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Numerical Solution of Monge-Ampere Equation

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We show that the numerical solution, of the fully non linear Monge-Ampre equation in two dimension, can be obtained by resolving an optimisation problem implying the resolution of a quasilinear Dirichlet problem. A gradient method is used. We give a no classical method to compute the gradient.

Key Words Monge-Ampre, finite elements, gradient method.

1. Introduction

In this paper we give a numerical solution of the following Monge-Ampre problem :

$$(\mathcal{P}_I)\left\{\begin{array}{l} det[D^2u] = f^2(x,u) \;\; x\in\Omega \\ u_{|\Gamma} = 0, u \text{ convex on } \bar{\Omega}. \end{array}\right.$$

Where Ω is a smooth convex and bounded domain in $\mathbb{R}^2$, $[D^2u]$ is the Hessian of u and $f \in \mathcal{C}^2(\bar{\Omega}\times\mathbb{R})$, $f > 0$ on $\bar{\Omega}\times\mathbb{R}$, and $\dfrac{\partial f}{\partial s}(x,s) \geq 0$.
The problem $(\mathcal{P}_I)$ has a unique strictly convex solution $u_I \in \mathcal{C}^2(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$(see[1]).

We propose a variationnal method for the approximation of the solution u_I of $(\mathcal{P}_I)$ as in [2]. We show that $(\mathcal{P}_I)$ is equivalent to the following problem :

$$(\mathcal{P}_{II})\min_{g\in V} J(g),$$

with

$$J(g) = \frac{1}{2}\int_\Omega [det[D^2u(g)] - f^2(x,u(g))]^2 dx$$

where $u(g)$ is solution of the Dirichlet problem

$$\mathcal{P}_g \begin{cases} -\Delta u + 2f(.,u) = -g \\ u_{|\Gamma} = 0 \end{cases}$$

and we show that $u_I = u(\tilde{g})$, where $\tilde{g} = Arg(minJ(g))$

In section 2 we prove the equivalence between $(\mathcal{P}_I)$ and $(\mathcal{P}_{II})$ and we use a Galerkin-finite elements to approximize the solution $u(g)$ of $(\mathcal{P}_g)$. In section 3 we give a non classical method to compute the gradient of the functional J. In the end we give a numerical test.

2. An equivalent problem

Let us consider the following assumptions

- (H_1) $f \in C^2(\overline{\Omega} \times \mathbb{R}) \cap W^{2,\infty}(\overline{\Omega} \times \mathbb{R})$.
- (H_2) $f(x,s) \geq \alpha_0 > 0$, $\forall s \in \mathbb{R}_-$, $\forall x \in \Omega$.
- (H_3) $\dfrac{\partial f}{\partial s}(x,s) > 0$, $\forall s \in \mathbb{R}_-$, $\forall x \in \Omega$.
- (H_4) $s \longmapsto f(.,s)$ is convex $\forall s \in \mathbb{R}_-$.

2.1. The Problem $(\mathcal{P}_{II})$

Let λ_1 and λ_2 be the eigenvalues of the matrix $[D^2u]$. We have

$$\begin{cases} \lambda_1 + \lambda_2 & = \Delta u_I, \\ \lambda_1 \lambda_2 & = f(.,u_I). \end{cases}$$

Then λ_1 and λ_2 are the solutions of

$$X^2 - \Delta u_I X + f^2(.,u_I) = 0.$$

So

$$(\Delta u_I)^2 - 4f^2(.,u_I) \geq 0.$$

Since u_I is convex and $f > 0$ we should have

$$\Delta u_I - 2f \geq 0.$$

If we put

$$\tilde{g} = \Delta u_I - 2f, \tag{2.1}$$

it is clear that u_I is solution of the following problem

$$\text{(2.2)} \qquad \mathcal{P}_{\tilde{g}} \begin{cases} -\Delta u + 2f(.,u) = -\tilde{g} \\ u_{|\Gamma} = 0 \end{cases}$$

To compute $\tilde{g}$, we consider the functional

$$\text{(2.3)} \qquad J(g) = \frac{1}{2}\int_\Omega [det[D^2u(g)] - f^2(x,u(g))]^2 dx$$

where $u(g)$ is the solution of the Dirichlet problem

$$\text{(2.4)} \qquad (\mathcal{P}_g) \begin{cases} -\Delta u + 2f(.,u) = -g \\ u_{|\Gamma} = 0. \end{cases}$$

We remark that J is well-defined if $u(g) \in W^{2,4}(\Omega)$ and $f^2(.,u(g)) \in L^2(\Omega)$. We recall the following result:

Theorem 2.1. *Under assumptions* (H_3) *and* $g \in L^2(\Omega)$ *the quasilinear elliptic problem* $(\mathcal{P}_g)$ *has a unique solution* $u(g) \in H_0^1(\Omega)$. (see[4]).

