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ℓ-Fuzzy Integrals of Multifunctions on Lattice

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The aim of the present paper is to introduce a new kind of integral called ℓ-fuzzy integral for multi-functions on lattice. Various kind of properties are shown for ℓ-fuzzy integral of multi-functions.

Key Words: Interval topology; order-complete; ℓ-fuzzy measure; multi-function; ℓ-fuzzy integral

1. Introduction

Sugeno [5] had introduced the concept of Fuzzy measures and Fuzzy integrals in 1974; subsequently other authors [4, 6-8] contributed a lot to the theory. However, they dealt with integrands which were all point-functions (point-valued).

Aumann [1] introduced integrals for multi-function with respect to Lebesgue measure in 1965. Using the approaches of Aumann *et al.*, Zhanand Wang [8] extended fuzzy integrals of Sugeno to integrals involving multi-functions and discussed many properties thereof.

In this paper, we introduce a kind of integrals, called ℓ-Fuzzy integrals of multifunction with respect to ℓ-fuzzy measure. These integrals are lattice-valued. We generalized almost all results of [4] in the weaker setting of a complete lattice with universal bounds.

2. Preliminaries

Since we proposed to discuss fuzzy integrals in the backdrop of lattice, it is quite in order to recall some definitions and results from the theory of lattice. a *lattice* L is a partially ordered set in which every pair of elements x and y have a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$. It is called an *order-complete* lattice if every set $M \subset L$, has a least upper bound and a greatest lower bound in L. If there exist two elements $l,\ u \in L$ such that $l \leq x \leq u$ for every $x \in L$, then l and u are called universal bounds; they are usually denoted respectively by 0 and 1. We assume that if $S \subset L$ and sup *sup* $\dot{S} = s$ exists, there exists a sequence $\{s_n\} \subset S$ such that $0 - \lim_n s_n = s$ (definition follows). The set R is a lattice of this type.

Let $\{x_n\} \subset L$ be a sequence; then we define

$$\lim_n \ \sup \ x_n = \bigwedge_{n=1}^{\infty} (\bigvee_{m=n}^{\infty} x_m)$$

and

$$\lim_n \ \inf \ x_n = \bigvee_{n=1}^{\infty} (\bigwedge_{m=n}^{\infty} x_m)\,,$$

provided sup(‘$\vee$′) and inf(‘$\wedge$′) exist.

Clearly $\lim_n \ \inf \ x_n \leq \lim_n \ \sup \ x_n$. When equality occurs we say that $\{x_n\}$ order-converges and write

$$0 - \lim x_n = \lim_n \ \inf \ x_n = \lim_n \ \sup \ x_n = x\,.$$

Then x is called the *order-limit* of $\{x_n\}$. If $x = 0 - \lim_n x_n$, then there exists sequences $\{u_n\}$ and $\{v_n\}$ in L such that

$$u_n {\uparrow}_{n=1}^{\infty} x \quad \text{ and } \quad v_n {\downarrow}_{n=1}^{\infty} x\,,$$

and $u_n \leq x_n \leq v_n$ for every n and conversely; $u_n {\uparrow}_{n=1}^{\infty} x$ means $u_n \leq u_{n+1}$ for every n and $\bigvee_{n=1}^{\infty} u_n = x$; and $v_n {\downarrow}_{n=1}^{\infty} x$ means the dual statement,([2], Chapter 8, §9, p. 244).

Among intrinsic topologies of an arbitrary lattice L (topologies born out of order-structure), mention must be made of order-topology and interval topology. In order-topology a subset $S \subset L$ is closed if and only if

$$\Big(\{x_n\} \subset S,\ 0 - \lim_n = x\Big) \ \Rightarrow \ x \in S\,.$$

On the other hand, in a lattice with universal bounds 0 and 1, interval topology is defined by taking closed intervals $[a, b],\ a,\ b \in L$ as sub-basis of closed sets.

In this connection we mention a theorem.

Theorem A. ([2], Th. 21, P. 251): *Every subset of a bidirected set which is closed in the interval topology is also closed in the order-topology.*

Obviously this results also holds in a lattice.

We mention another theorem due to Frink.

Theorem B. ([2], Th. 20, P. 250): *A lattice is compact in its interval topology if and only if it is order-complete.*

3. ℓ-Fuzzy measure, measurable multi-functions

Let L be an order-complete lattice with universal bounds throughout the paper, if not stated otherwise.

