

## Computing with Matrix Invariants <sup>1</sup>

*Vesselin Drensky*

*Presented by P. Kenderov*

We present an introduction to the theory of the invariants under the action of  $GL_n(\mathbb{C})$  by simultaneous conjugation of  $d$  matrices of size  $n \times n$ . Then we survey some results, old or recent, obtained by a dozen of mathematicians, on minimal sets of generators, the defining relations of the algebras of invariants and on the multiplicities of the Hilbert series of these algebras. The picture is completely understood only in the case  $n = 2$ . Besides, explicit minimal sets of generators are known for  $n = 3$  and any  $d$  and for  $n = 4$ ,  $d = 2$ . The multiplicities of the Hilbert series are obtained only for  $n = 3, 4$  and  $d = 2$ . For  $n > 2$  most of the concrete results are obtained with essential use of computers.

*AMS Subj. Classification:* Primary: 16R30; Secondary: 05E05, 05E10, 13A50, 13P10, 16P90, 16S15

*Key Words:* trace rings, invariant theory, matrix concominants, defining relations, Hilbert series, symmetric functions, Schur functions

### 1. Introduction to invariant theory

All considerations in this paper are over an arbitrary field  $K$  of characteristic 0. If not explicitly stated, all vector spaces and algebras are over  $K$ , and the algebras are unitary and commutative.

To get some idea about invariant theory, we start with the following well known result from the undergraduate algebra course.

Let  $n \geq 2$  be an integer and let

$$A = K[X] = K[x_1, \dots, x_n]$$

---

<sup>1</sup>This is an improved version of the talk of the author given at the Antalya Algebra Days VII on May 21, 2005. Partially supported by Grant MM-1106/2001 of the Bulgarian Foundation for Scientific Research.

be the algebra of polynomials in  $n$  variables. The algebra of symmetric polynomials  $A^{S_n} = K[X]^{S_n}$  consists of all polynomials  $f(x_1, \dots, x_n) \in A$  such that

$$f(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = f(x_1, \dots, x_n)$$

for all permutations  $\sigma$  in the symmetric group  $S_n$ .

**Theorem 1.1.**

- (i) *The algebra  $A^{S_n}$  is generated by the elementary symmetric functions*

$$\begin{aligned} e_1 &= x_1 + \dots + x_n = \sum x_i, \\ e_2 &= x_1x_2 + x_1x_3 + \dots + x_{n-1}x_n = \sum_{i < j} x_ix_j, \\ &\dots \\ e_n &= x_1 \cdots x_n. \end{aligned}$$

- (ii) *The presentation of the symmetric polynomials as polynomials of  $e_1, \dots, e_n$  is unique, i.e.*

$$A^{S_n} \cong K[y_1, \dots, y_n].$$

As a measure *how many are the symmetric polynomials in  $n$  variables* we may use the dimensions of the vector spaces  $A_k^{S_n}$  of the homogeneous symmetric polynomials of degree  $k$ . For example, for  $n = 3$ , the vector space  $A_k^{S_3}$  has a basis consisting of all products  $e_1^{k_1}e_2^{k_2}e_3^{k_3}$ , where  $k_1 + 2k_2 + 3k_3 = k$ . Instead, we consider the *generating function* of the sequence  $\dim(A_k^{S_n})$ , namely the *formal power series*

$$H(A^{S_n}, t) = \sum_{k \geq 0} \dim(A_k^{S_n}) t^k,$$

which is called the *Hilbert series* of  $A^{S_n}$ . Since the set of products

$$\{e_1^{k_1}e_2^{k_2} \cdots e_n^{k_n} \mid k_1, \dots, k_n \geq 0\}$$

is a basis of  $A^{S_n}$ , it is easy to see that

$$\begin{aligned} H(A^{S_n}, t) &= \sum_{k_j \geq 0} t^{k_1} t^{2k_2} \cdots t^{nk_n} \\ &= (1 + t + t^2 + \cdots)(1 + t^2 + t^4 + \cdots) \cdots (1 + t^n + t^{2n} + \cdots) = \prod_{i=1}^n \frac{1}{1 - t^i}. \end{aligned}$$

Let us consider another example.

