Numerical Solution of Monge-Ampere Equation

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We show that the numerical solution, of the fully non linear Monge-Ampre equation in two dimension, can be obtained by resolving an optimisation problem implying the resolution of a quasilinear Dirichlet problem. A gradient method is used. We give a no classical method to compute the gradient.

Key Words Monge-Ampre, finite elements, gradient method.

1. Introduction

In this paper we give a numerical solution of the following Monge-Ampre problem:

\[
\begin{cases}
\text{det} [D^2 u] = f^2(x, u) & x \in \Omega \\
u|_{\Gamma} = 0, u \text{ convex on } \Omega.
\end{cases}
\] (P_I)

Where \(\Omega\) is a smooth convex and bounded domain in \(\mathbb{R}^2\), \([D^2 u]\) is the Hessian of \(u\) and \(f \in C^2(\bar{\Omega} \times \mathbb{R})\), \(f > 0\) on \(\bar{\Omega} \times \mathbb{R}\), and \(\frac{\partial f}{\partial s}(x, s) \geq 0\).

The problem \((P_I)\) has a unique strictly convex solution \(u_I \in C^2(\bar{\Omega}) \cap W^{1,\infty}(\Omega)\) (see[1]).

We propose a variationnal method for the approximation of the solution \(u_I\) of \((P_I)\) as in [2]. We show that \((P_I)\) is equivalent to the following problem:

\[
(P_{II}) \min_{g \in V} J(g),
\]

with

\[
J(g) = \frac{1}{2} \int_{\Omega} [\text{det} [D^2 u(g)] - f^2(x, u(g))]^2 dx
\]
where \( u(g) \) is solution of the Dirichlet problem

\[
P_g \begin{cases} -\Delta u + 2f(., u) = -g \\ u|_{\Gamma} = 0 \end{cases}
\]

and we show that \( u_I = u(\tilde{g}) \), where \( \tilde{g} = \text{Arg}(\text{min}J(g)) \).

In section 2 we prove the equivalence between \((P_I)\) and \((P_{II})\) and we use a Galerkin-finite elements to approximize the solution \( u(g) \) of \((P_g)\). In section 3 we give a non classical method to compute the gradient of the functional \( J \). In the end we give a numerical test.

2. An equivalent problem

Let us consider the following assumptions

- \((H_1)\) \( f \in C^2(\overline{\Omega} \times \mathbb{R}) \cap W^{2,\infty}(\overline{\Omega} \times \mathbb{R}) \).
- \((H_2)\) \( f(x, s) \geq c_0 > 0, \forall s \in \mathbb{R}_-, \forall x \in \Omega \).
- \((H_3)\) \( \frac{\partial f}{\partial s}(x, s) > 0, \forall s \in \mathbb{R}_-, \forall x \in \Omega \).
- \((H_4)\) \( s \mapsto f(., s) \) is convex \( \forall s \in \mathbb{R}_- \).

2.1. The Problem \((P_{II})\)

Let \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of the matrix \([D^2u]\). We have

\[
\begin{cases} \lambda_1 + \lambda_2 = \Delta u_I, \\ \lambda_1 \lambda_2 = f(., u_I). \end{cases}
\]

Then \( \lambda_1 \) and \( \lambda_2 \) are the solutions of

\[
X^2 - \Delta u_I X + f^2(., u_I) = 0.
\]

So

\[
(\Delta u_I)^2 - 4f^2(., u_I) \geq 0.
\]

Since \( u_I \) is convex and \( f > 0 \) we should have

\[
\Delta u_I - 2f \geq 0.
\]

If we put

\[
(2.1) \quad \tilde{g} = \Delta u_I - 2f,
\]
it is clear that $u_I$ is solution of the following problem
\begin{equation}
\mathcal{P}_g \left\{ \begin{array}{l}
-\Delta u + 2f(\cdot, u) = -\tilde{g} \\
u|\Gamma = 0
\end{array} \right.
\end{equation}
To compute $\tilde{g}$, we consider the functional
\begin{equation}
J(g) = \frac{1}{2} \int_{\Omega} \left[ \det[D^2 u(g)] - f^2(x, u(g)) \right]^2 dx
\end{equation}
where $u(g)$ is the solution of the Dirichlet problem
\begin{equation}
\mathcal{P}_g \left\{ \begin{array}{l}
-\Delta u + 2f(\cdot, u) = g \\
u|\Gamma = 0.
\end{array} \right.
\end{equation}
We recall the following result:

**Theorem 2.1.** Under assumptions $(H_3)$ and $g \in L^2(\Omega)$ the quasilinear elliptic problem $(\mathcal{P}_g)$ has a unique solution $u(g) \in H^1_0(\Omega)$. (see[4]).

