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A Note on Erdös-Diophantine Graphs and Diophantine Carpets

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We give an effective construction for Erdös-Diophantine graphs and characterize the chromatic number of Diophantine carpets.

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1. Introduction

A Diophantine figure, see i.e. [4, 5, 6], is a set of points on the integer grid $\mathbb{Z}^2$ where all mutual Euclidean distances are integers. We also speak of *Diophantine graphs*. The vertices are points in $\mathbb{Z}^2$ (the coordinates) and the edges are labeled with the distance between the two adjacent vertices, which is integral. In this language a Diophantine figure is a complete Diophantine graph. Two Diophantine graphs are equivalent if they only differ by translation or rotation of vertices. Due to a famous theorem of Erdös and Anning [1] there are complete Diophantine graphs which are not contained in larger ones. We call them *Erdös-Diophantine graphs*. We will give a proof of this theorem as we need it for an algorithm later on.

Theorem 1.1. ***(Erdös, Anning 1945 [1])***
Infinitely many points in the plane with pairwise integral distances are collinear.

Proof. Let A, B, and C be three non collinear points and let $k = \max(\overline{AC}, \overline{BC})$. Then there are at most $4(k+1)^2$ points P such that the differences of Euclidean distances $\overline{PA} - \overline{PB}$ and $\overline{PB} - \overline{PC}$ are integers. We see this as follows: Due to the triangle inequality we have $|\overline{PA} - \overline{PB}| \leq \overline{AB} \leq k$. Thus

$|\overline{PA} - \overline{PB}| \in \{0, 1, \ldots, k\}$. So P is on one of $k+1$ hyperbolas. Analog we have that P is situated also on one of $k+1$ hyperbolas through B and C. Because two distinct hyperbolas intersect in at most 4 points, there are at most $4(k+1)^2$ points P. ■

A special class of Diophantine graphs are *Diophantine carpets* [2, 7]. These are planar triangulations of a subset of the integer grid.

2. Problems

The authors of [5] have posed some open problems for Diophantine graphs and Diophantine carpets which we would like to solve in this section.

2.1. Pythagorean triangles. Let us denote by $\chi(l)$ the number of all Pythagorean triangles with hypothenuse $l \in \mathbb{N}$. The question in [5] was to determine the asymptotic of the function $\chi(l)$ when $l \to \infty$.
Due to Jacobi (1828) we have $\chi(l) = d_{1,4}(l) - d_{3,4}(l)$ where $d_{r,n}(l)$ denotes the number of divisors (including 1 and l) of n which are congruent to r modulo n, see i.e. [9]. So $\chi(l) \in O(n^{\varepsilon})$ for $\varepsilon > 0$, see [8] for a deeper analysis of the divisor function.

2.2. Erdös-Diophantine triangles. Are there Erdös-Diophantine triangles and is there an effective algorithm to determine all integer points P which extend a given Diophantine triangle (= complete Diophantine graph of 3 points) to a complete Diophantine graph of 4 points?
For such an effective algorithm we can use Theorem 1. For the given integral points $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ we have the following system of equations for a forth point $P = (x, y)$ being the intersection of two hyperbolas.

$$\sqrt{(x-a_1)^2 + (y-a_2)^2} - \sqrt{(x-c_1)^2 + (y-c_2)^2} = d_1$$
$$\sqrt{(x-b_1)^2 + (y-b_2)^2} - \sqrt{(x-c_1)^2 + (y-c_2)^2} = d_2$$

Due to the proof of theorem 1 we have $-\overline{AC} \le d_1 \le \overline{AC}$ and $-\overline{BC} \le d_2 \le \overline{BC}$ for $d_1, d_2 \in \mathbb{Z}$. Thus we can solve the corresponding $(2\overline{AC}+1)(2\overline{BC}+1)$ equation systems to determine the possible points P.
To answer the first question we loop over all Heronian triples, which are not Pythagorean. (Pythagorean triples can be extended) Heronian triples are triples of edge lengths, which correspond to an triangle of rational area. The restriction in search to the Heronian triples comes from the fact that triangles in the $\mathbb{Z}^2$ lattice are always of rational area as the area is the half of the determinant:

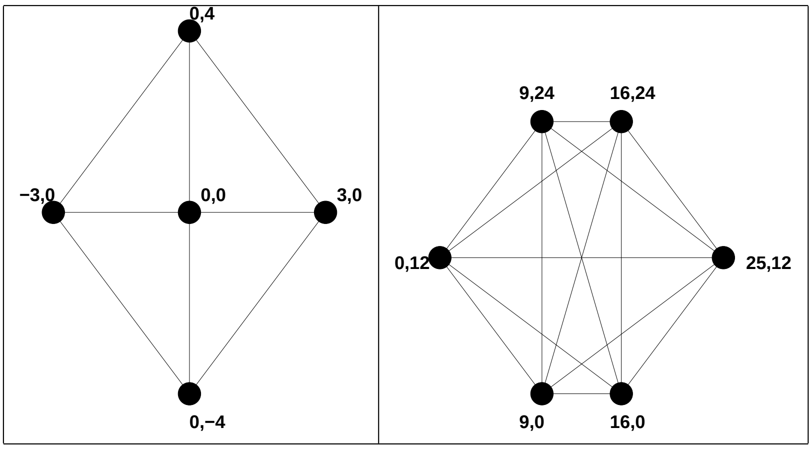
$$\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$