We consider interval topology on L.

Definition 3.1. Let Ω be a non-empty set, **A** be a σ-algebra of subsets of Ω and $(\Omega,\mathbf{A})$ be a measurable space. Let $\mu : \mathbf{A} \to L$ be a function satisfying the following properties:

(i) $\mu(\phi) = 0$;

(ii) $\mu(A) \le \mu(B)$ whenever $A,\ B \in \mathbf{A}$ and $A \subset B$;

(iii) If $\{A_n\} \subset \mathbf{A},\ A_n \subset A_{n+1},\ n = 1, 2, ...,$ then

$$\mu(\cup_{n=1}^{\infty} A_n) = 0 - \lim_n \mu(A_n)\,;$$

(iv) If $\{A_n\} \subset \mathbf{A},\ A_n \supset A_{n+1},\ n = 1, 2, ...,$ then

$$\mu(\cap_{n=1}^{\infty} A_n) = 0 - \lim_n \mu(A_n)\,.$$

We call μ lattice-fuzzy measure (or, in short, ℓ fuzzy measure).

Definition 3.2. A function $F : \Omega \to 2^L - \{\phi\}$ is called a
(i) *multifunction*; and a function $\sigma : \Omega \to L$ is called a
(ii) *selection* of F if for every $t \in \Omega,\ \sigma(t) \in F(t)$.

Definition 3.3. F is called a *measurable function* if for every closed set U (or equivalently for every open set U)of L,

$$F_U = \Big\{ t \in \Omega \mid F(t) \cap U \neq \phi \Big\} \in \mathbf{A}\,.$$

We assume that every measurable multifunction $F(\omega)$ admits a measurable selection function $\sigma(w)$ such that

$$\sigma(\omega) \in F(\omega)\,.$$

Let $S(F)$ denote the class of all measurable selections of F and let

$$g(w) = \sup_{\sigma(\omega) \in S(F)} \sigma(\omega) \, .$$

4. Integrals of multifunctions and their properties

Definition 4.1. For, $A \in \mathbf{A}$ and a measurable multifunction

$$F \; : \; \Omega \to 2^L - \{\phi\} \, ,$$

we define

$$\int_A F d\mu = \bigvee_{\alpha \in L} \Big[\alpha \wedge \mu(A \cap F_\alpha) \Big]$$

where $F_\alpha = \Big\{ \omega \in \Omega \mid F(\omega) \cap \Big[\alpha, \; 1 \Big] \neq \phi \Big\}$.

We call $\displaystyle\int_A F d\mu$ – the ℓ-fuzzy integral of F on A and instead of $\displaystyle\int_\Omega F d\mu$, we will write $\int F d\mu$.

Definition 4.2. For $A \in \mathbf{A}$ and a measurable single valued function $F : \Omega \to L$, we define

$$\int_A f d\mu = \bigvee_{\alpha \in L} \Big[\alpha \wedge \mu(A \cap f_\alpha) \Big] \, ,$$

where $f_\alpha = \Big\{ \omega \in \Omega \mid f(\omega) \geq \alpha \Big\}$.

We call $\int_A f d\mu$ - the ℓ-fuzzy integral of f on A.

Lemma 4.1. *Let L be an order-complete lattice with universal bounds. Then*

(i) $\displaystyle\int_\Omega F d\mu = \bigvee_{\alpha \in L} \Big[\alpha \wedge \mu(F_\alpha) \Big]$*;*

(ii) $\displaystyle\int_A F d\mu = 0$*, if* $\mu(A) = 0$*;*

(iii) If $A \subset B$, then $\int_A F d\mu \leq \int_B F d\mu$;

(iv) Let F and H be two multifunctions with closed-values such that $F(\omega) \subset H(\omega)$ for every $\omega \in \Omega$, then $\displaystyle\int F d\mu \leq \int H d\mu$*;*

(v) If f and f_n be point-functions for every n and

$$f_n \uparrow_{n=1}^{\infty} f \ (f_n \downarrow_{n=1}^{\infty} f), \ \textit{then} \int f_n d\mu \uparrow_{n=1}^{\infty} \int f d\mu \ (\int f_n d\mu \downarrow_{n=1}^{\infty} \int f d\mu);$$

(vi) If $0 - \lim_n f_n = f$, then $\int f\, d\mu = 0 - \lim_n \int f_n d\mu$.