We have the following result:

**Theorem 2.2.** Problems $(\mathcal{P}_I)$ and $(\mathcal{P}_{II})$ are equivalents
Proof. By (2.2) we have $u_I = u(\tilde{g})$ so $J(\tilde{g}) = 0$.

Let $\bar{g}$ a solution of $(\mathcal{P}_{II})$ then $J(\bar{g}) = 0$ so
\begin{equation}
\left\{ \begin{array}{l}
\det[D^2 u(\bar{g})] = f, \\
u(\bar{g})|\Gamma = 0.
\end{array} \right.
\end{equation}
Since $\Delta u(\bar{g}) = 2f + \bar{g} > 0$ and $\det[D^2 u(\bar{g})] > 0$ we have $u(\bar{g})$ is strictly convex and from the uniqueness of solution for $(\mathcal{P}_I)$ we get $u(\bar{g}) = u_I$.

**Remark 2.3** From the previous section we can deduce that the computation by finite elements method of $u_I$ is possible by resolving $(\mathcal{P}_g)$ if one has $\bar{g}$ for this purpose we resolve $(\mathcal{P}_{II})$.

3. **The numerical resolution of $(\mathcal{P}_{II})$**

To numerical resolve $(\mathcal{P}_{II})$ we start linearizing $(\mathcal{P}_g)$ by considering a sequence of linear problems which are resolved by finite elements method. To compute $\bar{g}$ we use a gradient method.

3.1. **Resolution of the problem $(\mathcal{P}_g)$**

3.1.1. **Linearisation of the problem $(\mathcal{P}_g)$**

We assume that
\[ g \in H^1_+(\Omega) = \{ v \in H^1(\Omega) / v \geq 0 \} \]
We consider a sequence of linear problems: Let \( u^0 \) a solution of

\[
\mathcal{P}_0 \begin{cases} 
-\Delta u^0 &= -g \\
u^0_{|\Gamma} &= 0.
\end{cases}
\]

We have, since \( g \geq 0 \), by standard maximum principle: \( u^0 \leq 0 \) and we have

\[
\|u^0\|_{H^1_0(\Omega)} \leq C\|g\|_2.
\]

Let us consider, for \( k = 1, \ldots, n \), the problem \( \mathcal{P}_k \) given by:

\[
\mathcal{P}_k \begin{cases} 
-\Delta u^k + 2\frac{\partial f(.,u^{k-1})}{\partial s}u^k &= F(u^{k-1}) \\
u^k_{|\Gamma} &= 0,
\end{cases}
\]

where

\[
F(u^{k-1}) = -g + 2\frac{\partial f(.,u^{k-1})}{\partial s}u^{k-1} - 2f(.,u^{k-1}).
\]

Assume that \( u^k \leq 0 \) and \( u^k \in H^1_0(\Omega) \) for \( 1 \leq k \leq n \), where \( u^k \) is the solution of \( (\mathcal{P}_k)_{1 \leq k \leq n} \).

We consider the problem in \( u^{n+1} \):

\[
\mathcal{P}_{n+1} \begin{cases} 
-\Delta u^{n+1} + 2\frac{\partial f(.,u^n)}{\partial s}u^{n+1} &= F(u^n) \\
u^{n+1}_{|\Gamma} &= 0
\end{cases}
\]

where

\[
F(u^n) = -g + 2\frac{\partial f(.,u^n)}{\partial s}u^n - 2f(.,u^n)
\]

Remark 3.1.

i) Since \(-g \leq 0\), \( u^n \leq 0 \), and \( f(.,u^n) \geq 0 \) we have with (H2) and (H3): \( F(u^n) \leq 0 \). So by the maximum principle \( u^{n+1} \leq 0 \).

ii) Since \( \Omega \) is a bounded domain of \( \mathbb{R}^2 \), by Sobolev imbedding theorem we have \( u^n \) and \( g \in L^4(\Omega) \). By assumption (H1) we have \( F(u^n) \in L^4(\Omega) \). So \( u^{n+1} \in W^{2,4}(\Omega) \cap W^{1,4}_0(\Omega) \).

iii) Using the coercivity of the variational formulation of \( (\mathcal{P}_{n+1}) \) we have:

\[
\|u^{n+1}\|_{H^1} \leq C\|F(u^n)\|_2 \leq C(\Omega, \|g\|_2, \|f\|_\infty, \|\partial_s f\|_\infty).
\]
3.1.2 The Convergence

We need the following lemma

**Lemma 3.2.** Let $L_n$ be the linear operator

$$L_n(\omega) = -\Delta \omega + 2 \frac{\partial f}{\partial s}(., u^n)\omega$$

if we note $\omega_n = u(g) - u^n$, where $u^n$ is the solution of ($\mathcal{P}_n$) and $u(g)$ of ($\mathcal{P}_g$).