**Example 1.2.** Let  $n = 3$ ,  $A = K[x_1, x_2, x_3]$  and let  $G$  be the subgroup of  $S_3$  generated by the cycle  $\rho = (123)$ . We denote by  $A^G$  the algebra of “*cyclicly symmetric polynomials*”,

$$A^G = \{f(x_1, x_2, x_3) \in A \mid f(x_2, x_3, x_1) = f(x_1, x_2, x_3)\}.$$

One can show that  $A^G$  is generated by the elementary symmetric functions  $e_1, e_2, e_3$  and one more polynomial

$$f_4 = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1,$$

which is not symmetric. The sum

$$a = (x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1) + (x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2)$$

of  $f_4$  and its “other half”  $x_1 x_2^2 + x_2 x_3^2 + x_3 x_1^2$ , is symmetric. The polynomial  $b = f_4(a - f_4)$  is also symmetric. Hence both  $a, b$  belong to  $K[e_1, e_2, e_3]$  and  $f_4$  satisfies the relation  $f_4^2 - af_4 + b = 0$  with coefficients

$$a = \alpha(e_1, e_2, e_3), b = \beta(e_1, e_2, e_3) \in K[e_1, e_2, e_3].$$

There exists a natural homomorphism  $\pi : K[y_1, y_2, y_3, y_4] \rightarrow A^G$  defined by

$$y_1 \rightarrow e_1, \quad y_2 \rightarrow e_2, \quad y_3 \rightarrow e_3, \quad y_4 \rightarrow f_4,$$

and

$$r(y_1, y_2, y_3, y_4) = y_4^2 - \alpha(y_1, y_2, y_3)y_4 + \beta(y_1, y_2, y_3)$$

belongs to the kernel of  $\pi$ . One can show that the ideal  $\text{Ker}(\pi)$  is generated by  $r(y_1, y_2, y_3, y_4)$ . Hence  $A^G$  has a basis

$$\{e_1^{k_1} e_2^{k_2} e_3^{k_3}, \quad e_1^{k_1} e_2^{k_2} e_3^{k_3} f_4 \mid k_1, k_2, k_3 \geq 0\}$$

and, if  $A_k^G$  is the vector space of the homogeneous cyclicly symmetric polynomials of degree  $k$ , then the Hilbert series of  $A^G$  is

$$\begin{aligned} H(A^G, t) &= \sum_{k \geq 0} \dim(A_k^G) t^k \\ &= \sum_{k_i \geq 0} t^{k_1} (t^2)^{k_2} (t^3)^{k_3} + \left( \sum_{k_i \geq 0} t^{k_1} (t^2)^{k_2} (t^3)^{k_3} \right) t^3 = \frac{1 + t^3}{(1 - t)(1 - t^2)(1 - t^3)}. \end{aligned}$$

Let  $V_n$  be the vector space with basis  $X = \{x_1, \dots, x_n\}$  and let the general linear group  $GL_n = GL_n(K) = GL(V_n)$  act canonically on  $V_n$ . If we identify  $g \in GL_n$  with the invertible matrix  $g = (g_{ij})$ ,  $g_{ij} \in K$ , then the action of  $g$  on  $x_j$  is defined by

$$g(x_j) = g_{1j}x_1 + \dots + g_{nj}x_n.$$

This action is extended *diagonally* on  $A = K[X] = K[x_1, \dots, x_n]$ :

$$g(f(x_1, \dots, x_n)) = f(g(x_1), \dots, g(x_n)), \quad g \in GL_n, \quad f(x_1, \dots, x_n) \in A.$$

**Definition 1.3.** Let  $G$  be any subgroup of  $GL_n$ . (One says that the group  $G$  is *linear*.) The polynomial  $f(x_1, \dots, x_n)$  is called a  $G$ -invariant if

$$g(f(x_1, \dots, x_n)) = f(x_1, \dots, x_n)$$

for all  $g \in G$ . We denote the algebra of  $G$ -invariants by  $A^G$ .

For example, if  $G$  is the subgroup of all permutational matrices

$$g_\sigma = \sum_{j=1}^n e_{\sigma(j),j}, \quad \sigma \in S_n,$$

where the  $e_{ij}$ 's are the matrix units, then  $G \cong S_n$  and the algebra of  $G$ -invariants consists of all symmetric polynomials in  $n$  variables.

For  $n = 3$  and  $G$  being the cyclic group of order 3 generated by the matrix

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

the algebra of invariants  $K[x_1, x_2, x_3]^G$  consists of all cyclicly symmetric polynomials.