P r o o f. (i) – (iv) follow immediately from the definition (4.1).

(v) If f and f_n be point-functions for every n and

$$f_n \uparrow_{n=1}^{\infty} f \ (f_n \downarrow_{n=1}^{\infty} f)/,, \ \text{then} \int f_n d\mu \uparrow_{n=1}^{\infty} \int f d\mu \ (\int f_n d\mu \downarrow_{n=1}^{\infty} \int f d\mu);$$

P r o o f. Since $f_n \uparrow_{n=1}^{\infty} f$,

$$\int f_n \ d\mu \leq \int f_{n+1} \ d\mu \tag{1}$$

$$\text{and} \int f_n \ d\mu \leq \int f \ d\mu \tag{2}$$

for every n. Moreover, if g be a point-function such that $f_n \leq g$ for every n, then

$$f_n \leq f \leq g \text{ holds for every } n$$

Accordingly,

$$\begin{cases} \int f_n \ d\mu \leq \int g \ d\mu \ , n = 1, \ 2, \ 3, \ldots \ \text{and} \\ \int f \ d\mu \leq \int g \ d\mu \end{cases} \tag{3}$$

From (2) and (3) it follows that

$$\int f_n d\mu \uparrow_{n=1}^{\infty} \int f d\mu \,.$$

Identically we can show that if $f_n \downarrow_{n=1}^{\infty} f$, then $\int f_n d\mu \downarrow_{n=1}^{\infty} \int f d\mu$.

(vi) If $0 - \lim_n f_n = f$, then $\int f d\mu = 0 - \lim_n \int f_n d\mu$.

P r o o f. There exist sequences $\{h_n\}$ such that

$$h_n \uparrow_{n=1}^{\infty} \ \text{ and } \ g_n \downarrow_{n=1}^{\infty}$$

and $h_n \leq f_n \leq g_n$ for every n, where $h_n = \bigwedge_{i=n}^{\infty} f_i$ and $g_n = \bigvee_{i=n}^{\infty} f_i$.

From (v) above,

$$\int h_n \ d\mu \uparrow_{n=1}^{\infty} \int f \ d\mu \text{ and } \int g_n \ d\mu \downarrow_{n=1}^{\infty} \int f \ d\mu \,.$$

However, $\int h_n \; d\mu \leq \int f_n \; d\mu \leq \int f_n \; d\mu$.

Therefore, $0 - \lim_n \int f_n d\mu = \int f d\mu$.

Theorem 4.1. *For measurable multifunction F,*

$$\int_A F \; d\mu = \int_\Omega \chi_A.F \; d\mu \,, \qquad A \in \mathbf{A}$$

where

$$(\chi_A.F)(\omega) = \begin{cases} F(\omega), \; \textit{if} & \omega \in \; A \\ \{0\}, \; \textit{if} & \omega \notin \; A \end{cases}$$

P r o o f.

$$\begin{aligned} \int_A F \; d\mu &= \bigvee_{\alpha \in \; L} \Big[\alpha \wedge \mu(A \cap F_\alpha)\Big]; \\ &= \bigvee_{\alpha \in \; L - \{0\}} \Big[\alpha \wedge \mu(A \cap F_\alpha)\Big] \vee \Big[0 \wedge \mu(A \cap F_0)\Big] \\ &= \bigvee_{\alpha \in \; L - \{0\}} \Big[\alpha \wedge \mu((\chi_A.F)_\alpha)\Big] \vee \Big[0 \wedge \mu((\chi_A.F)_0)\Big] \\ &= \bigvee_{\alpha \in \; L} \Big[\alpha \wedge \mu((\chi_A.F)_\alpha)\Big] \\ &= \int_\Omega \chi_A.F \; d\mu. \end{aligned}$$

This completes the proof. ■

5. Properties of integrals

Theorem 5.1. *Let F be a measurable multifunction with closed values; then*

(i) $\int F \; d\mu = \beta \wedge \mu(F_\beta)$, for some $\beta \in L$,

and

(ii) $g(\omega) \geq \beta \wedge \mu(F_\beta), \quad \omega \in \Omega$.

