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Fekete-Szegö Problem for Certain Subclass of Analytic Functions

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Making use of the Carlson-Shaffer linear operator, two new subclasses of analytic functions are defined and a upper bound is obtained for $\left|a_3 - \mu a_2^2\right|$ over these the classes.

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1. Introduction.

Let A denote the family of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$.Further, let S denote the class of functions which are univalent in U.A function $f(z)$belonging to A is said to be strongly starlike of order β and type α in U,and denoted by $\tilde{S}^*_\alpha(\beta)$ if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in U) \tag{1.2}$$

for some α $(0 \leq \alpha < 1)$ and β $(0 < \beta \leq 1)$.If $f(z) \in A$satisfies

$$\left|\arg\left(1 + \frac{zf"(z)}{f'(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (z \in U) \tag{1.3}$$

for some α $(0 \leq \alpha < 1)$ and β $(0 < \beta \leq 1)$,then we say that $f(z)$is strongly convex of order of order β and type α in U,and we denote by $\tilde{C}_{\alpha}(\beta)$ the class of all such functions (you can see also Srivastava and Owa [5]).

Now we define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n,$$

for

$$c \neq 0, -1, -2, ..., a \neq -1; \ z \in U$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(n+\lambda)}{\Gamma(\lambda)} = \left\{ \begin{array}{ll} 1; & n = 0 \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+n-1), & n \in \mathbb{N} = \{1, 2, ...\} \end{array} \right\}.$$

Carlson and Shaffer [4] introduced a linear operator $L(a, c)$,by

$$\begin{aligned} L(a, c)f(z) &= \phi(a, c; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} a_n z^n, \ \ z \in U, \end{aligned}$$

where $*$stands for the Hadamard product or convulation product of two power series $\varphi(z) = \sum\limits_{n=1}^{\infty} \varphi_n z^n$and $\pi(z) = \sum\limits_{n=1}^{\infty} \pi_n z^n$

defined by $\varphi * \pi = \pi * \varphi = \sum\limits_{n=1}^{\infty} \varphi_n \pi_n z^n$.We note that

$$L(a, a)f(z) = f(z), \quad L(2, 1)f(z) = zf'(z),$$

$$L(m+1, 1)f(z) = D^m f(z),$$

where $D^m f(z)$ is the Ruscheweyh derivative of $f(z)$ defined by Ruscheweyh [9] as

$$D^m f(z) = \frac{z}{(1-z)^{m+1}} * f(z), \quad m > -1.$$

Which is equivalently,

$$D^m f(z) = \frac{z}{m!}\frac{d^m}{dz^m}\{z^{m-1}f(z)\}.$$

With the aid of the Carlson and Shaffer linear operator $L(a,c)$,we say that a function $f(z)$belonging to A is to be in the class $P_\alpha(\beta,a,c)$if it satisfies

$$\left|\arg\left(\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (\mathrm{z} \in U) \tag{1.4}$$

for some α $(0 \le \alpha < 1), \beta$ $(0 < \beta \le 1)$ and for all $z \in U$.A function $f(z)$belonging to A is to be in the class $C_\alpha(\beta,a,c)$ if it satisfies

$$\left|\arg\left(1 + \frac{z(L(a,c)f(z))''}{(L(a,c)f(z))'} - \alpha\right)\right| < \frac{\pi}{2}\beta \quad (\mathrm{z} \in U) \tag{1.5}$$

for some α $(0 \le \alpha < 1), \beta$ $(0 < \beta \le 1)$ and for all $z \in U$. Note that

$$P_\alpha(\beta,a,a) \equiv \tilde{S}^*_\alpha(\beta)$$

and

$$C_\alpha(\beta,a,a) \equiv \tilde{C}_\alpha(\beta).$$

For the class S of analytic univalent functions, Fekete-Szegö [6] obtained the maximum value of $\left|a_3 - \mu a_2^2\right|$when μ is real. For various functions of S the upper bound for $\left|a_3 - \mu a_2^2\right|$is investigated by many different authors including [2, 7, 10-14].

