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A Generation of Foliated Semi-Symmetric Hypersurfaces in Four-Dimensional Euclidean Space [1]

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We characterize the foliated semi-symmetric hypersurfaces in Euclidean space as envelopes of two-parameter families of hyperplanes. We introduce an invariant of a two-dimensional surface in four-dimensional Euclidean space and prove that any two-dimensional surface with non-positive invariant generates a foliated semi-symmetric hypersurface.

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1. Introduction

Let (M^n, g) be an n-dimensional Riemannian manifold. We denote by T_pM^n the tangent space to M^n at a point $p \in M^n$ and by $\mathfrak{X}M^n$ - the algebra of all differentiable vector fields on M^n. The associated Levi-Civita connection of the metric g is denoted by ∇ and the Riemannian curvature tensor R is defined by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z; \quad X, Y, Z \in \mathfrak{X}M^n.$$

A *semi-symmetric space* is a Riemannian manifold (M^n, g), whose curvature tensor R satisfies the identity

$$R(X,Y) \cdot R = 0$$

for all $X, Y \in \mathfrak{X}M^n$. According to Szabo's classification [5] every locally irreducible semi-symmetric manifold belongs to one of the following three classes:

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1) "trivial" class, consisting of all locally symmetric spaces and all two-dimensional Riemannian manifolds;

2) "exceptional" class of all elliptic, hyperbolic, Euclidean and Kählerian cones;

3) "typical" class of all Riemannian manifolds foliated by Euclidean leaves of codimension two.

Foliated semi-symmetric spaces were studied by E. Boeckx, O. Kowalski and L. Vanhecke with respect to their metrics under the name Riemannian manifolds of conullity two [2].

A Riemannian manifold (M^n, g) is *of conullity two*, if at every point $p \in M^n$ the tangent space T_pM^n can be decomposed in the form

$$T_pM^n = \Delta_0(p) \oplus \Delta_0^{\perp}(p),$$

where $\dim \Delta_0(p) = n-2$, $\dim \Delta_0^{\perp}(p) = 2$ and $\Delta_0(p)$ is the nullity vector space of the curvature tensor R_p, i.e.

$$\Delta_0(p) = \{X \in T_pM^n \mid R_p(X,Y)Z = 0; \quad Y, Z \in T_pM^n\}.$$

The $(n-2)$-dimensional distribution $\Delta_0 : p \longrightarrow \Delta_0(p)$ is integrable and its integral manifolds are totally geodesic and locally Euclidean. So, (M^n, g) is foliated by Euclidean leaves of codimension two.

We study foliated semi-symmetric hypersurfaces in Euclidean space $\mathbb{E}^{n+1}$ with respect to their second fundamental tensors.

We denote the standard metric in $\mathbb{E}^{n+1}$ by g and its Levi-Civita connection by ∇'. Let ∇ be the induced connection on a hypersurface M^n in $\mathbb{E}^{n+1}$ and

$$h(X,Y) = g(AX,Y); \quad X, Y \in \mathfrak{X}M^n$$

be the second fundamental tensor of M^n with corresponding shape operator A.

Foliated semi-symmetric hypersurfaces in $\mathbb{E}^{n+1}$ are characterized in terms of the second fundamental tensor as follows:

Proposition 1.1. [4] *A hypersurface M^n in $\mathbb{E}^{n+1}$ is a foliated semi-symmetric space if and only if its second fundamental tensor h has the form*

$$h = \lambda\omega \otimes \omega + \mu(\omega \otimes \eta + \eta \otimes \omega) + \nu\eta \otimes \eta, \quad \lambda\nu - \mu^2 \neq 0,$$

where ω and η are unit one-forms; λ, μ and ν are functions on M^n.

The Euclidean leaves of the foliation are the integral submanifolds of the distribution Δ_0, determined by the one-forms ω and η as follows:

$$\Delta_0(p) = \{X \in T_pM^n \mid \omega(X) = 0,\ \eta(X) = 0\}, \quad p \in M^n.$$

In section 2 we characterize the foliated semi-symmetric hypersurfaces in $\mathbb{E}^{n+1}$ as envelopes of two-parameter families of hyperplanes (Theorem 2.3).

In section 3 we introduce an invariant K of a two-dimensional surface in four-dimensional Euclidean space $\mathbb{E}^4$ and prove the basic theorem in this paper:

Each two-dimensional surface in $\mathbb{E}^4$ with $K \leq 0$ generates a foliated semi-symmetric hypersurface.

2. Foliated semi-symmetric hypersurfaces as envelopes of two-parameter families of hyperplanes

Let M^n be a foliated semi-symmetric hypersurface in $\mathbb{E}^{n+1}$. Since the integral submanifolds of the distribution Δ_0 are totally geodesic submanifolds in $\mathbb{E}^{n+1}$, then they are locally parts of $(n-2)$-dimensional planes $\mathbb{E}^{n-2}$. It is easily seen that the foliated semi-symmetric hypersurfaces in $\mathbb{E}^{n+1}$ can be considered locally as developable two-parameter systems of planes of codimension three in the sense of the following

Definition 2.1. A hypersurface M^n in $\mathbb{E}^{n+1}$, which is a two-parameter system $\{\mathbb{E}^{n-2}(u,v)\}$, $(u,v) \in \mathcal{D}$ of planes of codimension three, defined in a domain $\mathcal{D} \subset \mathbb{R}^2$, is said to be developable, if the tangent space T_pM^n at all regular points p of an arbitrary fixed plane $\mathbb{E}^{n-2}(u,v)$ is one and the same.