P r o o f. (i)We have $\int F \; d\mu = \bigvee_{\alpha \in L} \Big[\alpha \wedge \mu(F_\alpha)\Big]$, where

$$F_\alpha = \Big\{\omega \in \Omega : F(\omega) \cap \Big[\alpha, 1\Big] \neq \phi\Big\}\,.$$

Let $\int F \; d\mu = t$. Then there exists$\{\alpha_n\} \subset L$ such that

$$0 - \lim_n \Big\{\alpha_n \wedge \mu(F_\alpha)\Big\} = t\,.$$

Since L is compact [vide theorem B], we can choose a monotone subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that $0-\lim_k \alpha_{n_k}=\beta$, for some $\beta\in L$.

Suppose $\alpha_{n_k}\uparrow_{k=1}^{\infty}\beta$, so that $\alpha_{n_k}\le\alpha_{n_{k+1}}\le\beta$ for every k.

This ensures that $F_\beta\subset F_{\alpha_{n_{k+1}}}\subset F_{\alpha_{n_k}}$ for every k.

Accordingly, $F_\beta=\cap_{k=1}^{\infty}F_{\alpha_{n_k}}$, and F_β is a closed set and $F_{\alpha_{n_k}}\downarrow_{k=1}^{\infty}F_\beta$.

By definition 3.1 (iii),

$$0-\lim_k \mu(F_{\alpha_{n_k}})=\mu(F_\beta)=\mu(\cap_{k=1}^{\infty}F_{\alpha_{n_k}})\,.$$

On the other hand, if $\alpha_{n_k}\downarrow_{k=1}^{\infty}\beta$, then $F_{\alpha_{n_k}}\uparrow F_\beta$ and $F_{\alpha_{n_k}}\subset F_\beta$, $\forall k$; and $F_\beta=\cup_{k=1}^{\infty}F_{\alpha_{n_k}}$.

Hence

$$0-\lim_k \mu(F_{\alpha_{n_k}})=\mu(\cup_{k=1}^{\infty}F_{\alpha_{n_k}})\le\mu(F_\beta)\,.$$

Therefore,

$$0-\lim_k \mu(F_{\alpha_{n_k}}))\le\mu(F_\beta)\,.$$

So, $t=0-\lim_k\Big[\alpha_{n_k}\wedge\mu(F_{\alpha_{n_k}})\Big]\le 0-\lim_k\Big[\alpha_{n_k}\wedge\mu(F_\beta)\Big]$ (by monotonicity of μ)

$$\begin{aligned}&= (0-\lim\alpha_{n_k})\wedge\mu(F_\beta)\\ &= \beta\wedge\mu(F_\beta)\le\bigvee_{\alpha\in L}\Big[\alpha\wedge\mu(F_\alpha)\Big]=t.\end{aligned}$$

Thus $t=\beta\wedge\mu(F_\beta)$ for some $\beta\wedge\mu(F_\beta)$ for some $\beta\in L$. This proves (i).

(ii) In view of (i) above,

$$\int F\ d\mu=\beta\wedge\mu(F_\beta),\ \text{for some}\beta\in L\,.$$

L is assumed to bear the interval topology.

Let $F(\omega)$ be a closed set, $\omega\in\Omega$ with respect to interval topology so that,

$$F(\omega)=\cap_{\gamma\in\Gamma}C_\gamma^\omega\,, \tag{1}$$

where every C_γ^ω is a finite union of closed intervals of the form $\Big[a,\ b\Big]$, $a,\ b\in L$, corresponding to ω.

Let $\sigma(\omega)\in F(\omega)$ be a selection of F. Then

$$\sigma(\omega)\in F(\omega)\subset C_\gamma^\omega \tag{2}$$

for every $\gamma\in\Gamma$.

However, for every γ, $C^{\omega}_{\gamma} = \cup_{i=1}^{p(\gamma)}\left[a^{\omega}_{n_i^{\gamma}},\ b^{\omega}_{n_i^{\gamma}}\right]$, where $p(\gamma)$ is a positive integer depending upon γ, $a^{\omega}_{n_i^{\gamma}},\ b^{\omega}_{n_i^{\gamma}} \in L,\ a^{\omega}_{n_i^{\gamma}} \leq b^{\omega}_{n_i^{\gamma}},\ n_i^{\gamma}$ are positive integers for every γ and i.

From (2), $\sigma(\omega) \geq a^{\omega}_{n_i^{\gamma}}$, for some i and γ.