Then $\exists \theta \in ]0, 1[\text{ such that}$

$$L_n(\omega_{n+1}) = -\frac{\partial^2 f}{\partial s^2}(., u^n + \theta \omega_n)\omega_n^2$$

**Proof.** We have

$$\begin{cases}
-\Delta u(g) + f(., u(g)) = -g \\
-\Delta u^{n+1} + \frac{\partial f}{\partial s}(., u^n)u^{n+1} = F(u^n)
\end{cases}$$

subtracting, with Taylor formula and $\theta \in ]0, 1[$, gives

$$-\Delta \omega_{n+1} + 2 \frac{\partial f}{\partial s}(., u^n)\omega_{n+1} = 2g(., u^n) - 2f(., u(g)) + 2 \frac{\partial f}{\partial s}(., u^n)\omega_n$$

$$= -\frac{\partial^2 f}{\partial s^2}(., u^n + \theta \omega_n)\omega_n^2.$$

Consider now the sequence $(u^n)_{n \in \mathbb{N}}$. We have the following:

**Proposition 3.3.**

$$u(g) \leq \cdots \leq u^{n+1} \leq u^n \leq \cdots \leq u^0 \leq 0.$$  

**Proof.**

(1) We first show that $u^{n+1} - u^n \leq 0$ Let $\omega_0 = u^1 - u^0$, we have

$$\begin{cases}
-\Delta u^1 + 2 \frac{\partial f}{\partial s}(., u^0)u^1 = F(u^0) \\
-\Delta u^0 = -g
\end{cases}$$

subtracting gives

$$-\Delta \omega_0 + 2 \frac{\partial f}{\partial s}(., u^0)\omega_0 = -2f(., u^0) \leq 0,$$
then by the maximum principle \( \omega_0 \leq 0 \). Subtracting \((P_{n+1})\) and \((P_n)\) we obtain under assumption \((H_4)\) and Lemma 3.3

\[
-\Delta \omega_n + 2 \frac{\partial f}{\partial s}(., u^n)\omega_n = -\frac{\partial g}{\partial s^2}(., u^{n-1} + \theta \omega_{n-1})\omega_{n-1}^2 \leq 0.
\]

By maximum principle we have \( \omega_n \leq 0 \).

(2) We show now that, \( u(g) \leq u^n \), \forall n.

By Lemma 3.3, we see that the function \( \omega_{n+1} = u(g) - u^{n+1} \) satisfy

\[
-\Delta \omega_{n+1} + 2 \frac{\partial f}{\partial s}(., u^n)\omega_{n+1} = -\frac{\partial^2 f}{\partial s^2}(., u^{n} + \theta \omega_{n})\omega_{n}^2 \leq 0
\]

and then \( \omega_{n+1} \leq 0 \).

Remark 3.4. Assumption \( H_3 \) with (3.6) and (3.8) implies that \( P_g \) has a solution which is giving by

\[
u(g) = \inf (u^n).
\]

Remark 3.5. Let \( E_{n+1} = u^{n+1} - u(g) \) the error between the solution of \((P_{n+1})\) and the solution of \((P_g)\). By Lemma 3.3, we have

\[
-\Delta E_{n+1} + 2 \frac{\partial f}{\partial s}(., u^n)E_{n+1} = \frac{\partial^2 f}{\partial s^2}(., u^n - \theta E_n)E_n^2
\]

so we obtain

\[
\|E_{n+1}\|_{H^1} \leq C\|E_n\|^2_{H^1}
\]

which implies that the convergence is quadratic.

3.2. Computation of the gradient of \( J \).

To compute the gradient of the functional \( J \) considered in (2.3), we consider the weak solution of \((P_g)\), and we assume that \( T_h \) is a triangulation of \( \Omega \). Thus \( J \) can be written

\[
J(g) = \frac{1}{2} \sum_{\ell=1}^{N} \int_{K_\ell} [\det[D^2u(g)] - f^2(x, u(g))]^2 dx
\]

where \( N \) is the number of triangles \( K_\ell \) and \( u(g) \) is given by :

\[
u(g) = \sum_{i=1}^{N_0} E_i(g) \varphi_i
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with \((\varphi_i)_{i=1,\ldots,N_0}\) the basic functions of Galerkin-finite elements of order two.