We have the following result:

Theorem 2.2. *Problems* $(\mathcal{P}_I)$ *and* $(\mathcal{P}_{II})$. *are equivalents*

Proof. By (2.2) we have $u_I = u(\widetilde{g})$ so $J(\widetilde{g}) = 0$.
Let $\overline{g}$ a solution of $(\mathcal{P}_{II})$ then $J(\overline{g}) = 0$ so

$$\begin{cases} det[D^2u(\overline{g})] = f, \\ u(\overline{g})_{|\Gamma} = 0. \end{cases}$$

Since $\Delta u(\overline{g}) = 2f + \overline{g} > 0$ and $det[D^2u(\overline{g}) > 0$ we have $u(\overline{g})$ is strictly convex and from the uniqueness of solution for $(\mathcal{P}_I)$ we get $u(\overline{g}) = u_I$. ■

Remark 2.3 From the previous section we can deduce that the computation by finite elements method of u_I is possible by resolving $(\mathcal{P}_{\widetilde{g}})$ if one has $\widetilde{g}$ for this purpose we resolve $(\mathcal{P}_{II})$.

3. The numerical resolution of $(\mathcal{P}_{II})$

To numerical resolve $(\mathcal{P}_{II})$ we start linearizing $(\mathcal{P}_g)$ by considering a sequence of linear problems which are resolved by finite elements method. To compute $\tilde{g}$ we use a gradient method.

3.1. Resolution of the problem $(\mathcal{P}_g)$

3.1.1. *Linearisation of the problem* $(\mathcal{P}_g)$ We assume that

$$g \in H_+^1(\Omega) = \{v \in H^1(\Omega)/v \geq 0\}$$

.
We consider a sequence of linear problems : Let u^0 a solution of

$$(3.1)\qquad \mathcal{P}_0 \begin{cases} -\Delta u^0 & = -g \\ u^0_{|\Gamma} & = 0. \end{cases}$$

We have, since $g \geq 0$, by standard maximum principle : $u^0 \leq 0$ and we have

$$(3.2)\qquad \|u^0\|_{H^1_0(\Omega)} \leq C\|g\|_2.$$

Let us consider, for $k = 1, \cdots n \cdots$, the problem $(\mathcal{P}_k)$ given by :

$$(3.3)\qquad \mathcal{P}_k \begin{cases} -\Delta u^k + 2\dfrac{\partial f}{\partial s}(., u^{k-1})u^k = F(u^{k-1}), \\ u^k_{|\Gamma} = 0, \end{cases}$$

where

$$F(u^{k-1}) = -g + 2\frac{\partial f}{\partial s}(., u^{k-1})u^{k-1} - 2f(., u^{k-1}).$$

Assume that $u^k \leq 0$ and $u^k \in H^1_0(\Omega)$ for $1 \leq k \leq n$, where u^k is the solution of $(\mathcal{P}_k)_{1\leq k\leq n}.$,
We consider the problem in u^{n+1} :

$$(3.4)\qquad \mathcal{P}_{n+1} \begin{cases} -\Delta u^{n+1} + 2\dfrac{\partial f}{\partial s}(., u^n)u^{n+1} = F(u^n) \\ u^{n+1}_{|\Gamma} = 0 \end{cases}$$

where

$$(3.5)\qquad F(u^n) = -g + 2\frac{\partial f}{\partial s}(., u^n)u^n - 2f(., u^n)$$

Remark 3.1.

i) Since $-g \leq 0$, $u^n \leq 0$, and $f(., u^n) \geq 0$ we have with (H_2) and (H_3): $F(u^n) \leq 0$. So by the maximum principle $u^{n+1} \leq 0$.

ii) Since Ω is a bounded domain of $\mathbb{R}^2$, by Sobolev imbedding theorem we have u^n and $g \in L^4(\Omega)$. By assumption (H_1) we have $F(u^n) \in L^4(\Omega)$. So $u^{n+1} \in W^{2,4}(\Omega) \cap W^{1,4}_0(\Omega)$.

iii) Using the coercivity of the variational formulation of $(\mathcal{P}_{n+1})$ we have :

$$(3.6)\qquad \|u^{n+1}\|_{H^1} \leq C\|F(u^n)\|_2 \leq C(\Omega, \|g\|_2, \|f\|_\infty, \|\partial_s f\|_\infty).$$

.