As γ varies over Γ, $\sigma(\omega) \geq a^{\omega}_{n_i^{\gamma}} \geq \beta \geq \beta \wedge \mu(F_\beta)$.

Therefore, $g(\omega) = (Sup_{\sigma \in S(F)}\sigma)(\omega) \geq \sigma(\omega) \geq \beta \wedge \mu(F_\beta)$.

Hence, $g(\omega) \geq \beta \wedge \mu(F_\beta),\ \omega \in \Omega$.

This proves (ii). ■

Theorem 5.2. *Let L be lattice and F be as in the preceding theorem. Then*

$$\int F\ d\mu = \int g\ d\mu\,.$$

Proof. We shall firstly show that $\int g\ d\mu \leq \int F\ d\mu$.

Let $A_\alpha = \left\{\omega \in L | g(\omega) \geq \alpha),\ \alpha \in L\right\}$. For $\omega \in A_\alpha$, we can choose a sequence $\left\{\sigma_n\right\}$ in $S(F)$ such that $0 - \lim_n \sigma_n(\omega) = g(\omega)$.

However, $\sigma_n(\omega) \in F(\omega)$, since $\sigma_n(\omega)$ is a selection and $F(\omega)$ is closed.

So, $g(\omega) \in F(\omega)$; however,$g(\omega) \geq \alpha$, and so, $F(\omega) \cap \left[\alpha,\ 1\right] \neq \phi$ i.e., $\omega \in F_\alpha$.

Hence $A_\alpha \subset F_\alpha$, for each $\alpha \in L$.

Therefore, $\int g\ d\mu \leq \int F\ d\mu$, by Lemma 1.4.1 (iv).

Next we shall show the reverse inequality.

By Theorem 5.1 (i), there exists $\beta \in L$ such that

$$\int F\ d\mu = \beta \wedge \mu(F_\beta).$$

We have

$$\left\{\omega | g(\omega) \geq \beta\right\} \supset F_\beta \tag{3}$$

Therefore,

$$\begin{aligned}\int g\ d\mu &= \bigvee_{\alpha \in L}\left[\alpha \wedge \mu(\left\{\omega | g(\omega) \geq \alpha\right\})\right] \\ &\geq \beta \wedge \mu(\left\{\omega | g(\omega) \geq \beta\right\}) \\ &\geq \beta \wedge \mu(F_\beta) = \int F\ d\mu, \quad \text{by (3).}\end{aligned}$$

Hence $\int F\ d\mu = \int g\ d\mu.$

The proof of theorem is complete. ■

Theorem 5.3. *Let F be as in the preceding theorem; then*

$$\int F\ d\mu = t,\ \ t > 0$$

if and only if

(i) $\beta \wedge \mu(F_\beta) \le t,\ \ \beta \in L$ and

(ii) there exists $\beta_0 \in L$ such that $\beta_o \wedge \mu(F_{\beta_o}) = t$.

Proof. Let $\int F\ d\mu = t$. We have, $\displaystyle\int F\ d\mu = \bigvee_{\alpha\in L}\Big[\alpha \wedge \mu(F_\alpha)\Big] = t,$ so that $\beta \wedge \mu(F_\beta) \le t$ for all $\beta \in L$.

On the other hand, by theorem 1.5.1 (i), there exists $\beta_o \in L$ such that $\int F\ d\mu = \beta_o \wedge \mu(F_{\beta_o})$.

Therefore, $\beta_o \wedge \mu(F_{\beta_o}) = t$. Conversely, if (i) holds, then $\int F\ d\mu \le t$. In addition,

$$t = \beta_o \wedge \mu(F_{\beta_o}) \le \bigvee_{\alpha\in L}\Big[\alpha \wedge \mu(F_\alpha)\Big] = \int F\ d\mu\,.$$

Hence, $\displaystyle\int F\ d\mu = t$. This proves the theorem. ■

Theorem 5.4. *If L be a distributive lattice, then for $c \in L$*

$$\int (c \vee F)\ d\mu = \int c\ d\mu \vee \int F\ d\mu\,,$$

where

$$(c \vee F)(\omega) = \Big\{c \vee f(\omega) | f(\omega) \in F(\omega)\Big\}$$

for every $\omega \in \Omega$.

