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The Connectivity of Squares of Box Graphs

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The aim of the paper is to study the connectivity and the edge-connectivity of square of the box graph $[B(G)]^2$ of a graph G with the help of connectivity and the edge-connectivity of the graph G and its inserted graph $I(G)$.

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1. Introduction

We consider ordinary graphs (finite, undirected,with no loops or multiple edges). Let G be a graph with vertex set V_G and edge set E_G. Each member of $V_G \cup E_G$ will be called an element of G. A graph G is called trivial graph if it has a vertex set with single vertex and a null edge set. If e be an edge of a graph G with end vertices x and y, then we denote the edge e, by $e = xy$. We introduce the notions of box graph $B(G)$, inserted graph $I(G)$ and square of a box graph $[B(G)]^2$ of a non-trivial graph G in [2].

There are two major measures how highly connected a graph can be, namely the connectivity and edge-connectivity.

The connectivity $k(G)$ of a graph G is the least number of vertices whose removal (along with all incident edges) disconnects G or reduces it to the trivial graph; a set of $k(G)$ vertices satisfying this condition is called a minimal separating vertex set of G. Moreover G is n-connected if and only if $k(G) \geq n$.

On the other hand, the edge-connectivity $\lambda(G)$ of a graph G is the least number of edges whose removal disconnects G; and a set of $\lambda(G)$ edges satisfying this condition is called a minimal separating edge set of G. Moreover G is is m-edge-connected if and only if $\lambda(G) \geq m$.

In §2, we recall some definitions and results to be used in this paper and construct square of box graph $[B(G)]^2$ for a non-trivial graph G.

In §3, we investigate the connectivity relationships between a graph and square of its box graph. In particular, we show that if $k(G) = n$, $n \geq 1$, and $\lambda(G) = m$, then $\lambda([B(G)]^2) \geq 2m$, and $k([B(G)]^2) \geq n + 2 + [\frac{n-2}{3}]$, where $[x]$ is the greatest integer not exceeding x.

2. Preliminaries

In this section at first we recall some definitions.

Definition 2.1 *For a graph G, the square of G i.e, G^2 is a graph with the property that there always exists a one-one correspondence between its vertices and the vertices of G such that two vertices of G^2 are adjacent if the corresponding vertices of G are joined by a path of length one or two. [5]*

Definition 2.2 *A graph can be constructed by inserting a new vertex on each edge of G, the resulting graph is called Box graph of G, denoted by $B(G)$. For an edge e of G, $\overline{e}$ denote the vertex of $B(G)$ corresponding to the edge e. [2]*

The graph $B(G)$ has the property that, there always exists a one-one correspondence between the vertices and the elements of G such that any two vertices of $B(G)$ are adjacent if and only if the corresponding elements of G are an edge and an incident vertex. Obviously $B(G)$ is a bipartite graph whose number of vertices is equal to the number of elements of G. Moreover if $V_G = \{v_1, v_2, ..., v_n\}$ and $E_G = \{e_1, e_2, ..., e_m\}$ then $V_{B(G)} = \{v_1, v_2, ..., v_n, \overline{e}_1, \overline{e}_2, ..., \overline{e}_m\}$.

Definition 2.3 *Let I_G be the set of all inserted vertices in $B(G)$. A graph $I(G)$ with vertex set I_G is called the inserted graph in which any two vertices are adjacent if they are joined by a path of length two in $B(G)$. Therefore if $E_G = \{e_1, e_2, ..., e_m\}$ then $I_G = V_{I(G)} = \{\overline{e}_1, \overline{e}_2, ..., \overline{e}_m\}$. [2]*

Now we construct square of box graph $[B(G)]^2$ for a non-trivial graph G as follows (Fig. 1).
Here $\otimes$ marked vertices are the newly inserted vertices of $B(G)$. The graph $[B(G)]^2$ has the property that the graphs G, $B(G)$ and $I(G)$ are edge disjoint subgraphs of $[B(G)]^2$.

Now we recall here some results related to connectivity and edge-connectivity, to which we shall have occasion to refer in what follows. Characterizations of n-connected graphs and m-edge-connected graphs are presented bellow [4].