3.1.2 *The Convergence* We need the following lemma

Lemma 3.2. *Let $\mathcal{L}_n$ be the linear operator*

$$\mathcal{L}_n(\omega) = -\Delta\omega + 2\frac{\partial f}{\partial s}(.,u^n)\omega$$

if we note $\omega_n = u(g) - u^n$, where u^n is the solution of $(\mathcal{P}_n)$ and $u(g)$ of $(\mathcal{P}_g)$. Then $\exists\, \theta \in]0,1[$ such that

$$\mathcal{L}_n(\omega_{n+1}) = -\frac{\partial^2 f}{\partial s^2}(.,u^n + \theta\omega_n)\omega_n^2 \tag{3.7}$$

P r o o f. We have

$$\begin{cases} -\Delta u(g) + f(.,u(g)) = -g \\ -\Delta u^{n+1} + \dfrac{\partial f}{\partial s}(.,u^n)u^{n+1} = F(u^n) \end{cases}$$

subtracting, with Taylor formula and $\theta \in]0,1[$, gives

$$\begin{aligned} -\Delta\omega_{n+1} + 2\frac{\partial f}{\partial s}(.,u^n)\omega_{n+1} &= 2g(.,u^n) - 2f(.,u(g)) + 2\frac{\partial f}{\partial s}(.,u^n)\omega_n \\ &= -\frac{\partial^2 f}{\partial s^2}(.,u^n + \theta\omega_n)\omega_n^2. \end{aligned}$$

■

Consider now the sequence $(u^n)_{n\in\mathbb{N}}$. We have the following:

Proposition 3.3.

$$u(g) \le \cdots \le u^{n+1} \le u^n \le \cdots \le u^0 \le 0. \tag{3.8}$$

P r o o f.

(1) We first show that $u^{n+1} - u^n \le 0$ Let $\omega_0 = u^1 - u^0$, we have

$$\begin{cases} -\Delta u^1 + 2\dfrac{\partial f}{\partial s}(.,u^0)u^1 = F(u^0) \\ -\Delta u^0 = -g \end{cases}$$

subtracting gives

$$-\Delta\omega_0 + 2\frac{\partial f}{\partial s}(.,u^0)\omega_0 = -2f(.,u^0) \le 0,$$

then by the maximum principle $\omega_0 \leq 0$. Subtracting $(\mathcal{P}_{n+1})$ and $(\mathcal{P}_n)$ we obtain under assumption (H_4) and Lemma3.3

$$-\Delta\omega_n + 2\frac{\partial f}{\partial s}(., u^n)\omega_n = -\frac{\partial g}{\partial s^2}(., u^{n-1} + \theta\omega_{n-1})\omega_{n-1}^2 \leq 0.$$

By maximum principle we have $\omega_n \leq 0$.

(2) We show now that, $u(g) \leq u^n$, $\forall\, n$.
By Lemma3.3, we see that the function $\omega_{n+1} = u(g) - u^{n+1}$ satisfy

$$-\Delta\omega_{n+1} + 2\frac{\partial f}{\partial s}(., u^n)\omega_{n+1} = -\frac{\partial^2 f}{\partial s^2}(., u^n + \theta\omega_n)\omega_n^2 \leq 0$$

and then $\omega_{n+1} \leq 0$.

■

Remark 3.4. Assumption H_3 with(3.6) and (3.8) implies that $\mathcal{P}_g$ has a solution which is giving by

$$u(g) = Inf(u^n).$$

Remark 3.5. Let $\mathcal{E}_{n+1} = u^{n+1} - u(g)$ the error between the solution of $(\mathcal{P}_{n+1})$ and the solution of $(\mathcal{P}(g))$. By Lemma3.3, we have

$$-\Delta\mathcal{E}_{n+1} + 2\frac{\partial f}{\partial s}(., u^n)\mathcal{E}_{n+1} = \frac{\partial^2 f}{\partial s^2}(., u^n - \theta\mathcal{E}_n)\mathcal{E}_n^2,$$

so we obtain

$$\|\mathcal{E}_{n+1}\|_{H^1} \leq C\|\mathcal{E}_n\|_{H^1}^2, \tag{3.9}$$

which implies that the convergence is quadratic.

