertions holds*

(i) $$r<-\tfrac{\alpha+1}{q},\quad s>m+(\alpha+1)\left(1-\tfrac{1}{p}\right)\quad \text{and}\quad p>1,$$

(ii) $$s-r>m+(\alpha+1)\left(1-\tfrac{1}{p}\right)\quad \text{and}\quad \tfrac{1}{q}+\tfrac{1}{p}\leq 1,$$

(iii) $$s-r>m+\tfrac{\alpha+1}{q}\quad \text{and}\quad p=1,$$

then $B_{\alpha,\sigma}$ is a bounded operator from $E_{\alpha}^{s,p}$ to $E_{\alpha}^{r,q}$.

Theorem 2. *Let $s,r,\lambda\in\mathbb{R}$, $1\leq p,q<\infty$ and $\sigma\in S_*^{1,m}$. Let $\varphi_{\lambda}\in C_*^1(\mathbb{R})$ such that $|\varphi_{\lambda}'(\xi)|\leq(1+|\xi|)^{-\lambda}$. If one of the following assertions holds*

(i) $$r<\tfrac{1}{2}-\tfrac{\alpha+1}{q},\quad s>m-\lambda+1+(\alpha+1)\left(1-\tfrac{1}{p}\right)\quad \text{and}\quad p>1,$$

(ii) $$s-r>m-\tfrac{\lambda}{2}+(\alpha+1)\left(1-\tfrac{1}{p}\right) \quad \textit{and} \quad \tfrac{1}{q}+\tfrac{1}{p}\le 1,$$

(iii) $$s-r>m-\tfrac{\lambda}{2}+\tfrac{\alpha+1}{q} \quad \textit{and} \quad p=1,$$

then $[B_{\alpha,\sigma}, I_{\varphi_\lambda}]$ *is a bounded operator from* $E_\alpha^{s,p}$ *to* $E_\alpha^{r,q}$.

This paper is organized as follows. In Section 2 we collect some harmonic analysis results related to the Bessel operator. Section 3 is devoted to the proof of Theorems 1 and 2. Some remarks concerning the continuity from $E_\alpha^{s,p}$ to itself are also given in this section.

2. Preparations

In this section we recall some basic results in harmonic analysis related to the Bessel operators (see [5]). All functions and spaces are defined on $\mathbb{R}$. For a Banach space E let $\|\cdot\|_E$ denotes its norm. We set $\mathcal{C}^k(\mathbb{R})=\mathcal{C}^k$, $L^p(\mathbb{R})=L^p$, etc... C_*^k denotes the space of even functions of class k... The spaces $\mathcal{S}_*$, $S_*^{1,m}$ and $E_\alpha^{s,p}$ will be as defined above. We denote $L_\alpha^p([0,+\infty[)$ the space of all functions f defined on $[0,+\infty[$ such that $\|f\|_{L_\alpha^p}<\infty$, where

$$\|f\|_{L_\alpha^p}=\begin{cases}\left(\displaystyle\int_0^\infty |f(x)|^p\, v_\alpha(x)\,dx\right)^{1/p} & \text{if} \quad 1\le p<\infty,\\ \operatorname*{ess\,sup}_{x\in[0,+\infty[} |f(x)| & \text{if} \quad p=\infty.\end{cases}$$

We recall that the Bessel operator is given by $L_\alpha=\frac{1}{x^{2\alpha+1}}\frac{d}{dx}\left(x^{2\alpha+1}\frac{d}{dx}\right)$ and satisfies the following properties

(1) $$(-L_\alpha)^k\left(j_\alpha(\lambda x)\right)=\lambda^{2k} j_\alpha(\lambda x),$$

(2) $$(-L_\alpha)^k=\sum_{i=1}^{2k} C_i x^{\alpha_i}\left(\frac{1}{x}\frac{d}{dx}\right)^i,$$

where $C_i\in\mathbb{R}$ and $\alpha_i\in\mathbb{N}$ with $\alpha_i\le i$. For instance

$$(-L_\alpha)^2=4(\alpha+2)(\alpha+1)\left(\frac{1}{x}\frac{d}{dx}\right)^2+4(\alpha+2)\left(\frac{1}{x}\frac{d}{dx}\right)^3+x^4\left(\frac{1}{x}\frac{d}{dx}\right)^4.$$

Noting that, we can obtain (2) by induction on k.