Theorem 2.4 *A graph G is n-connected (m-edge-connected) if and only if between every pair of distinct vertices there exist at least n disjoint (m edge-disjoint) paths.*

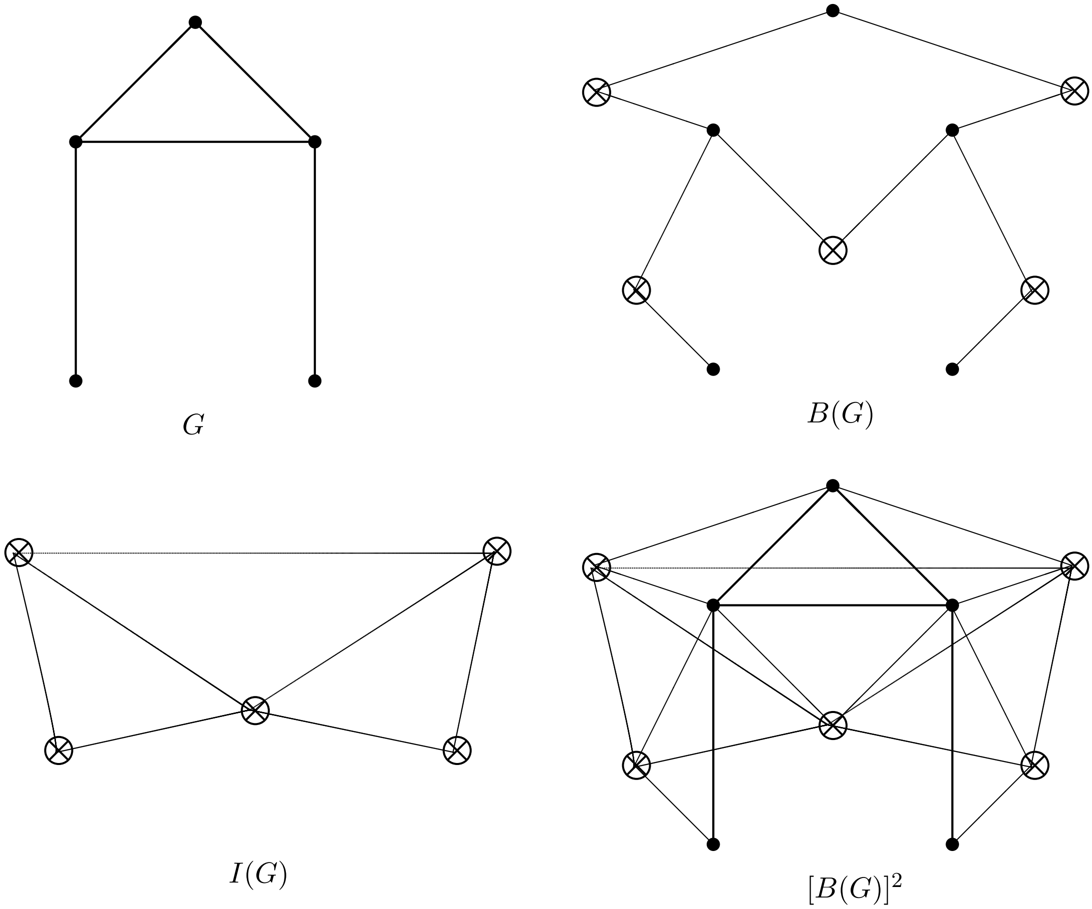

Figure 1

The next theorem is due to Adhikari and Pramanik [3].

Theorem 2.5 *If* $k(G_1) = n$ *and* $\lambda(G_2) = m$, *then* $k(I(G_1)) \geq n$ *and* $\lambda(I(G_1)) \geq 2n - 2$ *while* $k(I(G_2)) \geq m$ *and* $\lambda(I(G_2)) \geq 2m - 2$.

The next observation is due to Whitney [6]. We write min deg G to denote the smallest degree among the vertices of G.

Theorem 2.6 *For any graph* G, $k(G) \leq \lambda(G) \leq$ *min deg* G.