It is easy to see that the algebra of invariants is graded with respect to the usual degree of polynomials. This means that if the polynomial  $f = f_0 + f_1 + \dots + f_p$  is a  $G$ -invariant and  $f_k$  is its homogeneous component of degree  $k$ , then  $f_k$  is also  $G$ -invariant. If  $K[X]_k^G$  is the homogeneous component of degree  $k$  of  $K[X]^G$ , then one may define the Hilbert series of  $K[X]^G$  as the formal power series

$$H(K[X]^G, t) = \sum_{k \geq 0} \dim(K[X]_k^G) t^k.$$

We state of list of problems which are among the main problems in invariant theory. For more details, see some book on invariant theory, e.g. Dieudonné and Carrell [9] or Springer [58]; for computational aspects of invariant theory see Sturmfels [61].

Recall that if  $R$  is a finitely generated commutative algebra, then  $R$  is a homomorphic image of the polynomial algebra with the same number of generators. Hence  $R \cong K[y_1, \dots, y_m]/I$  for some  $m$  and for some ideal  $I$ . The equations  $f(y_1, \dots, y_m) = 0$ ,  $f \in I$ , are called *relations* of the algebra  $R$  with respect to the given set of generators. The elements of any generating set of the ideal  $I$  are called *defining relations* of  $R$ . A finitely generated algebra with a finite system of defining relations is called *finitely presented*.

#### Problems 1.4.

- (i) *For a given linear group  $G$ , find a set of generators of the algebra of  $G$ -invariants.*
- (ii) *Is it true that the algebra of invariants  $K[X]^G$  is finitely generated for any subgroup  $G$  of  $GL_n$ ?*
- (iii) *If  $K[X]^G$  is finitely generated, find its defining relations. Is  $K[X]^G$  always finitely presented?*
- (iv) *When  $K[X]^G$  is isomorphic to a polynomial algebra (i.e. has no defining relations with respect to a suitable system of generators)?*
- (v) *Calculate the Hilbert series of  $K[X]^G$ . Is it a rational function? What kind of properties of  $K[X]^G$  can be recovered from its Hilbert series?*

We give short comments on the solutions of the above problems in the case of finite groups. Problem 1.4 (i) is solved by the following theorem of Emmy Noether.

**Theorem 1.5.** (Endlichkeitssatz, [46]) *The algebra of invariants  $K[X]^G$  is finitely generated for any finite subgroup  $G$  of  $GL_n$ , by a set of invariants of degree  $\leq |G|$ , the order of the group  $G$ .*

It is well known that the linear operator of  $K[X]$  (called the *Reynolds operator*)

$$\rho : f(x_1, \dots, x_n) \rightarrow \frac{1}{|G|} \sum_{g \in G} g(f)$$

is a projection of  $K[X]$  onto  $K[X]^G$ , i.e.  $\rho^2 = \rho$  and  $\rho(K[X]) = K[X]^G$ . If we take  $f = x_1^{k_1} \cdots x_n^{k_n}$  and consider all monomials of degree  $\leq |G|$ , then we obtain a system of generators of  $K[X]^G$ .

Problem 1.4 (ii) was the main motivation for the Hilbert 14-th Problem from the famous list of 23 open problems presented by Hilbert at the International Congress of Mathematicians in Paris, 1900, [33]. Only in the late 1950's Nagata [44] found the first counterexamples to the 14-th Hilbert Problem. We recommend the recent survey by Freudenburg [29] for the state-of-the-art of the problem and for different approaches to construct other counterexamples.

**Theorem 1.6.** (Hilbert Basissatz, or Hilbert Basis Theorem, [32]) *Every ideal of  $K[y_1, \dots, y_m]$  is finitely generated.*

As an immediate consequence of the Hilbert Basissatz and the theorem of Emmy Noether we obtain that for finite groups  $G$  the algebra  $K[X]^G$  is finitely presented, i.e. the positive solution of Problem 1.4 (iii). Since the degree of the generators of  $K[X]^G$  is bounded, in principle, one can answer also the question which are the defining relations of  $K[X]^G$ .

Problem 1.4 (iv) is solved by the following theorem of Chevalley-Shephard-Todd [8, 55].