We call the planes $\mathbb{E}^{n-2}(u,v)$ *generators* of the hypersurface M^n.

Definition 2.1 is analogous to the definition of a developable ruled hypersurface [1]. The foliated semi-symmetric hypersurfaces can be considered as a generalization of the developable ruled hypersurfaces, characterized by a second fundamental tensor of the following form [3]:

$$h = \nu\, \eta \otimes \eta, \quad \nu \neq 0.$$

The non-developable ruled hypersurfaces form a special class of foliated semi-symmetric hypersurfaces, described in [4].

The developable hypersurfaces in $\mathbb{E}^{n+1}$ are characterized by

Lemma 2.2 [3] *A hypersurface M^n in Euclidean space $\mathbb{E}^{n+1}$ is developable if and only if it is the envelope of a one-parameter family of hyperplanes in $\mathbb{E}^{n+1}$.*

As a generalization of Lemma 2.2 we prove

Theorem 2.3. *A hypersurface M^n in Euclidean space $\mathbb{E}^{n+1}$ is locally a foliated semi-symmetric hypersurface if and only if it is the envelope of a two-parameter family of hyperplanes in $\mathbb{E}^{n+1}$.*

Proof. I. Let M^n be a foliated semi-symmetric hypersurface in $\mathbb{E}^{n+1}$. We consider M^n as a developable two-parameter system of planes of codimension three: $M^n = \{\mathbb{E}^{n-2}(u,v)\}$, $(u,v) \in \mathcal{D}$. M^n can be parameterized locally by $X = X(u,v,w^1,\dots,w^{n-2})$, where $\Delta_0 = \operatorname{span}\{\frac{\partial}{\partial w^\alpha}\}$, $\alpha = 1,\dots,n-2$. Let N be the unit normal vector field of M^n (N is determined up to a sign). If we fix $(u,v) \in \mathcal{D}$, then according to Definition 2.1, there exists a uniquely determined hyperplane $\mathbb{E}^n(u,v)$ - the constant tangent space to M^n at the regular points of the fixed generator $\mathbb{E}^{n-2}(u,v)$. The normal vector of $\mathbb{E}^n(u,v)$ is $N(u,v)$. Thus we obtain a two-parameter family $\mathcal{F} = \{\mathbb{E}^n(u,v)\}, (u,v) \in \mathcal{D}$ of hyperplanes in $\mathbb{E}^{n+1}$. The hyperplane $\mathbb{E}^n(u,v)$ is determined by the unit vector-valued function $N = N(u,v)$ and a scalar function $r = r(u,v)$, measuring the oriented distance from the origin O of $\mathbb{E}^{n+1}$ to the hyperplane $\mathbb{E}^n(u,v)$. Hence, the position vector $X = X(u,v,w^1,\dots,w^{n-2})$ of an arbitrary point $p \in M^n$ satisfies the equality

$$N\,X - r = 0. \tag{2.1}$$

Differentiating (2.1) with respect to u and v and using that the normal vector field N is orthogonal to the tangent space $\operatorname{span}\{X_u, X_v, X_\alpha\}$, $\alpha = 1,\dots,n-2$ of M^n, we get the equalities

$$\begin{aligned} &N\,X - r = 0;\\ &N_u\,X - r_u = 0;\\ &N_v\,X - r_v = 0, \end{aligned}$$

which imply that M^n is the envelope of the two-parameter family

$$\mathcal{F} = \{\mathbb{E}^n(u,v)\}, (u,v) \in \mathcal{D}\,.$$

II. Let M^n be a hypersurface in $\mathbb{E}^{n+1}$, which is the envelope of a two-parameter family $\mathcal{F} = \{\mathbb{E}^n(u,v)\}, (u,v) \in \mathcal{D}$ of hyperplanes. We denote by $l = l(u,v)$ the unit normal vector of $\mathbb{E}^n(u,v)$ (l is determined up to a sign) and by $r = r(u,v)$ - the oriented distance from the origin O of $\mathbb{E}^{n+1}$ to the hyperplane $\mathbb{E}^n(u,v)$. Hence, the position vector $X = X(u,v,w^\alpha)$, $\alpha = 1,\dots,n-2$ of an

arbitrary point p of the envelope M^n satisfies

$$\begin{aligned} &l\,X - r = 0;\\ &l_u\,X - r_u = 0;\\ &l_v\,X - r_v = 0. \end{aligned} \tag{2.2}$$

If l_u and l_v are linearly dependent vectors, then the equalities (2.2) determine a developable ruled hypersurface in $\mathbb{E}^{n+1}$. So, we assume that l_u and l_v are linearly independent. Then, fixing $(u,v) \in \mathcal{D}$, the equalities (2.2) determine an $(n-2)$-dimensional plane $\mathbb{E}^{n-2}(u,v)$, which is orthogonal to $\mathrm{span}\{l, l_u, l_v\}$. Hence, for each point $p \in M^n$ there exists an $(n-2)$-dimensional plane $\mathbb{E}^{n-2}(u,v)$, passing through p. Consequently, M^n lies on a two-parameter system $\{\mathbb{E}^{n-2}(u,v)\}, (u,v) \in \mathcal{D}$ of planes of codimension three.