3.2. Computation of the gradient of J.

To compute the gradient of the functional J considered in (2.3), we consider the weak solution of $(\mathcal{P}_g)$, and we assume that $\mathcal{T}_h$ is a triangulation of Ω. Thus J can be written

$$J(g) = \frac{1}{2}\sum_{\ell=1}^{N}\int_{K_\ell}[\det[D^2u(g)] - f^2(x, u(g))]^2dx \tag{3.10}$$

where N is the number of triangles K_ℓ and $u(g)$ is given by :

$$u(g) = \sum_{i=1}^{N_h^0}\mathcal{E}_i(g)\varphi_i \tag{3.11}$$

with $(\varphi_i)_{i=1,\cdots,N_h^0}$ the basic functions of Galerkin-finite elements of order two . If

$$\varphi_{i\ell}(x,y) = \alpha_{i\ell}x^2 + \beta_{i\ell}y^2 + \gamma_{i\ell}xy + \delta_{i\ell}x + \eta_{i\ell}y + \tau_{i\ell} \quad , i = 1 \cdots 6 \tag{3.12}$$

are the basic functions on K_ℓ, we have :

$$det[D^2u(g)]_{|K_\ell} = 4(\sum_{i=1}^{6} \mathcal{E}_i(g)\alpha_{i\ell})(\sum_{j=1}^{6} \mathcal{E}_j(g)\beta_{j\ell}) - (\sum_{k=1}^{6} \mathcal{E}_k(g)\gamma_{k\ell})^2 \tag{3.13}$$

and

$$J(g) = \frac{1}{2}\sum_{\ell=1}^{N}(A_\ell D_\ell^2(g) - 2B_\ell(g)D_\ell(g) + C_\ell(g)) \tag{3.14}$$

where

$$A_\ell = Area(K_\ell), \quad B_\ell(g) = \int_{K_\ell} f^2(x, u(g(x))dx$$

and

$$D_\ell(g) = det[D^2u(g)]_{|K_\ell}, \quad C_\ell(g) = \int_{K_\ell} f^4(x, u(g(x))dx$$

So if we write

$$g = \sum_{m=1}^{N_h} g_m\varphi_m \tag{3.15}$$

we have :

$$\frac{\partial J}{\partial g_m} = \sum_{\ell=1}^{N}\{(A_\ell D_\ell(g) - B_\ell(g))\frac{\partial D_\ell(g)}{\partial g_m} - D_\ell(g)\frac{\partial B_\ell}{\partial g_m}(g) + \frac{1}{2}\frac{\partial C_\ell(g)}{\partial g_m}\} \tag{3.16}$$

with

$$\frac{\partial B_\ell(g)}{\partial g_m} = \int_{K_\ell} \frac{\partial f^2}{\partial s}(., u(g))\Phi_{m,\ell}dx \tag{3.17}$$

$$\frac{\partial C_\ell(g)}{\partial g_m} = \int_{K_\ell} \frac{\partial f^4}{\partial s}(., u(g))\Phi_{m,\ell}dx. \tag{3.18}$$

Where

$$\Phi_{m,\ell}(x) = \sum_{i=1}^{6} \frac{\partial \mathcal{E}_i(g)}{\partial g_m}\varphi_{i\ell}(x) \tag{3.19}$$

and

$$\begin{aligned} \frac{\partial D_\ell(g)}{\partial g_m} &= 4\left(\sum_{i=1}^{6}\frac{\partial \mathcal{E}_i(g)}{\partial g_m}\alpha_{i\ell}\right)\Big(\sum_{j=1}^{6}\mathcal{E}_j(g)\beta_{j\ell}\Big) \\ &+ 4\Big(\sum_{i=1}^{6}\mathcal{E}_i(g)\alpha_{i\ell}\Big)\Big(\sum_{j=1}^{6}\frac{\partial \mathcal{E}_j(g)}{\partial g_m}\beta_{j\ell}\Big) \\ &- 2\Big(\sum_{k=1}^{6}\mathcal{E}_k(g)\gamma_{k\ell}\Big)\Big(\sum_{k=1}^{6}\frac{\partial \mathcal{E}_k(g)}{\partial g_m}\gamma_{k\ell}\Big) \end{aligned} \tag{3.20}$$

So to compute the gradient of J we need

$$\frac{\partial \mathcal{E}_i(g)}{\partial g_m}\ i = 1, \cdots, N_h^0,\ \ m = 1, \cdots, N_h \tag{3.21}$$

We consider then equation (2.4) and after partial derivation we have by (3.15)

$$\begin{cases} -\Delta\omega_m + 2\dfrac{\partial f}{\partial s}(., u(g))\omega_m = -\varphi_m, \\ \omega_{m|\Gamma} = 0. \end{cases} \tag{3.22}$$

