Mathematica
Balkanica

New Series Vol. 21, 2007, Fasc. 1-2

The Lifts of a Derivation Determined by D_{K_XY} and Their Applications

A. Gezer[1] *and A. A. Salimov*[2]

Presented by P. Boyvalenkov

The main purpose of the present paper is to define a derivation D_{K_XY} determined by a tensor field $K_XY \in \Im_1^1(M_n)$, where

$$(K_XY)Z = L_X(\nabla_Y Z) - \nabla_Y(L_X Z) - \nabla_{[X,Y]}Z$$

and to discuss relations between lifts of D_{K_XY} and lifts of already known.

AMS Subj. Classification: 53C07, 53B05, 55R100

Key Words: Tensor, bundle, complete lift, Lie derivation

1. Introduction

Let M_n be $n-$ dimensional differentiable manifold of class C^∞ , $T_q^p(M_n)$ its tensor bundle of type (p,q) , and π the natural projection $T_q^p(M_n) \to M_n$. Let x^j, $j = 1,...,n$ be local coordinates in neighborhood U of a point x of M_n. Then a tensor t of type (p,q) at $x \in M_n$ which is an element of $T_q^p(M_n)$ is expressible in the form $(x^j, t_{j_1...j_q}^{i_1...i_p}) = (x^j, x^{\bar{j}})$, $x^{\bar{j}} = t_{j_1...j_q}^{i_1...i_p}$, $\bar{j} = n+1,...,n+n^{p+q}$, whose $t_{j_1...j_q}^{i_1...i_p}$ are components of t with respect to the natural frame ∂_j . We may consider $(x^j, x^{\bar{j}})$ as local coordinates in a neighborhood $\pi^{-1}(U)$ of $T_q^p(M_n)$.

To a transformation of local coordinates of M_n : $x^{j'} = x^{j'}(x^j)$, there corresponds in $T_q^p(M_n)$ the coordinates transformation

$$(1.1) \qquad \begin{cases} x^{j'} = x^{j'}(x^j) \\ x^{\bar{j}'} = t_{j'_1...j'_q}^{i'_1...i'_p} = A_{i_1}^{i'_1}...A_{i_p}^{i'_p} A_{j'_1}^{j_1}...A_{j'_q}^{j_q} t_{j_1...j_q}^{i_1...i_p} = A_{(i)}^{(i')} A_{(j')}^{(j)} x^{\bar{j}}, \end{cases}$$

where

$$A^{(i')}_{(i)}A^{(j)}_{(j')} = A^{i'_1}_{i_1}...A^{i'_p}_{i_p}A^{j_1}_{j'_1}...A^{j_q}_{j'_q}, \qquad A^{i'_1}_{i_1} = \frac{\partial x^{i'}}{\partial x^{i}}, \qquad A^{j_1}_{j'_1} = \frac{\partial x^{j}}{\partial x^{j'}}$$

The Jacobian of (1.1) is given by the matrix

$$(1.2) \qquad \left(\frac{\partial x^{J'}}{\partial x^{J}}\right) = \begin{pmatrix} \frac{\partial x^{j'}}{\partial x^{j}} & \frac{\partial x^{j'}}{\partial x^{\bar{j}}} \\ \frac{\partial x^{\bar{j}'}}{\partial x^{j}} & \frac{\partial x^{\bar{j}'}}{\partial x^{\bar{j}}} \end{pmatrix} = \begin{pmatrix} A^{j'}_{j} & 0 \\ t^{(i)}_{(k)}\partial_j(A^{(i')}_{(i)}A^{(k)}_{(j')}) & A^{(i')}_{(i)}A^{(j)}_{(j')} \end{pmatrix},$$

where $J = (j, \bar{j}),\ J = 1, ..., n + n^{p+q},\ t^{(i)}_{(k)} = t^{i_1...i_p}_{k_1...k_q}$.

We denote by $\Im^p_q(M_n)$ the module over $F(M_n)$ of C^∞ tensor fields of type (p, q) ($F(M_n)$ is a ring of real-valued C^∞ functions on M_n). If $\alpha \in \Im^q_p(M_n)$, it is regarded, in a natural way, by contraction, as a function in $T^p_q(M_n)$, which we denote by $\imath\alpha$. If α has the local expression $\alpha = \alpha^{j_1...j_q}_{i_1...i_p}\partial_{j_1} \otimes ... \otimes \partial_{j_q} \otimes dx^{i_1} \otimes ... \otimes dx^{i_p}$ in a coordinate neighborhood $U(x^i) \subset M_n$, then $\imath\alpha = \alpha(t)$ has the local expression $\imath\alpha = \alpha^{j_1...j_q}_{i_1...i_p}t^{i_1...i_p}_{j_1...j_q}$ with respect to the coordinates $(x^i, x^{\bar{i}})$ in $\pi^{-1}(U)$.