Differentiating the first equality of (2.2) with respect to w^α, $\alpha = 1, \ldots, n-2$, we get

$$l\,X_\alpha = 0, \quad \alpha = 1, \ldots, n-2,$$

which implies that l is orthogonal to $\mathrm{span}\{X_1, \ldots, X_{n-2}\}$. Moreover, using (2.2) we can easily obtain that

$$l\,X_u = 0; \quad l\,X_v = 0.$$

Hence, the unit vector field $l = l(u,v)$ is orthogonal to the tangent space of M^n, spanned by $X_u, X_v, X_1, \ldots, X_{n-2}$, i.e. $l(u,v)$ is a normal vector field of M^n. Since $l(u,v)$ is constant at the points of a fixed generator $\mathbb{E}^{n-2}(u,v)$, then the two-parameter system $\{\mathbb{E}^{n-2}(u,v)\}$, $(u,v) \in \mathcal{D}$ is developable. Consequently, M^n lies on a developable two-parameter system of planes of codimension three, i.e. M^n is locally a foliated semi-symmetric hypersurface. ■

Now, let us consider a hypersurface M^n in $\mathbb{E}^{n+1}$, which is the envelope of a two-parameter family $\mathcal{F} = \{\mathbb{E}^n(u,v)\}$, $(u,v) \in \mathcal{D}$ of hyperplanes $\mathbb{E}^n(u,v)$, determined by a unit vector-valued function $l = l(u,v)$, $(u,v) \in \mathcal{D}$ and a scalar function $r = r(u,v)$, $(u,v) \in \mathcal{D}$. We assume that l, l_u and l_v are linearly independent. Then, locally there exist $n-2$ mutually orthogonal unit vectors $b_1(u,v), \ldots, b_{n-2}(u,v)$, which are orthogonal to $\mathrm{span}\{l, l_u, l_v\}$. Further we use the following functions

$$E(u,v) = g(l_u, l_u); \quad F(u,v) = g(l_u, l_v); \quad G(u,v) = g(l_v, l_v),$$

satisfying the inequalities $E > 0$; $G > 0$; $E\,G - F^2 > 0$. Then, we denote $W = \sqrt{E\,G - F^2}$.

Using that the position vector $X = X(u, v, w^\alpha)$, $\alpha = 1, \dots, n-2$ of an arbitrary point of M^n satisfies (2.2), we obtain that M^n is parameterized as follows:

(2.3)
$$X(u, v, w^\alpha) = r\, l + \frac{G\, r_u - F\, r_v}{W^2}\, l_u + \frac{E\, r_v - F\, r_u}{W^2}\, l_v + w^\alpha\, b_\alpha; \quad (u, v) \in \mathcal{D},\ w^\alpha \in \mathbb{R},$$

where the summation convention is assumed.

Let M^2_r be the two-dimensional surface, defined by $z^r = z^r(u, v)$, $(u, v) \in \mathcal{D}$, where
$$z^r(u, v) = r\, l + \frac{G\, r_u - F\, r_v}{W^2}\, l_u + \frac{E\, r_v - F\, r_u}{W^2}\, l_v.$$
Then, (2.3) can be written in the form
$$X(u, v, w^\alpha) = z^r(u, v) + w^\alpha\, b_\alpha(u, v); \quad (u, v) \in \mathcal{D},\ w^\alpha \in \mathbb{R}.$$

Remark . If we change the origin O of $\mathbb{E}^{n+1}$ with another point O', such that $\overrightarrow{O'O} = c$ (where c is a constant vector), then the vector-valued function $l = l(u, v)$ does not change, while the scalar function $r(u, v)$ changes into
$$r'(u, v) = r(u, v) + g(c, l).$$
The surface $M^2_{r'}$, corresponding to r', is determined by $z^{r'} = z^{r'}(u, v)$, $(u, v) \in \mathcal{D}$, where
$$z^{r'}(u, v) = z^r(u, v) + g(c, l)\, l + \frac{Gg(c, l_u) - Fg(c, l_v)}{W^2}\, l_u + \frac{Eg(c, l_v) - Fg(c, l_u)}{W^2}\, l_v.$$
On the other hand, with respect to the frame $\{l, l_u, l_v, b_\alpha\}$, $\alpha = 1, \dots, n-2$ the constant vector c is decomposed as follows
$$c = g(c, l)\, l + \frac{Gg(c, l_u) - Fg(c, l_v)}{W^2}\, l_u + \frac{Eg(c, l_v) - Fg(c, l_u)}{W^2}\, l_v + g(c, b_\alpha)\, b_\alpha.$$
If we denote by c_{pr} the projection of c over $\mathrm{span}\{l, l_u, l_v\}$, then the connection between the surfaces $M^2_{r'}$ and M^2_r is given by
$$z^{r'}(u, v) = z^r(u, v) + c_{pr}.$$