The following lemmas may be proved as immediate consequence of definitions:

Lemma 2.7 *$[B(X)]^2$ is a subgraph of $[B(A)]^2$ if and only if X is a subgraph of A.*

Lemma 2.8 *A necessary condition for $[B(G)]^2$ to be connected is that G is connected.*

Lemma 2.9 *For any grapg G and H, $[B(G \cap H)]^2 = [B(G)]^2 \cap [B(H)]^2$.*

3. Connectivity and edge-connectivity of $[B(G)]^2$

Before we prove our first theorem we observe that G is connected if and only if $[B(G)]^2$ is connected; and that in $[B(G)]^2$ a vertex of G is adjacent to at least min deg G vertices of $I(G)$.

Theorem 3.1 *If G is m-edge-connected, than $[B(G)]^2$ is $2m$-edge-connected.*

Proof. If $m = 0$, then theorem is clearly true. So assume $m \geq 1$. First we show between each pair x and y of distinct vertices of $[B(G)]^2$ belonging to $I(G)$ there exist at least $2m$ edge-disjoint paths. Therefore by Theorem 2.5, there exist at least $2m-2$ edge-disjoint paths in $I(G)$. Let x and y correspond to the edges $e_1 = ab$ and $e_2 = cd$ respectively. If e_1 and e_2 have a vertex in common, that is, if for example $d = b$, then the paths (x, b, y) and (x, a, b, c, y) are two edge-disjoint $x - y$ paths, and no edge of these paths belong to $I(G)$. In case e_1 and e_2 have no-vertex in common, $m \geq 1$ implies that there exist at least one $b - d$ path, say $(b = b_0, b_1, b_2, ..., b_n = d)$ in G, where n is a positive integer. Then $x - y$ paths $(x, b, b_1, b_2, ..., b_{n-1}, d, y)$ and $(x, a, b_1, b_2, ..., b_{n-1}, c, y)$ are edge-disjoint. Again no edge of these paths is in $I(G)$. Hence the assertion follows.

Next suppose a set S, $\mid S \mid \leq 2m-1$, of edges disconnect $[B(G)]^2$. Remove S and denote the resulting graph by H. In H all vertices of $I(G)$ must be in one of its component, say H_1. Let H_2 be another component of H. All vertices of H_2 are vertices of G, moreover the number of vertices of H_2 is at least 2. This contradicts the inequality $\mid S \mid \leq 2m - 1$, since in $[B(G)]^2$ there are at least 2 min deg G edges joining vertices of H_1 to vertices of H_2, and by Theorem 2.6, $2m \leq 2$ min deg G. ■

Corollary 3.2 *If G is m-connected, than $[B(G)]^2$ is $2m$-edge-connected.*

Proof. Since G is m-connected, then by Theorem 2.6, $k(G) \leq \lambda(G)$. This implies that G is m-edge-connected.

The equalities $k(K_{m+1}) = \lambda(K_{m+1}) = m$ and min deg $([B(K_{m+1})]^2) = 2m$ shows that the results of Theorem 3.1 and Corollary 3.2 are the best. ■

Theorem 3.3 *If G is m-edge-connected, $m \geq 1$, than $[B(G)]^2$ is $(m+1)$-connected.*

P r o o f. Suppose a set S consisting of s vertices of $[B(G)]^2$, $s \leq m$, disconnects $[B(G)]^2$. Let $S = S_1 \cup S_2$, where S_1 is the set of all elements of S which are vertices of $I(G)$, and $S_2 = S - S_1$. If $| S_1 |< m$, then the removal of S from $I(G)$ results in a connected graph. This and the fact that a vertex of G in $[B(G)]^2$ is adjacent to at least m vertices of $I(G)$ give rise to a contradiction. So $| S_1 |= m$ and $| S_2 |= 0$. But then every vertex of $I(G)$ being adjacent to two vertices of G in $[B(G)]^2$ gives rise to a contradiction again. This completes the proof of the theorem. ■

The results of Theorem 3.3 is best possible, too. Identify two copies of K_{m+1} at one vertex y and denote the resulting graph by G. The vertex y is a cut-vertex of G and $\lambda(G) = m$. The subgraph $I(G)$ of $[B(G)]^2$ has connectivity m. The m vertices which disconnect $I(G)$ together with the vertex y, disconnect $[B(G)]^2$. Hence $k([B(G)]^2) = m + 1$. The graph in Fig. 1 illustrates this for $m = 1$.

Next, we note that a vertex of $I(G)$ in $[B(G)]^2$ is adjacent with at least 2(min deg G - 1) other vertices of $I(G)$.