Where $\omega_m = \dfrac{\partial u(g)}{\partial g_m}$. Note too using (3.11) that

$$\frac{\partial u(g)}{\partial g_m} = \sum_{i=1}^{N_h^0}\frac{\partial \mathcal{E}_i(g)}{\partial g_m}\varphi_i$$

and (3.22) give

$$\omega_m = \sum_{i=1}^{N_h^0}\eta_i^m\varphi_i. \tag{3.23}$$

So resolving (3.22) by the use of the finite elements method we obtain with (3.23)

$$\frac{\partial \mathcal{E}_i(g)}{\partial g_m} = \eta_i^m \quad i = 1, \cdots, N_h^0\ \ m = 1, \cdots, N_h. \tag{3.24}$$

4. Numerical test

In order to test this method, we have shosen, an example of problem $(\mathcal{P}_I)$ which we know its explicet solution. The latter is shosen in order to be compared to the computed solution.

Example Example 4.1. We take Ω the unit disque and

$$f((x,y),u) = 4\,(3 + 2Log\,(1+u))\,(1+u)^2\,.$$

It's clear that f verifies all assuptions (H_1), $(H_2,)$ (H_3) and (H_4). The solution of $(\mathcal{P}_I)$ is the following

$$u_I(x,y) = e^{\left(x^2+y^2-1\right)} - 1.$$

In Fig. 4.1 and Fig. 4.2 we present the computed solution obtained by applying this method at 10 iterations compared to the exact solution u_I.

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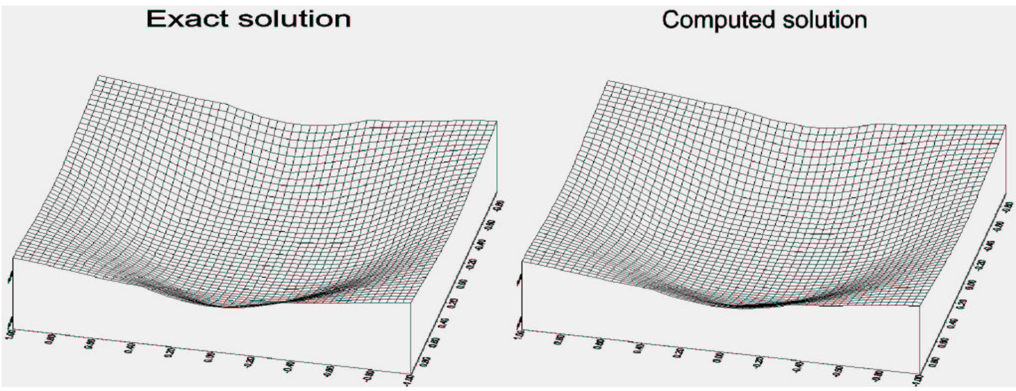

FIGURE 4.1. Graphs of u_I and the computed solution $u = u(x, y)$.

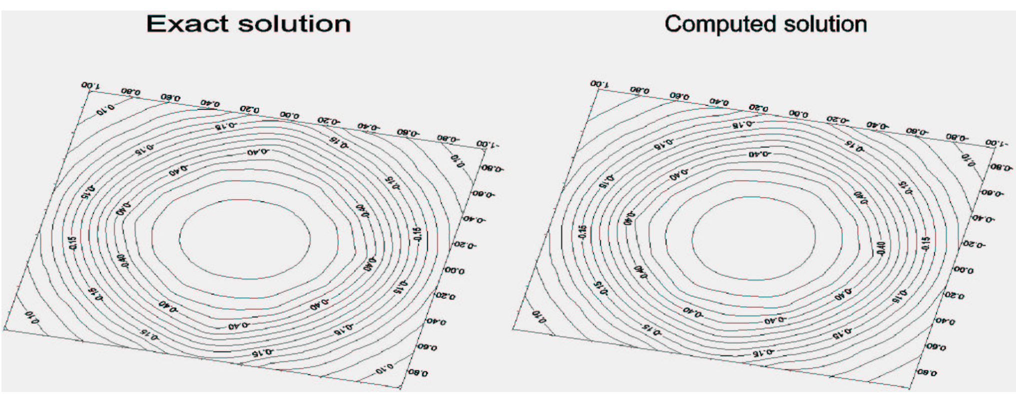

FIGURE 4.2. The contour plot of u_I and the computed solution.