3. Foliated semi-symmetric hypersurfaces in $\mathbb{E}^4$

We consider a three-dimensional foliated semi-symmetric hypersurface M^3 in $\mathbb{E}^4$ as the envelope of a two-parameter family $\mathcal{F} = \{\mathbb{E}^3(u,v)\}, (u,v) \in \mathcal{D}$ of hyperplanes in $\mathbb{E}^4$. Let $l = l(u,v), (u,v) \in \mathcal{D}$ be a vector-valued function and $r = r(u,v), (u,v) \in \mathcal{D}$ be a scalar function, determining the hyperplane $\mathbb{E}^3(u,v), (u,v) \in \mathcal{D}$. We assume that l, l_u and l_v are linearly independent. Then, locally M^3 is parameterized by

(3.1)
$$X(u,v,w) = r\,l + \frac{G\,r_u - F\,r_v}{W^2}\,l_u + \frac{E\,r_v - F\,r_u}{W^2}\,l_v + w\,b; \quad (u,v) \in \mathcal{D},\ w \in \mathbb{R},$$

where $b = b(u,v)$ is a unit vector field, orthogonal to $\mathrm{span}\{l, l_u, l_v\}$. The hypersurface M^3 is a developable two-parameter system $M^3 = \{\mathbb{E}^1(u,v)\}, (u,v) \in \mathcal{D}$ of lines $\mathbb{E}^1(u,v) = \mathrm{span}\{b(u,v)\}$. We shall find the tangent vector fields X_u, X_v and X_w of M^3.

With respect to the frame $\{l, l_u, l_v, b\}$ the derivatives of second order of $l(u,v)$ are presented as follows

$$\begin{aligned} l_{uu} &= \Gamma^1_{11}\,l_u + \Gamma^2_{11}\,l_v - E\,l + c_{11}\,b; \\ l_{uv} &= \Gamma^1_{12}\,l_u + \Gamma^2_{12}\,l_v - F\,l + c_{12}\,b; \\ l_{vv} &= \Gamma^1_{22}\,l_u + \Gamma^2_{22}\,l_v - G\,l + c_{22}\,b, \end{aligned} \tag{3.2}$$

where Γ^k_{ij} and c_{ij}, $i,j,k = 1,2$ are functions, given by

$$\Gamma^1_{11} = \frac{G\,E_u - 2F\,F_u + F\,E_v}{2W^2}; \qquad \Gamma^2_{11} = \frac{-F\,E_u + 2E\,F_u - E\,E_v}{2W^2};$$

$$\Gamma^1_{12} = \frac{G\,E_v - F\,G_u}{2W^2}; \qquad \Gamma^2_{12} = \frac{E\,G_u - F\,E_v}{2W^2};$$

$$\Gamma^1_{22} = \frac{-G\,G_u + 2G\,F_v - F\,G_v}{2W^2}; \qquad \Gamma^2_{22} = \frac{E\,G_v - 2F\,F_v + F\,G_u}{2W^2};$$

$$c_{11} = g(l_{uu}, b); \qquad c_{12} = g(l_{uv}, b); \qquad c_{22} = g(l_{vv}, b).$$

The derivatives of first order of the vector field $b(u,v)$ are expressed as follows:

$$\begin{aligned} b_u &= -\frac{c_{11}G - c_{12}F}{W^2}\,l_u - \frac{c_{12}E - c_{11}F}{W^2}\,l_v; \\ b_v &= -\frac{c_{12}G - c_{22}F}{W^2}\,l_u - \frac{c_{22}E - c_{12}F}{W^2}\,l_v. \end{aligned} \tag{3.3}$$

Using (3.1) and (3.2) we obtain the tangent vector fields of M^3:

$$
\begin{aligned}
X_u &= -a_{11}\, l_u - a_{12}\, l_v + p\, b;\\
X_v &= -a_{21}\, l_u - a_{22}\, l_v + q\, b;\\
X_w &= b,
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
a_{11} = \;& -r - \frac{G}{W^2}\, r_{uu} + \frac{F}{W^2}\, r_{uv} + \frac{1}{W^2}\left(G(\ln W)_u - \frac{1}{2}G_u\right) r_u\\
& + \frac{1}{W^2}\left(-F(\ln W)_u + F_u - \frac{1}{2}E_v\right) r_v + w\,(Gc_{11} - Fc_{12});\\
a_{12} = \;& \frac{F}{W^2}\, r_{uu} - \frac{E}{W^2}\, r_{uv} + \frac{1}{W^2}\left(-F(\ln W)_u + \frac{1}{2}E_v\right) r_u\\
& + \frac{1}{W^2}\left(E(\ln W)_u - \frac{1}{2}E_u\right) r_v + w\,(Ec_{12} - Fc_{11});\\
a_{21} = \;& -\frac{G}{W^2}\, r_{uv} + \frac{F}{W^2}\, r_{vv} + \frac{1}{W^2}\left(G(\ln W)_v - \frac{1}{2}G_v\right) r_u\\
& + \frac{1}{W^2}\left(-F(\ln W)_v + \frac{1}{2}G_u\right) r_v + w\,(Gc_{12} - Fc_{22});\\
a_{22} = \;& -r + \frac{F}{W^2}\, r_{uv} - \frac{E}{W^2}\, r_{vv} + \frac{1}{W^2}\left(-F(\ln W)_v + F_v - \frac{1}{2}G_u\right) r_u\\
& + \frac{1}{W^2}\left(E(\ln W)_v - \frac{1}{2}E_v\right) r_v + w\,(Ec_{22} - Fc_{12});\\
p = \;& c_{11}\,\frac{G\, r_u - F\, r_v}{W^2} + c_{12}\,\frac{E\, r_v - F\, r_u}{W^2};\\
q = \;& c_{12}\,\frac{G\, r_u - F\, r_v}{W^2} + c_{22}\,\frac{E\, r_v - F\, r_u}{W^2}.
\end{aligned}
$$

From (3.3) it follows that $b(u,v) = const = b_0$ if and only if $c_{ij} = 0,\ i,j = 1,2$.

If $b(u,v) = b_0$ for all $(u,v) \in \mathcal{D}$, i.e. $c_{ij} = 0,\ i,j = 1,2$, then $p = 0,\ q = 0$ and the tangent space of M^3 is spanned by the vector fields

$$
\begin{aligned}
X_u &= -a_{11}\, l_u - a_{12}\, l_v;\\
X_v &= -a_{21}\, l_u - a_{22}\, l_v;\\
X_w &= b,
\end{aligned}
$$

where $a_{ij} = a_{ij}(u,v)$. If $\det a_{ij} = 0$ for all $(u,v) \in \mathcal{D}$, then the vector fields X_u and X_v are linearly dependent and hence, all points of M^3 are irregular. So, we assume that there exists a domain $\mathcal{D}_0 \subset \mathcal{D}$, such that $(\det a_{ij})_{|\mathcal{D}_0} \neq 0$. Hence, ${M^3}_{|\mathcal{D}_0}$ is a regular hypersurface, parameterized by

$$X(u,v,w) = z^r(u,v) + w\, b_0,$$

which implies that ${M^3}_{|\mathcal{D}_0}$ is a cylindrical hypersurface.

Now, let us consider the non-cylindrical case, i.e. $(b_u, b_v) \neq (0,0)$ in a domain $\mathcal{D}_0 \subset \mathcal{D}$. Hence, there exist indeces $i,j \in \{1,2\}$, such that $c_{ij} \neq 0$. Then, $a_{ij} = a_{ij}(u,v,w)$ and a_{ij} is a linear function with respect to w.

From (3.4) it follows that the vector fields X_u, X_v and X_w are linearly independent if and only if $\det a_{ij} \neq 0$.

For an arbitrary $(u,v) \in \mathcal{D}_0$ the following subcases arise:

1. $\det a_{ij}(u,v,w) \neq 0$ for all $w \in \mathbb{R}$, i.e. there are no irregular points on the fixed generator $E^1(u,v)$.

2. There exists $w_0 \in \mathbb{R}$, such that $\det a_{ij}(u,v,w_0) = 0$, i.e. on the generator $E^1(u,v)$ there exists an irregular point of M^3, determined by $z^r(u,v) + w_0\, b(u,v)$. Since $\det a_{ij}(u,v,w)$ is a quadratic function with respect to w, then the number of the irregular points on $E^1(u,v)$ is either one or two.

We consider those foliated semi-symmetric hypersurfaces in $\mathbb{E}^4$, for which there exists an irregular point on each generator $E^1(u,v)$, $(u,v) \in \mathcal{D}_0$ $(\mathcal{D}_0 \subset \mathcal{D})$.

Let M^3 be a foliated semi-symmetric hypersurface, defined by (3.1), such that on each generator $E^1(u,v)$, $(u,v) \in \mathcal{D}_0$ $(\mathcal{D}_0 \subset \mathcal{D})$ there exists an irregular point, determined by $w_0 = f(u,v)$, $(u,v) \in \mathcal{D}_0$, where $f(u,v)$ is a function, satisfying

$$\det a_{ij}(u,v,f(u,v)) = 0. \tag{3.5}$$

We consider the two-parameter system M_0 of irregular points of M^3, parameterized by

$$Z(u,v) = z^r(u,v) + f(u,v)\, b(u,v), \quad (u,v) \in \mathcal{D}_0. \tag{3.6}$$

The two-parameter system of points, given by (3.6) can be one point (if $Z_u = 0$, $Z_v = 0$), or a curve c (if Z_u and Z_v are linearly dependent) or a two-dimensional surface M_0^2 (if Z_u and Z_v are linearly independent).