Next, we shall need the generalized translation operator T_x^α defined, for all $x,y\in[0,\infty[$ and suitable function f by

$$T_x^\alpha f(y)=\tfrac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)}\int_0^\pi f\left(\sqrt{x^2+y^2+2xy\cos\theta}\right)(\sin\theta)^{2\alpha}\,d\theta;$$

The above translation operator satisfies the following properties (see [5])

- $$T_x^\alpha\left(j_\alpha\left(\lambda\cdot\right)\right)(y)=j_\alpha\left(\lambda x\right)j_\alpha\left(\lambda y\right),$$
- $$\left\|T_x^\alpha f\right\|_{L_\alpha^p}\leq\left\|f\right\|_{L_\alpha^p}\qquad\text{for } 1\leq p\leq\infty,$$
- $$T_x^\alpha f\left(y\right)=\int_0^\infty f\left(t\right)W_\alpha\left(x,y,t\right)t^{2\alpha+1}dt,$$

where $t\rightarrow W_\alpha\left(x,y,t\right)$ is supported on $\left[\left|x-y\right|,x+y\right]$ and

$$\int_0^\infty W_\alpha\left(x,y,t\right)t^{2\alpha+1}dt=1.$$

As usual "C" denotes the constant with many vary from line to line. If $1\leq p\leq\infty$ then its conjugate is given by $p'=\frac{p}{p-1}$.

3. Proofs

3.1. Some estimates

The following propositions are useful:

Proposition 1. *Let $1\leq p\leq q\leq\infty$. Then there exists a constant $C>0$, such that for all function f defined on $[0,+\infty[\times[0,+\infty[$ with $f\left(\cdot,y\right)\in L_\alpha^p$ and $f\left(x,\cdot\right)\in L_\alpha^q$, one has*

$$\left(\int_0^\infty\left\|f\left(\cdot,y\right)\right\|_{L_\alpha^p}^q v_\alpha\left(y\right)dy\right)^{\frac{1}{q}}\leq C\left(\int_0^\infty\left\|f\left(x,\cdot\right)\right\|_{L_\alpha^q}^p v_\alpha\left(x\right)dx\right)^{\frac{1}{p}}.\tag{3}$$

Proof. Consider the operator

$$T\left(\left\{x\rightarrow\left\|f\left(x,\cdot\right)\right\|_{L_\alpha^q}\right\}\right)\subseteq\left\{y\rightarrow\left\|f\left(\cdot,y\right)\right\|_{L_\alpha^p}\right\}.$$

Then we obtain (3) by interpolation on T. Then it suffices, and is not difficult, to prove (3) for $p=1$ and for $p=q$. ■

Proposition 2. *Let $\sigma\in S_*^{1,m}$. Then, for all $k\in\mathbb{N}$, there exists a constant $C=C_{m,k}>0$ such that*

$$\left|\mathcal{F}_\mathcal{B}\left[\sigma\left(\cdot,\xi\right)\right](t)\right|\leq C\left(1+t^2\right)^{-k}\left(1+\xi^2\right)^m.\tag{4}$$

P r o o f. Using (1) and (2) together with $|j_\alpha(xt)| \le 1$, we obtain, for all $t \ge 1$,

$$\begin{aligned} & |\mathcal{F}_{\mathcal{B}}[\sigma(\cdot,\xi)](t)| \\ \le\ & Ct^{-2k}\sum_{i=1}^{2k}|C_i|\int_0^\infty x^{\alpha_i}\left|\left(\tfrac{1}{x}\tfrac{\partial}{\partial x}\right)^i\sigma(x,\xi)\right|v_\alpha(x)\,dx \\ \le\ & C't^{-2k}\left(1+\xi^2\right)^m\sum_{i=1}^{2k}\int_0^\infty\left(1+x^2\right)^{\frac{\alpha_i}{2}+\alpha+\frac{3}{2}}\left|\left(\tfrac{1}{x}\tfrac{\partial}{\partial x}\right)^i\sigma(x,\xi)\right|\tfrac{dx}{1+x^2} \\ \le\ & C''t^{-2k}\left(1+\xi^2\right)^m\sum_{i=1}^{2k}N^{1,m}_{\left[\frac{\alpha_i}{2}+\alpha+\frac{3}{2}\right]^++1,i,0}(\sigma). \end{aligned}$$