**Theorem 1.7.** *For a finite group  $G$ , the algebra of invariants  $K[X]^G$  is isomorphic to a polynomial algebra (in  $n = |X|$  variables) if and only if  $G \subset GL_n$  is generated by pseudo-reflections.*

Recall that the matrix  $g \in GL_n$  is called a *pseudo-reflection*, if it is similar to a matrix of the form

$$\text{diag}(\xi, 1, \dots, 1) = \begin{pmatrix} \xi & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

it is a *reflection* if  $\xi = -1$ . The symmetric group  $S_n$  is generated by the transpositions  $\tau_j = (1j)$ ,  $j = 2, \dots, n$ , and the images  $g_{\tau_j} = 1 - e_{11} - e_{jj} + e_{j1} + e_{1j}$  of  $\tau_j$  under the embedding of  $S_n$  into  $GL_n$  are reflections. Hence  $K[X]^{S_n}$  is isomorphic to a polynomial algebra. On the other hand, the matrix  $g$  from Example 1.2 (as well as its square  $g^2$ ) is similar to

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix},$$

where  $\varepsilon$  is a primitive third root of 1 in some extension of  $K$ . This implies that  $G = \{1, g, g^2\} \subset GL_3$  cannot be generated by pseudo-reflections and  $K[x_1, x_2, x_3]^G$  is not isomorphic to a polynomial algebra.

Finally, the following Molien formula [42] answers Problem 1.4 (v).

**Theorem 1.8.** *For any finite group  $G \subset GL_n$*

$$H(K[X]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gt)}.$$

Here  $\det(1 - gt)$  is the determinant of the  $n \times n$  matrix  $1 - gt$ . In particular, the Hilbert series  $H(K[X]^G, t)$  is a rational function. We refer to the book of Stanley [60] for the relationship between the algebraic properties of graded algebras and their Hilbert series.

The natural generalization of the case of finite linear groups is the case of reductive groups. Recall that, by the Maschke theorem, the finite subgroups  $G$  of  $GL_n$  are completely reducible. This means that if, for a suitable basis of  $V_n$ , the group  $G$  consists of matrices of the block triangular form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$

then, changing the basis we may assume that the matrices are of the block diagonal form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

A similar property holds for the class of reductive subgroups of  $GL_n$ . In particular, this holds for all classical groups ( $GL_k$ ,  $SL_k$ ,  $O_k$ ,  $SO_k$ ,  $Sp_k$ ,  $U_k$ ).

**Theorem 1.9.**

- (i) (Hilbert-Serre, see e.g. [9]) *For any reductive subgroup  $G$  of  $GL_n$  the algebra of invariants  $K[X]^G$  is finitely generated.*
- (ii) (Molien-Weyl formula, see [69]) *If  $G \subset GL_n(\mathbb{C})$  is compact, then one can define Haar measure on  $G$ , replace in the Molien formula the sum with an integral and obtain the formula for the Hilbert series of the algebra of invariants  $\mathbb{C}[X]^G$ .*

## 2. Matrix invariants

In the sequel, we do not use the symbol  $X$  for the set of variables  $\{x_1, \dots, x_n\}$  and we denote by  $X, X_i, Y, Y_i$ , etc.  $n \times n$  matrices. We start

with a special case of the main object of this paper. Let

$$X = (x_{ij}) = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

be an  $n \times n$  matrix with  $n^2$  algebraically independent entries  $x_{ij}$ . Such a matrix is called a *generic*  $n \times n$  matrix because every  $n \times n$  matrix  $A = (a_{ij})$  with entries  $a_{ij}$  from a commutative  $K$ -algebra  $S$  can be obtained from  $X$  by specialization of the variables  $x_{ij}$  (i.e. a homomorphism  $K[x_{ij}] = K[x_{ij} \mid i, j = 1, \dots, n] \rightarrow S$  defined by  $x_{ij} \rightarrow a_{ij}$ ,  $i, j = 1, \dots, n$ ). The group  $GL_n$  acts on  $X$  by conjugation:

$$gXg^{-1} = Y = (y_{ij}), \quad g \in GL_n,$$

where the  $y_{ij}$ 's are linear combinations of the  $x_{ij}$ 's. We define an action of  $GL_n$  on the algebra  $K[x_{ij}]$  by  $g : x_{ij} \rightarrow y_{ij}$ ,  $i, j = 1, \dots, n$ , and are interested in the algebra of invariants  $K[x_{ij}]^{GL_n}$ . The obvious invariants come from the Cayley-Hamilton theorem. If

$$f_X(t) = \det(1 - tX) = (-1)^n(t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n)$$

is the characteristic polynomial of  $X$ , then the coefficients  $a_1, \dots, a_n$  depend on the entries of  $X$ . In particular,  $a_1 = -\text{tr}(X)$  and  $a_n = (-1)^n \det(X)$ . The Cayley-Hamilton theorem states that  $f_X(X) = 0$ . Since  $gf_X(X)g^{-1} = f_X(gXg^{-1})$  for any  $g \in GL_n$ , we obtain that the coefficients  $a_1, \dots, a_n$  are  $GL_n$ -invariants. The following theorem is a partial case of Theorem 2.5 below, but probably was known much before it. It seems impossible to trace its origin.