2. Complete Lifts of Derivations

Let $A \in \Im^p_q(M_n)$. Then there is a unique vector field ${}^VA \in \Im^1_0(T^p_q(M_n))$ such that for $\alpha \in \Im^q_p(M_n)$

$${}^VA(\imath\alpha) = \alpha(A)o\pi = {}^V(\alpha(A)),$$

where ${}^V(\alpha(A))$ is the vertical lift of the function $\alpha(A) \in F(M_n)$. We call ${}^VA$ the vertical lift of $A \in \Im^p_q(M_n)$ to $T^p_q(M_n)$ (see [1]). The vertical lift ${}^VA$ has components of the form

$${}^VA = \begin{pmatrix} {}^VA^j \\ {}^VA^{\bar{j}} \end{pmatrix} = \begin{pmatrix} 0 \\ A^{i_1...i_p}_{j_1...j_q} \end{pmatrix}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T^p_q(M_n)$.

Let $\varphi \in \Im^1_1(M_n)$. We define a vector field $\gamma\varphi$ in $\pi^{-1}(U)$ by [2]

$$(2.1) \qquad \begin{cases} \gamma\varphi = \left(\displaystyle\sum_{\lambda=1}^{p} t^{i_1...m...i_p}_{j_1...j_q}\varphi^{i_\lambda}_{m}\right)\dfrac{\partial}{\partial x^{\bar{j}}}, & (p \geq 1,\ q \geq 0) \\ \tilde{\gamma}\varphi = \left(\displaystyle\sum_{\mu=1}^{q} t^{i_1...i_p}_{j_1...m...j_q}\varphi^{m}_{j_\mu}\right)\dfrac{\partial}{\partial x^{\bar{j}}}, & (p \geq 0,\ q \geq 1) \end{cases}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_q^p(M_n)$. From (1.2) we easily see that the vector fields $\gamma\varphi$ and $\tilde{\gamma}\varphi$ defined in each $\pi^{-1}(U)$ determine respectively global vertical vector fields in $T_q^p(M_n)$. We call $\gamma\varphi$ (or $\tilde{\gamma}\varphi$) the vertical-vector lift of the tensor field $\varphi \in \Im_1^1(M_n)$ to $T_q^p(M_n)$. From (2.1), we see that, $\gamma\varphi$ and $\tilde{\gamma}\varphi$ have respectively components

$$\gamma\varphi = \begin{pmatrix} 0 \\ \sum_{\lambda=1}^{p} t_{j_1...j_q}^{i_1...m...i_p} \varphi_m^{i_\lambda} \end{pmatrix} \tag{2.2}$$

$$\tilde{\gamma}\varphi = \begin{pmatrix} 0 \\ \sum_{\mu=1}^{q} t_{j_1...m...j_q}^{i_1...i_p} \varphi_{j_\mu}^{m} \end{pmatrix} \tag{2.3}$$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $\pi^{-1}(U) \subset T_q^p(M_n)$.

We now put $F(M_n) = \sum_{p,q=0}^{\infty} \Im_q^p(M_n)$, which is the direct sum of all tensor modules in M_n . A map $D : F(M_n) \to F(M_n)$ is a derivation in M_n , if

a) D is linear with respect to constant coefficients,

b) For all p, q , $D\Im_q^p(M_n) \subset \Im_q^p(M_n)$,

c) For all tensor fields T_1 and T_2 in M_n ,

$$D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes (DT_2),$$

d) D commutes with contraction.

For a derivation D in M_n , there exists a vector field P in M_n such that

$$Pf = Df, f \in F(M_n) \tag{2.4}$$

If we put

$$D(\partial_i) = Q_i^h \partial_h$$

in each coordinate neighborhood U of M_n , then the pair (P^h, Q_i^h) is called the components of the derivation D in U [3, p.26].