Here $[x]^+$ denotes the greatest integer less than or equal to x. On the other hand, for all $0<t<1$, we have

$$\begin{aligned} |\mathcal{F}_{\mathcal{B}}[\sigma(\cdot,\xi)](t)| &\le 2^k\left(1+t^2\right)^{-k}\int_0^\infty|\sigma(x,\xi)|\,v_\alpha(x)\,dx \\ &\le C\left(2^kN^{1,m}_{\left[\alpha+\frac{3}{2}\right]^++1,0,0}(\sigma)\right)\left(1+t^2\right)^{-k}\left(1+\xi^2\right)^m, \end{aligned}$$

which ends the proof. ■

3.2. Proof of Theorem 1.

• *The case* (i). Using Fubini's theorem and the properties of the translation operators we obtain

$$\begin{aligned} & \int_0^\infty\sigma(x,\xi)\,j_\alpha(x\xi)\,j_\alpha(x\eta)\,v_\alpha(x)\,dx \\ =\ & \int_0^\infty\sigma(x,\xi)\left(\int_0^\infty W_\alpha(\xi,\eta,t)\,j_\alpha(xt)\,t^{2\alpha+1}dt\right)v_\alpha(x)\,dx \\ =\ & \int_0^\infty W_\alpha(\xi,\eta,t)\,A_\sigma(t,\xi)\,t^{2\alpha+1}dt, \end{aligned}$$

where $A_\sigma(t,\xi)=\mathcal{F}_{\mathcal{B}}[\sigma(\cdot,\xi)](t)$. This leads to

$$\mathcal{F}_{\mathcal{B}}(B_{\alpha,\sigma}f)(\eta)=\int_0^\infty\mathcal{F}_{\mathcal{B}}f(\xi)\left[T^\alpha_\xi A_\sigma(\cdot,\xi)(\eta)\right]v_\alpha(\xi)\,d\xi. \tag{5}$$

Therefore, (5) and Hölder's inequality yield

$$|\mathcal{F}_{\mathcal{B}}(B_{\alpha,a}f)(\eta)|\le\|f\|_{E^{s,p}_\alpha}\left(\int_0^{+\infty}\left(1+\xi^2\right)^{-sp'}\left|T^\alpha_\xi A_\sigma(\cdot,\xi)(\eta)\right|^{p'}v_\alpha(\xi)\,d\xi\right)^{1/p'}.$$

Applying (4), we obtain

$$(6)\qquad \left|T_\xi^\alpha A_\sigma(\cdot,\xi)(\eta)\right| \leq C\left(1+\xi^2\right)^m \int_{|\xi-\eta|}^{\xi+\eta}\left(1+t^2\right)^{-k} W_\alpha(\xi,\eta,t)\, t^{2\alpha+1}dt$$
$$= C\left(1+\xi^2\right)^m \left|T_\xi^\alpha g_1(\eta)\right|,$$

where $g_1(t)=\left(1+t^2\right)^{-k}$. Since $\|T_x^\alpha g_1\|_{L_\alpha^\infty} \leq C\|g_1\|_{L_\alpha^\infty}$, then

$$\|B_{\alpha,\sigma}f\|_{E_\alpha^{r,q}} \leq C\|f\|_{E_\alpha^{s,p}}\left(\int_0^\infty \left(1+\eta^2\right)^{rq}\eta^{2\alpha+1}d\eta\right)^{1/q}\times \left(\int_0^\infty \left(1+\xi^2\right)^{(m-s)p'}\xi^{2\alpha+1}d\xi\right)^{1/p'}.$$

- *The case* (ii). Using (5) together with Hölder's inequality, we obtain

$$\|B_{\alpha,\sigma}f\|_{E_\alpha^{r,q}}^q \leq \|f\|_{E_\alpha^{s,p}}^q \int_0^\infty\left[\int_0^{+\infty}\left(1+\xi^2\right)^{-sp'}\left(1+\eta^2\right)^{rp'}\times \left|T_\xi^\alpha A_\sigma(\cdot,\xi)(\eta)\right|^{p'} v_\alpha(\xi)\,d\xi\right]^{q/p'} v_\alpha(\eta)\,d\eta.$$