In the present paper, we obtain sharp upper bound for $\left|a_3 - \mu a_2^2\right|$when $f$belonging to the class of functions defined as follows.

Definition: Let α $(0 \le \alpha < 1)$, $\beta > 0$and let $f \in A$.Then $\mathfrak{M}_\alpha(\beta,a,c)$ if and only if there exist $g \in P_\alpha(\beta,a,c)$such that

$$\mathrm{Re}\left(\frac{z(L(a,c)f(z))'}{L(a,c)g(z)}\right) > 0 \qquad (z \in U), \tag{1.6}$$

and $f \in K_\alpha(\beta,a,c)$ if and only if there exists $g \in C_\alpha(\beta,a,c)$ and satisfy condition (1.6) with $g(z) = z + b_2z^2 + b_3z^3 + ...$

Note that $\mathfrak{M}_0(\beta,a,a) \equiv \wp(\beta)$ is the class of close-to- convex functions defined in [2] and $\mathfrak{M}_0(1,a,a) \equiv \wp(1)$ is the class of nomalized close-to-convex functions defined by Kaplan [3]. $\mathfrak{M}_\alpha(\beta,a,a) \equiv M(\alpha,\beta)$ is the class $M(\alpha,\beta)$ introduced and studied by Frasin and Darus [8].

In the present paper we derive the generalization of the by Jahangiri [7] and Frasin et al.[8]

2. Main results

To establish our results, we shall require the following Lemma [1].

Lemma 2.1. Let $h \in P$,that is, h be analytic in U and be given by $h(z) = 1 + c_1 z + c_2 z^2 + ...$,and $\operatorname{Re} h(z) > 0$ for $z \in U$, then

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1^2|}{2}. \tag{2.1}$$

Theorem 2.2. $f(z) \in \mathfrak{M}_\alpha(\beta, a, c)$and be given by (1.1). Then for $0 \leq \alpha < 1$, $\beta \geq 1$ and $\mu \geq 1$,we have the sharp inequality

$$|a_3 - \mu a_2^2| \leq \frac{6\beta^2[\mu c^2(a+1) - ac(c+1)] + \alpha\beta^2[ac(c+1)(8-2\alpha) - 3\mu c^2(a+1)]}{3a^2(a+1)(2-\alpha)(1-\alpha)^2}$$

$$+ \frac{(2\beta + 1 - \alpha)[3\mu c^2(a+1) - 2ac(c+1)]}{3a^2(a+1)(1-\alpha)}. \tag{2.2}$$

P r o o f. Let $f(z) \in \mathfrak{M}_\alpha(\beta, a, c)$It follows from (1.6) that

$$z(L(a,c)f(z))' = \{L(a,c)g(z)\}q(z), \tag{2.3}$$

for $z \in U$, with $q \in P$ given by $q(z) = 1 + q_1 z + q_2 z^2 + ...$Equating coefficients, we obtain

$$2a_2 \frac{a}{c} = q_1 + \frac{a}{c} b_2, \qquad 3a_3 \frac{a(a+1)}{c(c+1)} = \frac{a(a+1)}{c(c+1)} b_3 + \frac{a}{c} b_2 q_1 + q_2. \tag{2.4}$$

Also, it follows from (1.4) that

$$z(L(a,c)g(z))'^{\beta}, \tag{2.5}$$

where

$$z \in U, \ p \in P \text{ and } p(z) = 1 + p_1 z + p_2 z^2 + ...$$

Thus equating coefficients, we obtain

$$(1-\alpha)\frac{a}{c} b_2 = \beta p_1, \quad (2-\alpha)\frac{a(a+1)}{c(c+1)} b_3 = \beta\{p_2 + \frac{\beta(3-\alpha) + \alpha - 1}{2(1-\alpha)} p_1^2\}. \tag{2.6}$$