**Theorem 2.1.** *With respect to the above action of  $GL_n$ , the algebra of invariants  $K[x_{ij}]^{GL_n}$  is generated by the coefficients of the characteristic polynomial of  $X$ .*

Up to a sign, the coefficients of the characteristic polynomial are equal to the elementary symmetric polynomials evaluated on the eigenvalues of the matrix. Let

$$p_k = x_1^k + \cdots + x_n^k$$

be the  $k$ -th power symmetric function. Since the characteristic of  $K$  is equal to 0, the Newton formulas

$$p_k - e_1 p_{k-1} + \cdots + (-1)^{k-1} e_{k-1} p_1 + (-1)^k k e_k = 0, \quad k = 1, \dots, n,$$

allow to express the elementary symmetric polynomials  $e_1, \dots, e_n$  in terms of the power symmetric polynomials. Hence  $K[x_1, \dots, x_n]^{S_n}$  is generated by  $p_1, \dots, p_n$ .



If  $\xi_1, \dots, \xi_n$  are the eigenvalues of the matrix  $X$  in the algebraic closure of the field of rational functions  $K(x_{ij})$ , then  $X$  is similar to the upper triangular matrix

$$\begin{pmatrix} \xi_1 & * & \cdots & * \\ 0 & \xi_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_n \end{pmatrix}.$$

(It is known that  $X$  is similar even to the diagonal matrix  $\text{diag}(\xi_1, \dots, \xi_n)$  with the same diagonal entries as the above matrix.) The coefficients of the characteristic polynomial  $f_X(X)$  are

$$a_1 = -e_1(\xi_1, \dots, \xi_n), a_2 = e_2(\xi_1, \dots, \xi_n), \dots, a_n = (-1)^n e_n(\xi_1, \dots, \xi_n).$$

Similarly,

$$p_1(\xi_1, \dots, \xi_n) = \text{tr}(X), p_2(\xi_1, \dots, \xi_n) = \text{tr}(X^2), \dots, p_n(\xi_1, \dots, \xi_n) = \text{tr}(X^n).$$

In this way, as an immediate consequence of Theorem 2.1, we obtain:

**Corollary 2.2.** *For the above action of  $GL_n$  on  $K[x_{ij}]$ , the algebra  $K[x_{ij}]^{GL_n}$  is generated by the traces  $\text{tr}(X), \text{tr}(X^2), \dots, \text{tr}(X^n)$  of the powers of  $X$ .*

Now we consider the more general situation, when the group  $GL_n$  acts on several generic matrices. We fix an integer  $d \geq 2$ . Let

$$X_i = \left( x_{pq}^{(i)} \right), \quad p, q = 1, \dots, n, \quad i = 1, \dots, d,$$

be  $d$  generic  $n \times n$  matrices and let

$$\Omega = \Omega_{nd} = K[x_{pq}^{(i)} \mid p, q = 1, \dots, n, \quad i = 1, \dots, d]$$

be the polynomial algebra in  $dn^2$  variables. The action of  $GL_n$  on  $\Omega$  is as in the case of one generic matrix. If

$$gX_i g^{-1} = Y_i = (y_{pq}^{(i)}), \quad g \in GL_n,$$

then we define

$$g : x_{pq}^{(i)} \rightarrow y_{pq}^{(i)}, \quad p, q = 1, \dots, n, \quad i = 1, \dots, d.$$

This action of  $GL_n$  on  $\Omega$  is called the action of  $GL_n$  by *simultaneous conjugation* on  $d$  generic  $n \times n$  matrices. Here are the main problems concerning the algebra of matrix invariants  $\Omega_{nd}^{GL_n}$ .