Combining (6) and Peetre's inequality, we get

$$(7)\quad \left(1+\eta^2\right)^r\left|T_\xi^\alpha A_\sigma(\cdot,\xi)(\eta)\right| \leq C2^{|r|}\left(1+\xi^2\right)^{m+r}\left(1+(\xi-\eta)^2\right)^{|r|}\left|T_\xi^\alpha g_1(\eta)\right|.$$

We take into account that $W_\alpha(\xi,\eta,.)$ is supported on $[|\xi-\eta|,\xi+\eta]$, then $|\xi-\eta|\leq t$, implies the lift-hand side of (7) is bounded by

$$C2^{|r|}\left(1+\xi^2\right)^{m+r}\left|T_\xi^\alpha g_2(\eta)\right|,$$

where $g_2(t)=\left(1+t^2\right)^{|r|-k}$, with k is at our disposal. Taking $k>|r|+\frac{\alpha+1}{q}$ then Proposition 2 leads to

$$\int_0^\infty\left[\int_0^\infty\left(1+\xi^2\right)^{(m-s+r)p'}\left(T_\xi^\alpha g_2(\eta)\right)^{p'}\xi^{2\alpha+1}d\xi\right]^{q/p'}\eta^{2\alpha+1}d\eta$$

$$\leq \left(\int_0^\infty\left(1+\xi^2\right)^{(m-s+r)p'}\left\|T_\xi^\alpha g_2\right\|_{L_\alpha^q}^{p'}\xi^{2\alpha+1}d\xi\right)^{q/p'}$$

$$\leq \|g_2\|_{L_\alpha^q}^q\left(\int_0^\infty\left(1+\xi^2\right)^{(m-s+r)p'}\xi^{2\alpha+1}d\xi\right)^{q/p'}.$$

Hence we have the desired result.

• *The case* (iii). We shall proceed as above. So, using Peetre's inequality, we obtain

$$\|B_{\alpha,\sigma}f\|^q_{E^{r,p}_\alpha} \leq \int_0^\infty \left[\int_0^\infty \left(1+\eta^2\right)^r |\mathcal{F}_\mathcal{B} f(\xi)| \left|T^\alpha_\xi A_\sigma(\cdot,\xi)(\eta)\right| v_\alpha(\xi)\, d\xi\right]^q v_\alpha(\eta)\, d\eta \leq 2^{q|s|}$$

$$\times \int_0^\infty \left[\int_0^\infty \frac{\left(1+\xi^2\right)^s}{\left(1+\eta^2\right)^{s-r}} \left(1+(\xi-\eta)^2\right)^{|s|} |\mathcal{F}_\mathcal{B} f(\xi)| \left|T^\alpha_\xi A_\sigma(\cdot,\xi)(\eta)\right| v_\alpha(\xi)\, d\xi\right]^q v_\alpha(\eta)\, d\eta$$

Now, as in (6) and since $W_\alpha(\xi,\eta,.)$ is supported on $[|\xi-\eta|,\xi+\eta]$, we obtain by Peetre's inequality

$$\begin{aligned} &\left(1+(\xi-\eta)^2\right)^{|s|} \left|T^\alpha_\xi A_\sigma(.,\xi)(\eta)\right| \\ \leq\ & C\left(1+\xi^2\right)^m \int_{|\xi-\eta|}^{\xi+\eta} \left(1+t^2\right)^{|s|-k} W_\alpha(\xi,\eta,t)\, t^{2\alpha+1} dt \\ \leq\ & C2^m \left(1+\eta^2\right)^m T^\alpha_\xi g_3(\eta), \end{aligned}$$

where $g_3(t) = \left(1+t^2\right)^{|s|-k+m}$. Hence for $k > |s|+m$, it holds

$$\|B_{\alpha,\sigma}(f)\|^q_{E^{r,q}_\alpha} \leq C \|g_3\|^q_{L^\infty_\alpha} \|f\|^q_{E^{s,1}_\alpha} \int_0^\infty \left(1+\eta^2\right)^{(r-s+m)q} \eta^{2\alpha+1} d\eta.$$

■

R e m a r k 1. Under the hypotheses of Theorem 1, with $m < -(\alpha+1)$. $B_{\alpha,\sigma}$ becomes a bounded operator on $E^{s,p}_\alpha$.