From (2.4) and (2.6), we have

$$a_3 - \mu a_2^2 = \frac{c(c+1)}{3a(a+1)}\left[q_2 - \frac{1}{2}q_1^2\right]$$

$$+\frac{2ac(c+1)-3\mu c^2(a+1)}{12a^2(a+1)}q_1^2 + \frac{\beta c(c+1)}{3a(a+1)(2-\alpha)}\left[p_2 - \frac{1}{2}p_1^2\right]$$
$$+\frac{\beta^2\{6[ac(c+1)-\mu c^2(a+1)]+\alpha[ac(c+1)(2\alpha-8)+3\mu c^2(a+1)]\}}{12a^2(a+1)(2-\alpha)(1-\alpha)^2}p_1^2$$

$$+\frac{\beta[2ac(c+1)-3\mu c^2(a+1)]}{6a^2(a+1)(1-\alpha)}p_1q_1. \tag{2.7}$$

Assume that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\mathrm{Re}(a_3 - \mu a_2^2)$, so from (2.7) and by using Lemma 2.1 and letting

$$p_1 = 2re^{i\theta},\ \ q_1 = 2Re^{i\phi};\ \ 0 \le r \le 1,\ 0 \le R \le 1;\ \ 0 \le \theta < 2\pi; 0 \le \phi < 2\pi,$$

we obtain

$$3\,\mathrm{Re}(a_3 - \mu a_2^2) = \frac{c(c+1)}{a(a+1)}\,\mathrm{Re}\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{2ac(c+1)-3\mu c^2(a+1)}{4a^2(a+1)}\,\mathrm{Re}\,q_1^2$$
$$+\frac{\beta c(c+1)}{a(a+1)(2-\alpha)}\,\mathrm{Re}\left(p_2 - \frac{1}{2}p_1^2\right)$$

$$+\frac{\beta^2\{6[ac(c+1)-\mu c^2(a+1)]+\alpha[ac(c+1)(2\alpha-8)+3\mu c^2(a+1)]\}}{4a^2(a+1)(2-\alpha)(1-\alpha)^2}\,\mathrm{Re}\,p_1^2$$
$$+\frac{\beta[2ac(c+1)-3\mu c^2(a+1)]}{2a^2(1-\alpha)(a+1)}\,\mathrm{Re}\,p_1q_1.$$

$$3\,\mathrm{Re}(a_3 - \mu a_2^2) \le \frac{2c(c+1)}{a(a+1)}(1-R^2) + \frac{2ac(c+1)-3\mu c^2(a+1)}{a^2(a+1)}R^2\cos 2\phi$$
$$+\frac{2\beta c(c+1)}{a(a+1)(2-\alpha)}(1-r^2)$$

$$+\frac{\beta^2\{6[ac(c+1)-\mu c^2(a+1)]+\alpha[ac(c+1)(2\alpha-8)+3\mu c^2(a+1)]\}}{a^2(a+1)(2-\alpha)(1-\alpha)^2}r^2\cos 2\theta$$

$$+\frac{2\beta[2ac(c+1)-3\mu c^2(a+1)]}{a^2(1-\alpha)(a+1)}rR\cos(\theta+\phi).$$

$$3\,\mathrm{Re}(a_3-\mu a_2^2)\leq\frac{3\mu c^2(a+1)-4ac(c+1)}{a^2(a+1)}R^2+\frac{2\beta[3\mu c^2(a+1)-2ac(c+1)]}{a^2(a+1)(1-\alpha)}rR$$

$$+\frac{6\beta^2[\mu c^2(a+1)-ac(c+1)]+\alpha\beta^2[ac(c+1)(8-2\alpha)-3\mu c^2(a+1)]-2\beta ac(c+1)(1-\alpha)^2}{a^2(a+1)(2-\alpha)(1-\alpha)^2}r^2$$