Proof.

$$\int (c \vee F)\ d\mu = \bigvee_{\alpha\in L}\Big[\alpha \wedge \mu((c \vee F)_\alpha)\Big]$$

However,
$$\begin{aligned}(c \vee F)_\alpha &= \Big\{\omega \in \Omega | (c \vee F)(\omega) \ge \alpha\Big\}\\ &= \Big\{\omega \in \Omega | c \vee f(\omega)(\omega) \ge \alpha,\ \ f(\omega) \in F(\omega)\Big\}\\ &\supset \Big\{\omega \in \Omega | c \ge \alpha\Big\} \cup \Big\{\omega \in \Omega | f(\omega)(\omega) \ge \alpha,\ \ f(\omega) \in F(\omega)\Big\}\end{aligned}$$

$\Rightarrow (c \vee F)_\alpha \supset \Big\{\omega \in \Omega | c \ge \alpha\Big\}$

and $(c \vee F)_\alpha \supset \Big\{\omega \in \Omega | f(\omega)(\omega) \ge \alpha,\ \ f(\omega) \in F(\omega)\Big\},$

and so, $\mu((c \vee F)_\alpha) \geq \mu(\Big\{\omega \in \Omega | c \geq \alpha\Big\})$, (monotonicity of μ)
$\Rightarrow \bigvee_{\alpha \in L}\Big[\alpha \wedge \mu((c \vee F)_\alpha)\Big] \geq \bigvee_{\alpha \in L}\Big[\alpha \wedge \mu(\Big\{\omega \in \Omega | c \geq \alpha\Big\})\Big]$
$\Rightarrow \int (c \vee F)\ d\mu \geq \int c\ d\mu$.

Similarly, $\int (c \vee F)\ d\mu \geq \int F\ d\mu$.

Therefore,

$$\int (c \vee F)\ d\mu \geq \int c\ d\mu \vee \int F\ d\mu\,. \tag{4}$$

On the other hand,

$$\begin{aligned}
\int c\ d\mu \vee \int F\ d\mu &\geq \Big[\alpha \wedge \mu(\Omega)\Big] \vee \Big[\alpha \wedge \mu(F_\alpha)\Big], \\
&= \Big[(\alpha \wedge \mu(\Omega)) \vee \alpha\Big] \wedge \Big[(\alpha \wedge \mu(\Omega)) \vee \mu(F_\alpha)\Big] \text{ by distributivity} \\
&= \alpha \wedge \Big[(\alpha \wedge \mu(\Omega)) \vee \mu(F_\alpha)\Big], \quad (\text{since } \alpha = (\alpha \wedge \mu(\Omega) \vee \alpha)) \\
&= \Big[(\alpha \wedge \mu(\Omega))\Big] \vee \Big[\alpha \wedge \mu(F_\alpha)\Big], \quad \text{Distributivity} \\
&= \alpha \wedge \Big[\mu(\Omega) \vee \mu(F_\alpha)\Big], \quad \text{Distributivity} \\
&= \alpha \wedge \mu(\Omega) \\
&\geq \alpha \wedge \mu\Big\{\omega \in \Omega \mid (c \vee f(\omega)) \geq \alpha,\ f(\omega) \in F(\omega)\Big\}.
\end{aligned}$$

Therefore,

$$\int c\ d\mu \vee \int F\ d\mu \geq \bigvee_{\alpha \in L}\Big[\alpha \wedge \mu\Big\{\omega \in \Omega \mid (c \vee f(\omega)) \geq \alpha,\ f(\omega) \in F(\omega)\Big\}\Big]$$

$$= \int (c \vee F)\ d\mu\,. \tag{5}$$

From (4) and (5),

$$\int (c \vee F)\ d\mu = \int c\ d\mu \vee \int F\ d\mu\,.$$

This completes the proof. ■

6. Convergence Theorems

For a sequence of multi-functions $\{F_n\}$, we define $\lim_n \sup F_n$ and $\lim_n \inf F_n$ pointwise i.e.,

$$(\lim_n \sup F_n)(\omega) = (\lim_n \sup)\ F_n, \quad \omega \in \Omega$$

and

$$(\lim_n \ \inf \ F_n)(\omega) = (\lim_n \ \inf) \ F_n, \ \omega \in \Omega \, .$$

where

$$\lim_n \ sup \ F_n \ (\omega) = \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} \ F_n(\omega)$$

and

$$\lim_n \ inf \ F_n \ (\omega) = \cup_{m=1}^{\infty} (\cap_{n=m}^{\infty} \ F_n(\omega)) \, .$$