Let α be an element of $\Im_p^q(M_n)$ with local expression $\alpha = \alpha_{i_1...i_p}^{j_1...j_q} \partial_{j_1} \otimes ... \otimes \partial_{j_q} \otimes dx^{i_1} \otimes ... \otimes dx^{i_p}$. Then we see that $D\alpha$ have components of the form

$$D\alpha : (P^m \partial_m \alpha_{i_1...i_p}^{j_1...j_q} + \sum_{\mu=1}^{q} \alpha_{i_1...i_p}^{j_1...m...j_q} Q_m^{j_\mu} - \sum_{\mu=1}^{q} \alpha_{i_1...m...i_p}^{j_1...j_q} Q_{i_\lambda}^{m})$$

in M_n , P^h being the components of $P \in \Im_0^1(M_n)$ given by (2.4).

Let D be a derivation in M_n . Then there is a unique vector field ${}^cD \in \Im_0^1(T_q^p(M_n))$ such that for $\alpha \in \Im_p^q(M_n)$ [1] .

$$ {}^cD(\imath\alpha) = \imath(D\alpha). $$

We call ${}^cD$ the complete lift of D to $T_q^p(M_n)$. ${}^cD$ has components

$$ {}^cD = \begin{pmatrix} p^j \\ \sum_{\mu=1}^{q} t^{i_1\ldots i_p}_{j_1\ldots m\ldots j_q}\varphi^m_{j_\mu} - \sum_{\lambda=1}^{p} t^{i_1\ldots m\ldots i_p}_{j_1\ldots j_q}\varphi^{i_\lambda}_m \end{pmatrix} \tag{2.5} $$

with respect to the coordinates $(x^j, x^{\bar{j}})$ in $T_q^p(M_n)$ [2] .

Let L_V denote Lie derivation with respect to V . Then, from

$$ L_V\alpha^{j_1\ldots j_q}_{i_1\ldots i_p} = V^m\partial_m\alpha^{j_1\ldots j_q}_{i_1\ldots i_p} - \sum_{\lambda=1}^{q}(\partial_m V^{i_\lambda})\alpha^{j_1\ldots m\ldots j_q}_{i_1\ldots i_p} + \sum_{\mu=1}^{p}(\partial_{i_\mu}V^m)\alpha^{j_1\ldots j_q}_{i_1\ldots m\ldots i_p} $$

we see that the Lie derivation L_V is having components $L_V : (V^h, -\partial_i V^h)$. Using (2.5), we have

$$ {}^c(L_V) = {}^cV $$

where ${}^cV$ is the complete lift of vector field V to $T_q^p(M_n)$ [4] :

$$ {}^cV = \begin{pmatrix} {}^cV^j \\ {}^cV^{\bar{j}} \end{pmatrix} = \begin{pmatrix} V^j \\ \sum_{\lambda=1}^{p} t^{i_1\ldots m\ldots i_p}_{j_1\ldots j_q}\partial_m V^{i_\lambda} - \sum_{\mu=1}^{q} t^{i_1\ldots i_p}_{j_1\ldots m\ldots j_q}\partial_{j_\mu}V^m \end{pmatrix}. \tag{2.6} $$

Let now ∇ be affine connection in M_n and ∇_V denote covariant derivation with respect to V . By similar devices, we see that the derivation ∇_V having components

$$ \nabla_V : (V^h, V^s\Gamma^h_{si}) $$

and

$$ {}^c(\nabla_V) = {}^cV - \gamma(\breve{\nabla}V) + \tilde{\gamma}(\breve{\nabla}V) $$

where $\breve{\nabla}$ is a new affine connection in M_n defined by

$$ \breve{\nabla}_V W = \nabla_W V + [V, W]\,,\ \forall V, W \in \Im_0^1(M_n). \tag{2.7} $$

In particular, if ∇ is a symmetric affine connection, then

$$ {}^c(\nabla_V) = {}^HV, \tag{2.8} $$

where ${}^HV$ is the horizontal lift of the vector field $V \in \Im_0^1(M_n)$ to $T_q^p(M_n)$ [4] :

$$ {}^HV = \begin{pmatrix} V^J \\ \hline V^s(\sum_{\mu=1}^{q} \Gamma_{sj_\mu}^m t_{j_1 \dots m \dots j_q}^{i_1 \dots i_p} - \sum_{\lambda=1}^{p} \Gamma_{sm}^{i_\lambda} t_{j_1 \dots j_q}^{i_1 \dots m \dots i_p}) \end{pmatrix}. $$