$$\frac{2c(c+1)[\beta-\alpha+2]}{a(a+1)(2-\alpha)}=\psi(r,R). \tag{2.8}$$

Letting α, β and μ fixed and differentiating $\psi(r,R)$ partially when $0\leq\alpha<1$, $\beta\geq 1$ and $\mu\geq 1$,we observe that

$$\psi_{rr}\psi_{RR}-\psi_{rR}^2=4\{4\beta a^2c^2(c+1)^2[4\beta+2+\alpha(2\alpha\beta+2\alpha-4-7\beta)]$$

$$-3\beta\mu ac^3(a+1)(c+1)[6\beta+2+\alpha(2\alpha\beta+2\alpha-4-8\beta)]\}<0 \tag{2.9}$$

Therefore, the maximum of $\psi(r,R)$occurs on the boundaries. Thus the desired inequality follows by observing that

$$\psi(r,R)\leq\psi(1,1)$$

$$=\frac{\beta^2\{6[\mu c^2(a+1)-ac(c+1)]+\alpha[ac(c+1)(8-2\alpha)-3\mu c^2(a+1)]\}}{a^2(a+1)(2-\alpha)(1-\alpha)^2}$$

$$+\frac{(2\beta+1-\alpha)[3\mu c^2(a+1)-2ac(c+1)]}{a^2(a+1)(1-\alpha)}. \tag{2.10}$$

The equality for (2.2) is attained when $p_1=q_1=2i$ and $p_2=q_2=-2$.

Letting $\alpha=0$ and $a=c=1$in Theorem 2.2 we have the result given by Jahangiri [7].

Corollary 2.3. Let $f(z)\in\wp(\beta)$ and be given by (1.1). Then for $\beta\geq 1$ and $\mu\geq 1$,we have the sharp inequality

$$\left|a_3-\mu a_2^2\right|\leq\beta^2(\mu-1)+\frac{(2\beta+1)(3\mu-2)}{3}. \tag{2.11}$$

Letting $a = c = 1$in Theorem 2.2 we have the result given by Frasin and Darus [8].

Corollary 2.4. Let $f(z) \in M(\alpha, \beta)$and be given by (1.1). Then for $0 \leq \alpha < 1, \beta \geq 1$ and $\mu \geq 1$,we have the sharp inequality

$$\left|a_3 - \mu a_2^2\right| \leq \frac{6\beta^2[\mu - 1] + \alpha\beta^2\,[8 - 2\alpha - 3\mu]}{3\,(1-\alpha)^2\,(2-\alpha)} + \frac{(2\beta + 1 - \alpha)\,(3\mu - 2)}{3\,(1-\alpha)}. \tag{2.12}$$

Theorem 2.5. $f(z) \in K_\alpha(\beta, a, c)$and be given by (1.1). Then for $0 \leq \alpha < 1$, $\beta \geq 1$ and $\mu \geq 1$,we have the sharp inequality

$$\left|a_3 - \mu a_2^2\right| \leq$$

$$\frac{6\beta^2[3\mu c^2(a+1) - 4ac(c+1)] + \alpha\beta^2[ac(c+1)(32-8\alpha) - 9\mu c^2(a+1)]}{36a^2(a+1)(2-\alpha)(1-\alpha)^2}$$

$$+\frac{(\beta+1-\alpha)[3\mu c^2(a+1) - 2ac(c+1)]}{3a^2(a+1)(1-\alpha)}. \tag{2.13}$$

P r o o f. Let $f(z) \in K_\alpha(\beta, a, c)$.It follows from (1.5) that

$$z(L(a,c)g(z))\text{''} + (1-\alpha)\{L(a,c)g(z)\}' = \{L(a,c)g(z)\}'^{\beta}, \tag{2.14}$$

where

$$z \in U,\ p \in P,\ \text{and}\ p(z) = 1 + p_1 z + p_2 z^2 + \ldots.$$

Thus equating coefficients, we obtain

$$2(1-\alpha)\frac{a}{c}b_2 = \beta p_1,$$

$$3(2-\alpha)\frac{a(a+1)}{c(c+1)}b_3 = \beta(p_2 + \frac{\beta(3-\alpha)+\alpha-1}{2(1-\alpha)}p_1^2) \tag{2.15}$$