If

\[
\varphi_{il}(x,y) = \alpha_{il}x^2 + \beta_{il}y^2 + \gamma_{il}xy + \delta_{il}x + \eta_{il}y + \tau_{il}, \quad i = 1 \cdots 6
\]

are the basic functions on \(K_\ell\), we have:

\[
det[D^2u(g)]_{|K_\ell} = 4\left(\sum_{i=1}^{6} E_i(g)\alpha_{il}\right)\left(\sum_{j=1}^{6} E_j(g)\beta_{jl}\right) - \left(\sum_{k=1}^{6} E_k(g)\gamma_{kl}\right)^2
\]

and

\[
J(g) = \frac{1}{2} \sum_{\ell=1}^{N} (A_\ell D^2_{\ell}(g) - 2B_\ell(g)D_\ell(g) + C_\ell(g))
\]

where

\[A_\ell = \text{Area}(K_\ell), \quad B_\ell(g) = \int_{K_\ell} f^2(x,u(g(x)))dx\]

and

\[C_\ell(g) = \int_{K_\ell} f^4(x,u(g(x)))dx\]

So if we write

\[
g = \sum_{m=1}^{N_h} g_m \varphi_m
\]

we have:

\[
\frac{\partial J}{\partial g_m} = \sum_{\ell=1}^{N} \left\{(A_\ell D_\ell(g) - B_\ell(g)) \frac{\partial D_\ell(g)}{\partial g_m} - D_\ell(g) \frac{\partial B_\ell(g)}{\partial g_m}(g) + \frac{1}{2} \frac{\partial C_\ell(g)}{\partial g_m}\right\}
\]

with

\[
\frac{\partial B_\ell(g)}{\partial g_m} = \int_{K_\ell} \frac{\partial f^2}{\partial s}(.u(g))\Phi_{m,\ell}dx
\]

\[
\frac{\partial C_\ell(g)}{\partial g_m} = \int_{K_\ell} \frac{\partial f^4}{\partial s}(.u(g))\Phi_{m,\ell}dx.
\]

Where

\[
\Phi_{m,\ell}(x) = \sum_{i=1}^{6} \frac{\partial E_i(g)}{\partial g_m}\varphi_{il}(x)
\]
and
\[
\frac{\partial D_i(g)}{\partial g_m} = 4 \left( \sum_{i=1}^{6} \frac{\partial \mathcal{E}_i(g)}{\partial g_m} \alpha_i \right) \left( \sum_{j=1}^{6} \mathcal{E}_j(g) \beta_j \right) + 4 \left( \sum_{i=1}^{6} \mathcal{E}_i(g) \alpha_i \right) \left( \sum_{j=1}^{6} \frac{\partial \mathcal{E}_j(g)}{\partial g_m} \beta_j \right) - 2 \left( \sum_{k=1}^{6} \mathcal{E}_k(g) \gamma_{kl} \right) \left( \sum_{k=1}^{6} \frac{\partial \mathcal{E}_k(g)}{\partial g_m} \gamma_{kl} \right)
\]
(3.20)

So to compute the gradient of $J$ we need
\[
\frac{\partial \mathcal{E}_i(g)}{\partial g_m} \mid i = 1, \cdots, N_h^0, \ m = 1, \cdots, N_h
\]
(3.21)

We consider then equation (2.4) and after partial derivation we have by (3.15)
\[
\begin{cases}
-\Delta \omega_m + 2 \frac{\partial f}{\partial s} (\cdot, u(g)) \omega_m = -\varphi_m, \\
\omega_{m\mid\Gamma} = 0.
\end{cases}
\]
(3.22)

Where $\omega_m = \frac{\partial u(g)}{\partial g_m}$. Note too using (3.11) that
\[
\frac{\partial u(g)}{\partial g_m} = \sum_{i=1}^{N_h^0} \frac{\partial \mathcal{E}_i(g)}{\partial g_m} \varphi_i
\]
and (3.22) give
\[
\omega_m = \sum_{i=1}^{N_h^0} \eta_i^m \varphi_i.
\]
(3.23)

So resolving (3.22) by the use of the finite elements method we obtain with (3.23)
\[
\frac{\partial \mathcal{E}_i(g)}{\partial g_m} = \eta_i^m \ i = 1, \cdots, N_h^0 \ m = 1, \cdots, N_h.
\]
(3.24)

4. Numerical test

In order to test this method, we have chosen, an example of problem $(\mathcal{P}_1)$ which we know its explicit solution. The latter is chosen in order to be compared to the computed solution.
Example 4.1. We take $\Omega$ the unit disk and
\[ f((x, y), u) = 4(3 + 2\log(1 + u))(1 + u)^2. \]
It’s clear that $f$ verifies all assumptions $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$. The solution of $(P_I)$ is the following
\[ u_I(x, y) = e^{(x^2 + y^2 - 1)} - 1. \]
In Fig. 4.1 and Fig. 4.2 we present the computed solution obtained by applying this method at 10 iterations compared to the exact solution $u_I$.

References


Figure 4.1. Graphs of $u_I$ and the computed solution $u = u(x, y)$.

Figure 4.2. The contour plot of $u_I$ and the computed solution.