Compute in the next step all possible embeddings of such a triple into $\mathbb{Z}^2$ and let an implementation of the above described algorithm search for possible fourth nodes. If we fail to find a fourth node we found a Erdös-Diophantine triangle. We experimentally noticed that there are very rare, but we found seven examples with edge lengths:

$$\begin{array}{rrr} (2066, & 1803, & 505) \\ (2549, & 2307, & 1492) \\ (3796, & 2787, & 2165) \\ (4083, & 2425, & 1706) \\ (4426, & 2807, & 1745) \\ (4801, & 2593, & 2210) \\ (4920, & 4177, & 985)\,. \end{array}$$

This is a complete list of Erdös-Diophantine triangles having an edge of length ≤ 5000.

2.3. Further Erdös-Diophantine Graphs. In [4] the following two Diophantine figures were depicted, which the author believed to be Erdös-Diophantine graphs.



With the above algorithm we checked their conjectures, and proved them.

2.4. Erdös-Diophantine Tetrahedrons. In [5] Pythagorean-Diophantine pyramids were defined as sets of four points with integer coordinates, integral edge lengths and three faces being Pythagorean triangles. They asked for Erdös Pythagorean-Diophantine pyramids. We slightly generalize their definition and search for tetrahedrons with coordinates in $\mathbb{Z}^3$ and integral edge lengths, integral face areas and integral volume. These objects are called *Diophantine tetrahedrons*. In the case that there is no further point in $\mathbb{Z}^3$ having integral distance to the vertices of the tetrahedron, we call it *Erdös Diophantine tetrahedron*.

For four points $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$, and $D = (d_1, d_2, d_3)$ of a Diophantine tetrahedron we have the following system of equations for a fifth point $P = (x, y, z)$ being the intersection of three hyperboloids.

$$\sqrt{(x-a_1)^2+(y-a_2)^2+(z-a_3)^2}-\sqrt{(x-d_1)^2+(y-d_2)^2+(z-d_3)^2}=e_1$$
$$\sqrt{(x-b_1)^2+(y-b_2)^2+(z-b_3)^2}-\sqrt{(x-d_1)^2+(y-d_2)^2+(z-d_3)^2}=e_2$$
$$\sqrt{(x-c_1)^2+(y-c_2)^2+(z-c_3)^2}-\sqrt{(x-d_1)^2+(y-d_2)^2+(z-d_3)^2}=e_3$$

Using a variant of the above algorithm we did an extensive search and found several solutions. We give the coordinates of B, C, D where the first point A is always the origin. We found the following Erdös-Diophantine Tetrahedrons:

$$\begin{pmatrix} 396 & 132 & 99 \\ 288 & -84 & 0 \\ 176 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 432 & 144 & 108 \\ 336 & -48 & 20 \\ 297 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 528 & 396 & 121 \\ 468 & 204 & -423 \\ 144 & 108 & -135 \end{pmatrix},$$

$$\begin{pmatrix} 540 & 180 & 135 \\ 336 & 252 & 0 \\ 400 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 624 & 468 & 0 \\ 648 & 360 & -189 \\ 660 & 264 & -77 \end{pmatrix}, \begin{pmatrix} 672 & 104 & 0 \\ 672 & 0 & 0 \\ 600 & 0 & 135 \end{pmatrix},$$

$$\begin{pmatrix} 672 & 104 & 0 \\ 672 & -104 & 0 \\ 600 & 0 & 135 \end{pmatrix}, \begin{pmatrix} 672 & 153 & 104 \\ 672 & 0 & 104 \\ 672 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 672 & 153 & 104 \\ 672 & -153 & 104 \\ 672 & 0 & 0 \end{pmatrix}.$$

2.5. Chromatic number of Diophantine carpets

We now examine the coloring problem for Diophantine carpets. Clearly the chromatic number is 1 iff the carpet consists of union of non-connected triangles. Given a Diophantine carpet $\mathcal{C}$ we define a graph $\mathcal{C}^*$ by replacing the triangles by nodes which are adjacent iff the corresponding triangles share a common side. This is the dual graph of the planar graph without a node for the outer face. As C is a triangulation, each node in C^* has maximal degree 3. A graph G is bipartite iff it contains no odd cycle [3]. For the remaining cases the chromatic number is 3. As the chromatic number is according to the theorem of Brooks[3] bound by the maximal degree.

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