From (2.4) and (2.15) and proceeding as in the proof of Theorem 2.2, we get

$$3\,\mathrm{Re}(a_3 - \mu a_2^2) \leq \frac{3\mu c^2(a+1) - 4ac(c+1)}{a^2(a+1)}R^2 + \frac{c(c+1)[2(\beta-3\alpha)+12]}{3a(a+1)(2-\alpha)} +$$

$$\frac{6\beta^2[3\mu c^2(a+1)-4ac(c+1)]+\alpha\beta^2[ac(c+1)(32-8\alpha)-9\mu c^2(a+1)]-8\beta ac(c+1)(1-\alpha)^2}{12a^2(a+1)(2-\alpha)(1-\alpha)^2}r^2$$

$$+\frac{\beta[3\mu c^2(a+1)-2ac(c+1)]}{a^2(a+1)(1-\alpha)}rR=\psi(r,R). \tag{2.16}$$

Letting α,β and μ fixed and differentiating $\psi(r,R)$ partially when $0\leq\alpha<1,\beta\geq 1$and $\mu\geq 1$we have

$$\begin{aligned}\psi_{rr}\psi_{RR}-\psi_{rR}^2=2\{4\beta a^2c^2(c+1)^2[18\beta+8+\alpha(8\alpha\beta+8\alpha-16-29\beta)]\\ -3\mu\beta ac^3(a+1)(c+1)[24\beta+8+\alpha(8a\beta+8\alpha-16-32\beta)]\}<0\end{aligned} \tag{2.17}$$

Therefore the maximum of $\psi(r,R)$occurs on the boundaries. Thus the desired inequality (2.13) follows by observing that

$$\psi(r,R)\leq\psi(1,1)=$$

$$\frac{6\beta^2[3\mu c^2(a+1)-4ac(c+1)]+\alpha\beta^2[ac(c+1)(32-8\alpha)-9\mu c^2(a+1)]}{12a^2(a+1)(2-\alpha)(1-\alpha)^2}$$

$$+\frac{(\beta+1-\alpha)[3\mu c^2(a+1)-2ac(c+1)]}{a^2(a+1)(1-\alpha)}. \tag{2.18}$$

The equality in (2.13) is attained on choosing $p_1=q_1=2i$ and $p_2=q_2=-2$. This completes the proof of Theorem 2.5. ∎

Letting $\alpha=0$ and $a=c=1$in Theorem 2.5 we have the result given by Jahangiri [7].

Corollary 2.6. Let $f(z)\in K_0(\beta,1,1)$ and be given by (1.1). Then for $\beta\geq 1$ and $\mu\geq 1$,we have the sharp inequality

$$\left|a_3-\mu a_2^2\right|\leq\frac{1}{12}\left[(3\mu-2)(\beta+2)^2-2\beta^2\right]. \tag{2.19}$$

Letting $a=c=1$in Theorem 2.5 we have the result given by Frasin and Darus [8].

Corollary 2.7. Let $f(z)\in K_\alpha(\beta,1,1)$and be given by (1.1). Then for $0\leq\alpha<1,\beta\geq 1$and $\mu\geq 1$,we have the sharp inequality

$$\left|a_3-\mu a_2^2\right| \leq \frac{6\beta^2\left(3\mu-4\right)+\alpha\beta^2\left(32-8\alpha-9\mu\right)}{36\left(1-\alpha\right)^2\left(2-\alpha\right)}+\frac{\left(\beta+1-\alpha\right)\left(3\mu-2\right)}{3\left(1-\alpha\right)}.$$

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