3.3. Proof of Theorem 2.

We shall give the proof first for the case $\lambda = 1$.

• *The case* (i). The use of (5) gives

$$\mathcal{F}_\mathcal{B}\left(B_{\sigma,\alpha} I_{\varphi_\lambda} f\right)(\eta) = \int_0^{+\infty} \varphi_\lambda(\xi)\, \mathcal{F}_\mathcal{B} f(\xi) \left[T^\alpha_\xi A_\sigma(\cdot,\xi)(\eta)\right] v_\alpha(\xi)\, d\xi$$

and

$$\mathcal{F}_\mathcal{B}\left(I_{\varphi_\lambda} B_{\sigma,\alpha} f\right)(\eta) = \varphi_\lambda(\eta) \int_0^{+\infty} \mathcal{F}_\mathcal{B} f(\xi) \left[T^\alpha_\xi A_\sigma(\cdot,\xi)(\eta)\right] v_\alpha(\xi)\, d\xi.$$

Then, it holds

$$\mathcal{F}_\mathcal{B}\left(\left[B_{\sigma,\alpha}, I_{\varphi_\lambda}\right] f\right)(\eta) =$$

$$\int_0^{+\infty} (\xi - \eta) \int_0^1 \varphi_\lambda' (t (\xi - \eta) + \eta) \, dt \mathcal{F}_{\mathcal{B}} f (\xi) \left[T_\xi^\alpha A_\sigma (\cdot, \xi) (\eta) \right] v_\alpha (\xi) \, d\xi.$$

The elementary estimate

$$(1 + |\eta + t (\xi - \eta)|)^{-1} \leq (1 + |\eta|)^{-1} (1 + t |\xi - \eta|)$$

(for $0 \leq t \leq 1, \xi \geq 0, \eta \geq 0$) together with Hölder's inequality imply that $|\mathcal{F}_{\mathcal{B}} ([B_{\sigma,\alpha}, I_{\varphi_\lambda}] f) (\eta)|$ is bounded by

(8)
$$C \|f\|_{E_\alpha^{s,p}} \frac{1}{1+\eta} \left\{ \int_0^{+\infty} \frac{|\xi-\eta|^{p'}}{(1+\xi^2)^{sp'}} (1 + |\xi - \eta|)^{p'} \left| T_\xi^\alpha A_\sigma (\cdot, \xi) (\eta) \right|^{p'} \xi^{2\alpha+1} d\xi \right\}^{1/p'} .$$

Next, the same argument given in the proof of Theorem 1 yields

(9) $$|\xi - \eta| (1 + |\xi - \eta|) \left| T_\xi^\alpha A_\sigma (.,\xi) (\eta) \right| \leq C \left(1 + \xi^2\right)^m |T_\xi^\alpha g_4 (\eta) |,$$

where $g_4 (t) = t (1 + t) \left(1 + t^2\right)^{-k}$. The contraction property of the translation operator leads to estimate (8) by the desired term, i.e.

$$C \|f\|_{E_\alpha^{s,p}} (1 + \eta)^{-1} \left(\int_0^\infty \left(1 + \xi^2\right)^{(m-s)p'} \xi^{2\alpha+1} d\xi \right)^{1/p'} .$$

- *The case* (ii). Combining relations (8) and (9), we obtain

$$\|\mathcal{F}_{\mathcal{B}} [B_{\sigma,\alpha}, I_\varphi] f\|_{E_\alpha^{r,q}}^q \leq CV \|f\|_{E_\alpha^{s,p}}^q ,$$

where

$$V = \int_0^{+\infty} \left[\int_0^{+\infty} \frac{\left(1+\xi^2\right)^{(m-s)p'} \left(1+\eta^2\right)^{rp'}}{(1+\eta)^{p'}} \left| T_\xi^\alpha g_4 (\eta) \right|^{p'} \xi^{2\alpha+1} d\xi \right]^{q/p'} \eta^{2\alpha+1} d\eta.$$