It is easy to see that

$$\lim_n \ \sup \ F_n = \Big\{ \alpha \in L \,|\, \alpha = 0 - \lim_k \alpha_{n_k}, \ \alpha_{n_k} \in F_{n_k}, \ \{n_k\} \subset \{n_k\} \text{ is a subsequence} \Big\}$$

and

$$\lim_n \ \inf \ F_n = \Big\{ \alpha \in L \mid \alpha = 0 - \lim_n \ \alpha_n, \ \alpha_n \in F_n \Big\} \, .$$

Theorem 6.1. *Let $\Big\{F_n\Big\}$ be a sequence of closed valued multi-functions from Ω to $2^L - \Big\{\phi\Big\}$ with $\lim_n \ sup \ F_n$ and $\lim_n \ inf \ F_n$ as closed set (with respect to the order topology). Then the following hold:*

(i) $$\lim_n \ \sup \ \int F_n \ d\mu \leq \int \lim_n \ \sup \ F_n \ d\mu$$

and

(ii) $$\int \lim_n \ \inf \ F_n \ d\mu \leq \lim_n \ \inf \ \int F_n \ d\mu \, .$$

P r o o f. (i) Let $t_n = \int F_n \ d\mu \in L$ and $t = \lim_n \ \sup \ \int F_n \ d\mu = \lim_n \ \sup \ t_n$. There exists, therefore, a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ $[t_{n_k} \downarrow \ t]$ such that

$$t = 0 - \lim_k t_{n_k} \, .$$

By theorem 5.2,

$$\int f_{n_k} \ d\mu = \int F_{n_k} \ d\mu = t_{n_k} \, , \tag{1}$$

where $f_{n_k} = \sup\Big\{\sigma_{n_k} \in S(F_{n_k})\Big\} \in S(F_{n_k})$.

However, $\{f_{n_k}\} \subset L^{\Omega}$ and L^{Ω} is compact with respect to the interval topology of L, since L is complete. So, $\{f_{n_k}\}$ possesses a convergent subsequences, say, $\{f_{n_{k_m}}\}, \ m = 1, \ 2, \ 3, ...$

$$n_{k_1} \ < n_{k_2} \ < \cdots < n_{k_m} \ < \cdots$$

The same is, therefore, a subsequence of $\{f_n\}$ and the corresponding sequence $\{F_{n_{k_m}}\}$ is a subsequence of $\{F_{n_k}\}$. Now,

$$\int f_{n_{k_m}}\ d\mu = \int F_{n_{k_m}}\ d\mu = t_{n_{k_m}} \quad \text{for all} \quad m\,, \tag{2}$$

and $\{t_{n_{k_m}}\}$ is a subsequence of $\{t_{n_k}\}$.

Accordingly,

$$\begin{aligned} t = 0 - \lim_m t_{n_{k_m}} &= 0 - \lim \int F_{n_{k_m}}\ d\mu \\ &= 0 - \lim \int f_{n_{k_m}}\ d\mu, \text{ by (2)} \\ &= \int \left(0 - \lim_m f_{n_{k_m}}\right)\ d\mu, \text{ by [Lemma 4.1.(vi)]} \\ &\leq \int 0 - \lim_n\ \sup\ F_n\ d\mu, \text{ by [Lemma 4.1.(iv)]} \end{aligned}$$

Therefore, $\lim_n\ \sup \int F_n\ d\mu \leq \int 0 - \lim_n\ \sup\ F_n\ d\mu$.

(ii) Let

$$E_n = \cap_{i=1}^{\infty};\ n = 1,\ 2,\ 3, \ldots \tag{3}$$

Then E_n is closed and

$$E_n \uparrow_{n=1}^{\infty}\ \cup_{n=1}^{\infty}(\cap_{i=n}^{\infty} F_i) = \lim_n inf\ F_n\,.$$

Which is a closed set by supposition.