3. Derivations D_{K_XY} and Formulas on Lie Derivations

When a derivation D in M_n satisfies the condition $Df = 0$ for any $f \in F(M_n)$, D determines an element $\varphi \in \Im_1^1(M_n)$ in such a way that $DX = \varphi X$, $\forall X \in \Im_0^1(M_n)$. In such a case, D is denoted by D_φ and called the derivation determined by φ . From $D_\varphi f = 0$ and $D_\varphi X = \varphi X$, we easily verify that the D_φ has local components

$$ D_\varphi : (0, \varphi_i^h),\ P^h = 0,\ Q_i^h = \varphi_i^h, \tag{3.1} $$

where φ_i^h are local components of φ in M_n . Then from (2.2), (2.3), (2.5) and (3.1), we have

$$ {}^c(D_\varphi) = \begin{pmatrix} 0 \\ \sum_{\mu=1}^{q} t_{j_1 \dots m \dots j_q}^{i_1 \dots i_p} \varphi_{j_\mu}^m - \sum_{\mu=1}^{q} t_{j_1 \dots j_q}^{i_1 \dots m \dots i_p} \varphi_m^{i_\lambda} \end{pmatrix} = \tilde{\gamma}\varphi - \gamma\varphi $$

or

$$ {}^c(D_\varphi) = \tilde{\gamma}\varphi - \gamma\varphi $$

The Lie derivative $L_X\nabla$ of symmetric affine connection ∇ with respect to $X \in \Im_0^1(M_n)$ is, by definition, an element of $\Im_2^1(M_n)$ such that

$$ (L_X\nabla)(Y,Z) = L_X(\nabla_Y Z) - \nabla_Y(L_X Z) - \nabla_{[X,Y]}Z $$

for any $Y, Z \in \Im_0^1(M_n)$. We now denote by K_XY the tensor field of type (1.1), defined by

$$ (K_xY)Z = (L_X\nabla)(Y,Z) = [L_X, \nabla_Y]\, Z - \nabla_{[X,Y]}Z \tag{3.2} $$

This (3.2) reduces to

$$K_X Y = [L_X, \nabla_Y] - \nabla_{[X,Y]} \tag{3.3}$$

which is an equation in terms of derivations. If we take the complete lifts of both sides in (3.3), we have

$$\tilde{\gamma}(K_X Y) - \gamma(K_X Y) = {}^c(D_{K_X Y}) = {}^c[L_X, \nabla_Y] - {}^c(\nabla_{[X,Y]}) \tag{3.4}$$

Taking account of [2]

$$[{}^cD_1, {}^cD_2] = {}^c[D_1, D_2]$$

for any derivations D_1 and D_2 . From (3.4), we have

$$\tilde{\gamma}(K_X Y) - \gamma(K_X Y) = {}^c(D_{K_X Y}) = [{}^c(L_X), {}^c(\nabla_Y)] - {}^c(\nabla_{[X,Y]})$$
$$= \left[{}^cX, {}^HY\right] - {}^H[X,Y]$$

Thus we have

Theorem 1.

$$\left[{}^cX, {}^HY\right] = {}^H[X,Y] + \tilde{\gamma}(K_X Y) - \gamma(K_X Y)$$

for any $X, Y \in \Im_0^1(M_n)$*, where* $K_X Y$ *denotes the tensor field of type*(1.1) *defined by (3.2).*

An infinitesimal transformation defined by vector field $X \in \Im_0^1(M_n)$ is said to be an infinitesimal affine transformation with affine connection ∇ , if $L_X \nabla = 0$. Then, from (3.2) and Theorem 1 we have

Theorem 2. *Let X be an infinitesimal affine transformation in M_n. Then*

$$\left[{}^cX, {}^HY\right] = {}^H[X,Y].$$

Let ∇ is a Riemannian connection in M_n and $\nabla X = 0$. Then $L_X g = 0$, i.e. X is a infinitesimal isometry or a Killing vector field. We next have $L_X \nabla = 0$ as a consequence of $L_X g = 0$. Since ${}^cX = {}^HX$ ($\nabla X = 0$), we have

Theorem 3. *Let X be a vector field with vanishing Riemannian covariant derivative. Then*

$$\left[{}^HX, {}^HY\right] = {}^H[X,Y],$$

i.e. the operation of taking the horizontal lift $^H : \Im_0^1(M_n) \to \Im_0^1(T_q^p(M_n))$ *is a homomorphism.*