Theorem 3.4 *If G is m-connected, $m \geq 1$, than $[B(G)]^2$ is $(m+2+[\frac{m-2}{3}])$-connected.*

P r o o f. Since G is m-connected, then by Theorem 2.6, $k(G) \leq \lambda(G)$. This implies that G is m-edge-connected. Now by Theorem 3.3 $[B(G)]^2$ is $(m+1)$- connected. Hence for $m = 1$, the theorem is true. So assume $m \geq 2$. Suppose there exist a set S consisting of $s = m+2+[\frac{m-2}{3}]$ or less vertices of $[B(G)]^2$ whose removal from $[B(G)]^2$ results in a disconnected graph H. Suppose $S_1 \subset S$ consists of those vertices of S belonging to $I(G)$ and $S_2 = S - S_1$.

If $| S_1 |\leq m-1$, then the removal of S_1 from $I(G)$ results in a connected graph. This together with the fact that in $[B(G)]^2$ each vertex of G adjacent to m vertices of $I(G)$ contradicts the fact that H is a disconnected graph. Thus $| S_1 |\geq m \geq 2$. Form this we conclude that

$$(1) \qquad | S_2 |=| S | - | S_1 |\leq s - m = 1 + \left[\frac{m-2}{3}\right] \leq m - 1$$

Since H is disconnected, $| S_2 |\geq 2$. Hence:

$$(2) \qquad 2 \leq | S_2 |\leq m - 1 \, .$$

Therefore, the removal of S_2 from G results in a connected graph.

Now remove S from $[B(G)]^2$ and denote the connected subgraph containing all remaining vertices of G (and possibly some vertices of $I(G)$) by H_1 and let H_2 denote the rest of the resulting graph H. The graph H_2 contains at least one vertex, say x. The first inequality in (2) implies that

(3) $$\mid S_1 \mid \leq m - 1 + \left[\frac{m-2}{3}\right].$$

From (3) and the note preceding Theorem 3.3 we get

(4) $$2m - 2 - m + 1 - \left[\frac{m-2}{3}\right] \geq 1\,.$$

Hence x is adjacent to another vertex y of $I(G)$ in H_2. The vertices x and y correspond to two adjacent edges in G. These two edges are incident with three vertices in G which must belong to S_2. Hence

(5) $$\mid S_1 \mid \leq s - 3 = m - 2 + \left[\frac{m-2}{3}\right].$$

Again, from (5) and the note preceding the Theorem 3.3 , we obtain

(6) $$2m - 2 - m + 2 - \left[\frac{m-2}{3}\right] \geq 2.$$

Therefore, besides y the vertex x adjacent to another vertex z of $I(G)$ in H_2. The vertices x, y and z correspond to three edges X, Y an$Z$ respectively of G. Since the edge X is adjacent to both Y and Z, one of the graphs in Fig. 2 must be subgraph of G.

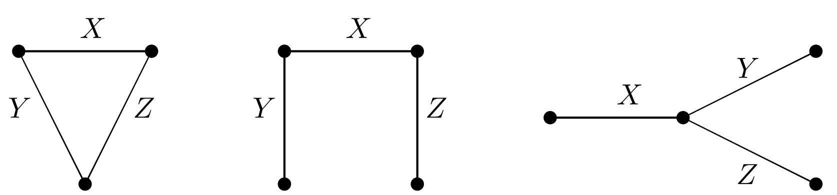

Figure 2

In each case there are at least $3m - 6$ edges in G, different from X, Y and Z which are adjacent to X, Y or Z. Hence, in addition to x, y, and z there are at least $3m-6$ vertices in $I(G)$ which are adjacent to the vertices x, y or z. Therefore we have

$$(7)\qquad 3m-6-(s-3)=2m-4-\left[\frac{m-2}{3}\right]\geq m-2\,.$$

Now (7) implies that at least $m-2$ vertices of $I(G)$ are left which are adjacent to x, y or z in H_2. These vertices correspond to $m-2$ edges of G adjacent to X, Y or Z. These $m-2$ edges together with the edges X, Y and Z are adjacent with at least $[\frac{m-2}{3}]$ vertices of G which must belong to S_2. Hence the set S contains at least $m+3+[\frac{m-2}{3}]$ vertices. Since this number is greater than s, the theorem must hold. ■

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