By theorem 5.2,

$$\int E_n\ d\mu = \int g_n\ d\mu \tag{4}$$

and

$$\int (\lim\ inf\ F_n)\ d\mu = \int h\ d\mu \tag{5}$$

where

$$\begin{aligned} g_n &= \sup\Big\{f_n \mid f_n \in S(E_n)\Big\} \\ h &= \sup\Big\{f_n \mid f_n \in S(\lim_n inf\ F_n)\Big\}\,. \end{aligned}$$

However, $\Big\{E_n\Big\}$ monotonically increases to $\lim_n\ \inf\ F_n$ and as such $g_n \uparrow_{n=1}^{\infty}\ h$. Thus we obtain, in view of Lemma 4.1.(v)

$$\begin{aligned} \int h\ d\mu &= \lim_n \int g_n\ d\mu = \lim_n inf \int g_n\ d\mu \\ &= \int (\lim_n inf F_n)\ d\mu, \qquad \text{by (5)}, \end{aligned} \tag{6}$$

it follows from (iv) of the lemma 4.1, that

$$\int E_i \; d\mu \leq \int F_i \; d\mu, \; i \geq n$$

and so

$$\inf_{i \geq n} \int E_i \; d\mu \leq \inf_{i \geq n} \int F_i \; d\mu, \; i \geq n\,.$$

This gives

$$\begin{aligned}
&\sup_{n \geq 1} (\inf_{i \geq n} \int F_i \; d\mu) \geq \sup_{n \geq 1} (\inf_{i \geq n} \int E_i \; d\mu) \\
&\Rightarrow \; \lim_n \; \inf \int F_n \; d\mu \geq \lim_n \; \inf \int E_n \; d\mu \\
&= \lim_n \; \inf \int g_n \; d\mu \;, \quad \text{by (4)} \\
&= \int h \; d\mu \\
&= \int (\lim_n \; \inf \; F_n) \; d\mu \;, \quad \text{by (6)} \\
&\Rightarrow \; \int (\lim_n \; \inf \; F_n) \; d\mu \leq \lim_n \; \inf \int F_n) \; d\mu.
\end{aligned}$$

Theorem 6.2. *Let $\{F_n\}$ be sequence of closed-valued multi-functions from Ω to $2^L - \{\phi\}$. If F be a closed-valued multi-functions from Ω to $2^L - \{\phi\}$ such that $\lim_n F_n = F$. Then*

$$\lim_n \int F_n \; d\mu = \int F \; d\mu\,.$$

P r o o f. By the given condition,

$$F = \lim_n F_n = \lim_n inf \; F_n = \lim_n \; \sup \; F_n\,.$$

Now,

$$\begin{aligned}
\int F \; d\mu \;\; &= \;\; \int \; \lim_n \; \inf \; F_n \; d\mu \\
&\leq \;\; \lim_n \; \inf \int F_n \; d\mu, \quad \text{by the preceding theorem} \\
&\leq \;\; \lim_n \; \sup \int F_n \; d\mu \\
&\leq \;\; \int \lim_n \; \sup F_n \; d\mu = \int F \; d\mu.
\end{aligned}$$

Therefore, $\int F \; d\mu = \lim_n \; \inf \int F_n \; d\mu = \lim_n \; \sup \int F_n \; d\mu = \lim_n \int F_n \; d\mu\,.$

This concludes the proof. ∎

References

[1] J. Aumann, Integrals of Set-valued functions, *J. Math. Anal. Appl.*, **12** (1965), 1-12.

[2] G. Birkhoff, Latice theory, *J. American Mathematical Society Colloquium Publications*, **25**, Providence, R.I., 1967.

[3] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Springer, Berlin, 1977.

[4] S. J. Cho, B.S. Lee, G. M. Lee, D.S. Kim, Fuzzy integrals for set-valued mappings, *Fuzzy Sets and Systems* **117** (2001), 333-337.

[5] M. Sugeno, *Theory of Fuzzy Integrals and its Applications*, PhD Dissertation, Tokyo Inst. of Tech., 1974.

[6] D. Zhang, and C. Guo, Fuzzy integrals of set-valued mappings and fuzzy mappings,*Fuzzy Sets and Systems,* **75** (1995), 103-109.

[7] D. Zhang, and C. Guo, Generalized fuzzy integrals of Set-Valued functions, *Fuzzy Sets and Systems,* **76** (1995), 365-373.

[8] D Zhang, and Z. Wang, On Set-Valued fuzzy integrals, *Fuzzy Sets and Systems*, **56** (1993), 237-241.

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