If M_0 consists only of one point z_0, then for each $(u,v) \in \mathcal{D}_0$ the line $E^1(u,v)$ of M^3 passes through the point z_0 and hence, M^3 can be parameterized as follows

$$X(u,v,w) = z_0 + w\, b(u,v),$$

which implies that M^3 is a conical hypersurface with vertex z_0.

If M_0 is a curve c, then for each $(u,v) \in \mathcal{D}_0$ the line $E^1(u,v)$ of M^3 passes through one point of the curve c and hence, M^3 can be parameterized by

$$X(u,v,w) = z(v) + w\, b(u,v), \tag{3.7}$$

where $c : z = z(v)$ is the curve, formed by the irregular points of M^3. The equality (3.7) implies that M^3 is a one-parameter system of conical two-dimensional surfaces, whose verteces lie on the curve c.

We shall consider the general case, in which the two-parameter system of irregular points, determined by (3.6), is a two-dimensional surface M_0^2. The tangent space of M_0^2 is spanned by the vector fields Z_u and Z_v. Since the function $f(u,v)$ satisfies the equality (3.5), then the vector fields $Z_{u|\mathrm{span}\{l_u,l_v\}}$ and $Z_{v|\mathrm{span}\{l_u,l_v\}}$ are linearly dependent. Hence, there exists a unit vector field $n(u,v) \in \mathrm{span}\{l_u, l_v\}$, such that the tangent vector fields of M_0^2 are expressed as

$$\begin{aligned} Z_u &= p_0\, n + (p + f_u)\, b; \\ Z_v &= q_0\, n + (q + f_v)\, b \end{aligned}$$

for some functions p_0 and q_0. So, the tangent space of M_0^2 is $\mathrm{span}\{Z_u, Z_v\} = \mathrm{span}\{n, b\}$. Consequently, the generators $E^1(u,v) = \mathrm{span}\{b(u,v)\}$ of the hypersurface M^3 are tangent to the surface M_0^2. Hence, the foliated semi-symmetric hypersurface M^3 can be considered as a two-parameter system of lines, tangent to the surface of irregular points.

Let $n^\perp(u,v)$ be a unit vector field orthogonal to $n(u,v)$, such that $\mathrm{span}\{n, n^\perp\} = \mathrm{span}\{l_u, l_v\}$. Since the tangent space of the surface M_0^2 is $\mathrm{span}\{n, b\}$, then the normal space of M_0^2 is $\mathrm{span}\{n^\perp, l\}$. Using that $b_u, b_v \in \mathrm{span}\{l_u, l_v\}$, we get that b_u and b_v are decomposed in the form

$$\begin{aligned} b_u &= \beta_1\, n + \gamma_1\, n^\perp; \\ b_v &= \beta_2\, n + \gamma_2\, n^\perp, \end{aligned} \tag{3.8}$$

where $\beta_1\, n$ and $\beta_2\, n$ are the tangent components of b_u and b_v, respectively, while $\gamma_1\, n^\perp$ and $\gamma_2\, n^\perp$ are the normal components of b_u and b_v, respectively. The equalities (3.8) imply that the normal components of b_u and b_v are collinear.

Let $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}\ (\mathcal{D} \subset \mathbb{R}^2)$ be a two-dimensional surface in $\mathbb{E}^4$. We denote by $\widetilde{\nabla}$ the induced connection on M^2 and by σ - the second fundamental form of M^2. Let $b(u,v)$ be an arbitrary tangent vector field of M^2. Applying the Gauss formula, we get

$$\nabla'_{z_u} b = \widetilde{\nabla}_{z_u} b + \sigma(z_u, b);$$
$$\nabla'_{z_v} b = \widetilde{\nabla}_{z_v} b + \sigma(z_v, b).$$

We call a tangent line of M^2, determined by a tangent vector field b, *a tangent of asymptotic type*, if $\sigma(z_u, b)$ and $\sigma(z_v, b)$ are collinear.

So, the considerations above give us the following

Proposition 3.1. *Let M^3 be a foliated semi-symmetric hypersurface in $\mathbb{E}^4$ with a two-dimensional surface M_0^2 of irregular points. Then, the generators of M^3 are tangents of asymptotic type of M_0^2.*

We shall prove that conversely, each two-dimensional surface in $\mathbb{E}^4$, admitting tangents of asymptotic type, generates a foliated semi-symmetric hypersurface in $\mathbb{E}^4$.

Let $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}\ (\mathcal{D} \subset \mathbb{R}^2)$ be a two-dimensional surface in $\mathbb{E}^4$ with second fundamental form σ. The tangent space of M^2 is $\mathrm{span}\{z_u, z_v\}$. Let e_1 and e_2 be two mutually orthogonal unit normal vector fields of M^2. With respect to the frame $\{z_u, z_v, e_1, e_2\}$ the derivatives of second order of $z(u,v)$ are presented as

$$\begin{aligned} z_{uu} &= \Gamma^1_{11}\, z_u + \Gamma^2_{11}\, z_v + c^1_{11}\, e_1 + c^2_{11}\, e_2; \\ z_{uv} &= \Gamma^1_{12}\, z_u + \Gamma^2_{12}\, z_v + c^1_{12}\, e_1 + c^2_{12}\, e_2; \\ z_{vv} &= \Gamma^1_{22}\, z_u + \Gamma^2_{22}\, z_v + c^1_{22}\, e_1 + c^2_{22}\, e_2, \end{aligned} \tag{3.9}$$

where Γ^k_{ij} and c^k_{ij}, $i,j,k = 1,2$ are functions on M^2.