On the other hand, Peetrs's inequality and the estimate

$$(1 + \eta)^{-1} \leq (1 + \xi)^{-1} (1 + |\xi - \eta|)$$

lead to

$$\left(1 + \eta^2\right)^r (1 + \eta)^{-1} \left| T_\xi^\alpha g_4 (\eta) \right| \leq C \left(1 + \xi^2\right)^r (1 + \xi)^{-1} \left| T_\xi^\alpha g_5 (\eta) \right| ,$$

where $g_5(t) = t(1+t)^2\left(1+t^2\right)^{|r|-k}$. Now, Proposition 1 and the choice of $k > |r| + 2 + \frac{\alpha+1}{q}$ give the correct bound, i.e.

$$V \le \left\{ \int_0^\infty \left(1+\xi^2\right)^{(m-s+r)p'} \frac{1}{(1+\xi)^{p'}} \left(\int_0^\infty \left|T_\xi^\alpha g_5(\eta)\right|^q \eta^{2\alpha+1} d\eta \right)^{p'/q} \xi^{2\alpha+1} d\xi \right\}^{q/p'}$$
$$\le C \|g_5\|_{L_\alpha^q}^q \left[\int_0^\infty \left(1+\xi^2\right)^{(m-s+r)p'} (1+\xi)^{-p'} \xi^{2\alpha+1} d\xi \right]^{q/p'}.$$

- *The case* (iii). It is easy to obtain the following estimates

$$\left|\mathcal{F}_{\mathcal{B}}\left(\left[B_{\sigma,\alpha}, I_\varphi\right]\right) f(\eta)\right|$$
$$\le C(1+\eta)^{-1} \left[\int_0^{+\infty} \left(1+\xi^2\right)^s \left|\mathcal{F}_{\mathcal{B}} f(\xi)\right| \left(1+\xi^2\right)^{m-s} T_\xi^\alpha g_4(\eta) \cdot \xi^{2\alpha+1} d\xi \right] \le$$
$$C' 2^{|m-s|} \frac{\left(1+\eta^2\right)^{m-s}}{(1+\eta)} \left\{ \int_0^{+\infty} \frac{\left(1+\xi^2\right)^s}{\left(1+(\xi-\eta)^2\right)^{-|m-s|}} \left|\mathcal{F}_{\mathcal{B}}(f)(\xi)\right| \left|T_\xi^\alpha g_4(\eta)\right| \xi^{2\alpha+1} d\xi \right\}$$
$$\le C'' \|g_6\|_{L_\alpha^\infty} \|f\|_{E_\alpha^{s,1}} \left(1+\eta^2\right)^{m-s} (1+\eta)^{-1},$$

where $g_6(t) = t(1+t)\left(1+t^2\right)^{|m-s|-k}$. The choice of $k > 2 + |m-s|$ together with the conditions in (iii) give the desired result. ■

Remark 2. The proof of the case $\lambda \neq 1$ is similar to the above one with adequately changes by using the following inequality

$$(1+|\eta + t(\xi-\eta)|)^{-\lambda} \le 2^{|\lambda|} (1+|\eta|)^{-\lambda} (1+t|\xi-\eta|)^{|\lambda|}.$$

We omit the details.

Remark 3. Let $\lambda < 0$. If φ_λ satisfies the property

$$\left|\varphi_\lambda(\xi) - \varphi_\lambda(\eta)\right| \le A|\xi-\eta|^{-\lambda},$$

then Theorem 2 is also valid. For example φ_λ belongs to the Lipschitz space Λ_λ.

Remark 4. Under the hypotheses of Theorem 2. Define

$$a = \tfrac{\lambda}{2} - \alpha - 1, \quad b = \lambda - \alpha - \tfrac{3}{2} \quad \text{and} \quad c = \tfrac{\lambda}{2} - (\alpha+1)p'.$$

If one of the following holds:

(i) $$\lambda \geq 1, \quad m < a \quad \text{and} \quad p \geq 1,$$

(ii) $$\lambda < 1, \quad m < b \quad \text{and} \quad p \geq 1,$$

(iii) $$\lambda < 1, \quad b < m < c, \quad p = 1 \quad \text{and} \quad p \geq 2,$$

(iv) $$\lambda < 1, \quad m > \max(b, c) \quad \text{and} \quad p = 1,$$

then $[B_{\alpha,\sigma}, I_{\varphi_\lambda}]$ is a bounded operator on $E_\alpha^{s,p}$.

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