Theorem 4. *Let* $X, Y \in \Im_0^1(M_n)$ *and* $F, G \in \Im_1^1(M_n)$*, let* R *and* $K_X Y$ *denote the curvature tensor field of the connection* ∇ *and the tensor field of type* (1.1) *defined by (3.2), respectively. Then*

(a). $[\tilde{\gamma}F - \gamma F, \tilde{\gamma}G - \gamma G] = \tilde{\gamma}\,[F, G] - \gamma\,[F, G]$

(b). $[^c X, \tilde{\gamma}F - \gamma F] = \tilde{\gamma}(L_X F) - \gamma(L_X F)$

(c). $\left[^H X, \tilde{\gamma}F - \gamma F\right] = \tilde{\gamma}(L_X F + (\nabla X)F - F(\nabla X)) - \gamma(L_X F + (\nabla X)F - F(\nabla X))$

P r o o f. *(a).* We can easily verify that

$$[D_F, D_G] = D_{[F,G]} \tag{3.5}$$

for any $F, G \in \Im_1^1(M_n)$, where $[F, G] = FoG - GoF$. If we take the complete lift of both sides of (3.5), we have

$$[\tilde{\gamma}F - \gamma F, \tilde{\gamma}G - \gamma G] = [^c(D_F), {}^c(D_G)] = {}^c\,[D_F, D_G]$$

$$= {}^c(D_{[F,G]}) = \tilde{\gamma}\,[F, G] - \gamma\,[F, G]$$

(b). We consider a derivative $[L_X, D_F] = L_X D_F - D_F L_X$. Since

$$[L_X, D_F]\, f = L_X D_F f - D_F L_X f = -D_F(Xf) = 0$$

for any $f \in F(M_n)$, we denote by $D_{[L_X, D_F]}$ the derivation defined by $[L_X, D_F] \in \Im_1^1(M_n)$. Then from equation

$$[L_X, D_F] = D_{[L_X, D_F]}$$

We have

$$[^c X, \tilde{\gamma}F - \gamma F] = {}^c\,[L_X, D_F] = {}^c(D_{[L_X, D_F]}) \tag{3.6}$$

Taking account of

$$(L_X F)Y = L_x(D_F Y) - D_F(L_X Y) = L_X(FY) - F(L_X Y)\,,$$

we have $L_X F = [L_X, D_F]$. Then from (3.6) we have

$$[^c X, \tilde{\gamma}F - \gamma F] = {}^c(D_{[L_X, D_F]}) = {}^c(D_{L_X F}) = \tilde{\gamma}(L_X F) - \gamma(L_X F)$$

(c). From *(a), (b)*, (2.7) and (2.8), we have

$$\begin{aligned}\left[{}^HX, \tilde{\gamma}F - \gamma F\right] &= [{}^cX + \tilde{\gamma}(\nabla X) - \gamma(\nabla X), \tilde{\gamma}F - \gamma F] \\ &= [{}^cX, \tilde{\gamma}F - \gamma F] + [\tilde{\gamma}(\nabla X) - \gamma(\nabla X), \tilde{\gamma}F - \gamma F] \\ &= [{}^cX, {}^c(D_F)] + \left[{}^c(D_{(\nabla X)}), {}^c(D_F)\right] \\ &= \tilde{\gamma}(L_XF) - \gamma(L_XF) + \tilde{\gamma}\,[(\nabla X), F] - \gamma\,[(\nabla X), F] \\ &= \tilde{\gamma}(L_XF + (\nabla X)F - F(\nabla X)) - \gamma(L_XF + (\nabla X)F - F(\nabla X))\end{aligned}$$

Acknowledgment: This paper is supported by The Scientific & Technological Council of Turkey. pr. No: 105T551.

References

[1] Ledger A., Yano K., Almost complex structures on tensor bundle, *J. Dif. Geom.* **1**, (1967), 355-368.

[2] Cengiz N., Salimov A.A., Complete lifts of derivations to tensor bundles, *Bol. Soc. Mat. Mexicana* **8** (2002), (3), 75-82.

[3] Yano K., and Ishihara S., *Tangent and Cotangent Bundles,* Marcel Dekker, Inc., New York (1973).

[4] Salimov A.A., A new method in the theory of liftings of tensor fields in a tensor bundle, *Russian Math. (Iz, Vuz)* **38** (1994), (3), 67-73.

Department of Mathematics,
Faculty of Arts and Sciences,
Atatürk University,
25240, Erzurum, TURKEY
E-mail: gezeraydin@hotmail.com

Received 08.09.2005