We denote

$$\Delta_1 = \begin{vmatrix} c^1_{11} & c^1_{12} \\ c^2_{11} & c^2_{12} \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} c^1_{11} & c^1_{22} \\ c^2_{11} & c^2_{22} \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} c^1_{12} & c^1_{22} \\ c^2_{12} & c^2_{22} \end{vmatrix}.$$

Lemma 3.2. *The tangent vector $b = \lambda\, z_u + \mu\, z_v$ of M^2 determines a tangent of asymptotic type if and only if*

$$\Delta_1 \lambda^2 + \Delta_2 \lambda\, \mu + \Delta_3 \mu^2 = 0.$$

P r o o f. Using equalities (3.9) we obtain

$$\sigma(b, z_u) = \left(\lambda\, c^1_{11} + \mu\, c^1_{12}\right) e_1 + \left(\lambda\, c^2_{11} + \mu\, c^2_{12}\right) e_2;$$
$$\sigma(b, z_v) = \left(\lambda\, c^1_{12} + \mu\, c^1_{22}\right) e_1 + \left(\lambda\, c^2_{12} + \mu\, c^2_{22}\right) e_2.$$

Hence, $\sigma(b, z_u)$ and $\sigma(b, z_v)$ are collinear if and only if

$$\begin{vmatrix} \lambda\, c^1_{11} + \mu\, c^1_{12} & \lambda\, c^2_{11} + \mu\, c^2_{12} \\ \lambda\, c^1_{12} + \mu\, c^1_{22} & \lambda\, c^2_{12} + \mu\, c^2_{22} \end{vmatrix} = 0.$$

The last equality is equivalent to $\Delta_1\lambda^2 + \Delta_2\lambda\,\mu + \Delta_3\mu^2 = 0$. ∎

The functions $E(u,v) = g(z_u, z_u)$, $F(u,v) = g(z_u, z_v)$ and $G(u,v) = g(z_v, z_v)$ determine the first fundamental form of M^2. Further we denote

$$L(u,v) = \frac{2\Delta_1}{W}; \quad M(u,v) = \frac{\Delta_2}{W}; \quad N(u,v) = \frac{2\Delta_3}{W},$$

where $W = \sqrt{E\,G - F^2}$.

Lemma 3.3. *The function* $K = \dfrac{L\,N - M^2}{E\,G - F^2}$ *is an invariant of the surface* M^2.

P r o o f. Let

$$\begin{aligned} u &= u(\overline{u}, \overline{v}); \\ v &= v(\overline{u}, \overline{v}), \end{aligned} \qquad (\overline{u}, \overline{v}) \in \overline{\mathcal{D}},\ \overline{\mathcal{D}} \subset \mathbb{R}^2$$

be a smooth change of the parameters of M^2 and $J = u_{\overline{u}}\, v_{\overline{v}} - u_{\overline{v}}\, v_{\overline{u}}$ $(J \neq 0)$. If $\overline{E} = g(z_{\overline{u}}, z_{\overline{u}})$, $\overline{F} = g(z_{\overline{u}}, z_{\overline{v}})$ and $\overline{G} = g(z_{\overline{v}}, z_{\overline{v}})$ are the coefficients of the first fundamental form of M^2 with respect to the parameters $\overline{u}, \overline{v}$, then $\overline{E}\,\overline{G} - \overline{F}^2 = J^2(E\,G - F^2)$. Let

$$\sigma(z_{\overline{u}}, z_{\overline{u}}) = \overline{c}^1_{11}\, e_1 + \overline{c}^2_{11}\, e_2;$$
$$\sigma(z_{\overline{u}}, z_{\overline{v}}) = \overline{c}^1_{12}\, e_1 + \overline{c}^2_{12}\, e_2;$$
$$\sigma(z_{\overline{v}}, z_{\overline{v}}) = \overline{c}^1_{22}\, e_1 + \overline{c}^2_{22}\, e_2.$$

Then, the functions $\overline{c}^k_{ij}$ and c^k_{ij}, $i, j, k = 1, 2$ are connected as follows

$$\begin{aligned} \overline{c}^k_{11} &= u^2_{\overline{u}}\, c^k_{11} + 2u_{\overline{u}}\, v_{\overline{u}}\, c^k_{12} + v^2_{\overline{u}}\, c^k_{22}; \\ \overline{c}^k_{12} &= u_{\overline{u}}\, u_{\overline{v}}\, c^k_{11} + (u_{\overline{u}}\, v_{\overline{v}} + u_{\overline{v}}\, v_{\overline{u}})\, c^k_{12} + v_{\overline{u}}\, v_{\overline{v}}\, c^k_{22}; \qquad k = 1, 2. \\ \overline{c}^k_{22} &= u^2_{\overline{v}}\, c^k_{11} + 2u_{\overline{v}}\, v_{\overline{v}}\, c^k_{12} + v^2_{\overline{v}}\, c^k_{22}; \end{aligned} \tag{3.10}$$

Using equalities (3.10) we obtain that the relation between the functions $\overline{\Delta}_i$ and Δ_i, $i = 1, 2, 3$ is given by

$$\begin{aligned}
\overline{\Delta}_1 &= J\left(u_{\overline{u}}^2\,\Delta_1 + u_{\overline{u}}\,v_{\overline{u}}\,\Delta_2 + v_{\overline{u}}^2\,\Delta_3\right);\\
\overline{\Delta}_2 &= J\left(2u_{\overline{u}}\,u_{\overline{v}}\,\Delta_1 + \left(u_{\overline{u}}\,v_{\overline{v}} + u_{\overline{v}}\,v_{\overline{u}}\right)\Delta_2 + 2v_{\overline{u}}\,v_{\overline{v}}\,\Delta_3\right);\\
\overline{\Delta}_3 &= J\left(u_{\overline{v}}^2\,\Delta_1 + u_{\overline{v}}\,v_{\overline{v}}\,\Delta_2 + v_{\overline{v}}^2\,\Delta_3\right).
\end{aligned}$$

Hence, $\dfrac{\overline{L}\,\overline{N} - \overline{M}^2}{\overline{E}\,\overline{G} - \overline{F}^2} = \dfrac{L\,N - M^2}{E\,G - F^2}$, i.e. K is invariant under a change of the parametrization of M^2.

It is easy to check that the function K does not depend on the choice of the normal frame $\{e_1, e_2\}$ of M^2. Consequently, K is an invariant of the surface M^2. ■

Using Lemma 3.2 we obtain

Proposition 3.4. *A two-dimensional surface M^2 in $\mathbb{E}^4$ admits tangents of asymptotic type if and only if $K \leq 0$.*

Now we shall give a construction of foliated semi-symmetric hypersurfaces in $\mathbb{E}^4$.

Theorem 3.5. *Each two-dimensional surface in $\mathbb{E}^4$ with $K \leq 0$ generates a foliated semi-symmetric hypersurface.*

Proof. Let $M^2 : z = z(u, v)$, $(u, v) \in \mathcal{D}$ $(\mathcal{D} \subset \mathbb{R}^2)$ be a two-dimensional surface in $\mathbb{E}^4$, for which $K \leq 0$. According to Proposition 3.4, there exist tangents of asymptotic type on M^2. Let $b(u, v) = \lambda\, z_u + \mu\, z_v$ be a tangent vector field on M^2, such that $\sigma(z_u, b)$ and $\sigma(z_v, b)$ are collinear. Hence, the vector fields b_u and b_v are decomposed in the form

$$\begin{aligned}
b_u &= a_{11}\, z_u + a_{12}\, z_v + p\, e;\\
b_v &= a_{21}\, z_u + a_{22}\, z_v + q\, e,
\end{aligned}$$

where $e = e(u, v)$ is a unit normal vector field of M^2; p, q, a_{ij}, $i, j = 1, 2$ are functions on M^2. We denote by $e^{\perp}(u, v)$ the unit normal vector field of M^2, which is orthogonal to $e(u, v)$.

Now, let us consider the hypersurface M^3 in $\mathbb{E}^4$, defined by

$$X(u, v, w) = z(u, v) + w\, b(u, v); \quad (u, v) \in \mathcal{D},\ w \in \mathbb{R}.$$

M^3 is a two-parameter system of lines, which are tangents of asymptotic type of the surface M^2. The tangent space of M^3 is spanned by the vector fields

$$X_u = (1 + w\, a_{11})\, z_u + w\, a_{12}\, z_v + w\, p\, e;$$
$$X_v = w\, a_{21}\, z_u + (1 + w\, a_{22})\, z_v + w\, q\, e;$$
$$X_w = \lambda\, z_u + \mu\, z_v.$$

Since the vector field $e^{\perp}(u,v)$ is orthogonal to $\mathrm{span}\{X_u, X_v, X_w\}$, then $e^{\perp}(u,v)$ is a normal vector field of the hypersurface M^3. Moreover, the normal vector field $e^{\perp}(u,v)$ of M^3 is constant at the points of an arbitrary fixed generator $\mathbb{E}^1(u,v) = \mathrm{span}\{b(u,v)\}$, i.e. M^3 is a developable two-parameter system of lines. Hence, M^3 is a foliated semi-symmetric hypersurface. ■

Remark . If $L = M = N = 0$, then any line on M^2 is a tangent of asymptotic type and hence, any one-parameter family of lines on M^2 generates a foliated semi-symmetric hypersurface. If $K = 0$ and $(L, M, N) \neq (0, 0, 0)$, then there exists only one family of tangents of asymptotic type on M^2 and M^2 generates one foliated semi-symmetric hypersurface. If $K < 0$, then on M^2 there exist two families of tangents of asymptotic type and M^2 generates two foliated semi-symmetric hypersurfaces.

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