**Problem 2.3.** *Describe the algebraic properties of  $\Omega_{nd}^{GL_n}$ . In particular, find concrete sets of generators and defining relations, calculate the Hilbert series, etc.*

We refer e.g. to the books by Procesi [47], Formanek [26], and Drensky and Formanek [14] as a further reading on invariant theory of matrices.

Since the action of  $GL_n$  on  $\Omega_{nd}$  is reductive, Classical Invariant Theory gives:

**Theorem 2.4.** *The algebra  $\Omega_{nd}^{GL_n}$  is finitely generated.*

Traditionally, a result giving the explicit generators of the algebra of invariants of a linear group  $G$  is called a *first fundamental theorem of the invariant theory of  $G$*  and a result describing the relations between the generators is a *second fundamental theorem*. As in the case of a single generic matrix, the coefficients of the Cayley-Hamilton polynomial of any matrix expressed in terms of  $X_1, \dots, X_d$  is  $GL_n$ -invariant. In particular, all traces  $\text{tr}(X_{i_1} \cdots X_{i_k})$  are  $GL_n$ -invariant. A more sophisticated result than the previous theorem states:

**Theorem 2.5.** (First Fundamental Theorem of Matrix Invariants) *The algebra  $\Omega_{nd}^{GL_n}$  is generated by all traces  $\text{tr}(X_{i_1} \cdots X_{i_k})$ , where  $i_1, \dots, i_k = 1, \dots, d$ .*

It is difficult to judge about the priority in the proof of this theorem. (See the history of generic matrices in the paper by Formanek [27].) The proof follows from general facts on invariant theory of  $GL_n$  and can be found in the book by Gurevich [30] and in the papers by Sibirskii [57] and Procesi [48]. Although it was stated as well known by Kirillov [36], it seems that the understanding of the importance of the theorem is a result of its rediscovery by Procesi [48].

Combining both theorems, we are looking for a concrete finite set of generators of  $\Omega_{nd}^{GL_n}$ . We recall one of the most important theorems in the theory of algebras with polynomial identities.

**Theorem 2.6.** (Nagata-Higman Theorem) *If the (nonunitary and noncommutative) algebra  $R$  is nil of bounded index  $\leq n$ , i.e.  $r^n = 0$  for all  $r \in R$ , then  $R$  is nilpotent, i.e. there exists an  $N = N(n)$  such that  $r_1 \cdots r_N = 0$  for all  $r_1, \dots, r_N \in R$ .*

The Nagata-Higman theorem was established in 1953 by Nagata [43] for algebras over a field of characteristic 0 and then in 1956 generalized by Higman [31] when  $\text{char } K = p > n$ . Much later it was discovered that this theorem was first established in 1943 by Dubnov and Ivanov [22] but their paper was overlooked by the mathematical community.

The class of nilpotency  $N(n)$  in the Nagata-Higman theorem is related in the following nice way with invariant theory of matrices.

**Theorem 2.7.** (Formanek [24], Procesi [48, 49], Razmyslov [52]) *Let  $N(n)$  be the class of nilpotency in the Nagata-Higman theorem. Then the algebra of invariants  $\Omega_{nd}^{GL_n}$  is generated by the traces  $\text{tr}(X_{i_1} \cdots X_{i_m})$  of degree  $\leq N(n)$ . For  $d$  sufficiently large this bound is sharp.*

It is important to know the exact value of the class of nilpotency  $N(n)$  in the Nagata-Higman theorem. The upper bound given in the proof of Higman [31] is  $N(n) \leq 2^n - 1$ . The best known upper bound is due to Razmyslov [52]. Applying *trace polynomial identities* of matrices, he obtained the bound  $N(n) \leq n^2$ . The proof of the theorem of Razmyslov is given also in his book [53] or in the book by Formanek [26]. For a lower bound, Kuzmin [37] showed that  $N(n) \geq \frac{1}{2}n(n+1)$ . A proof of the result of Kuzmin may be found also in the books by Drensky and Formanek [14] or by Kanel-Belov and Rowen [35]. Hence

$$\frac{n(n+1)}{2} \leq N(n) \leq n^2.$$

**Problem 2.8.** *Find the exact value  $N(n)$  of the class of nilpotency of nil algebras of index  $n$  over a field of characteristic 0.*

**Conjecture 2.9.** (Kuzmin [37]) *The exact value  $N(n)$  of the class of nilpotency of nil algebras of index  $n$  over a field of characteristic 0 is*

$$N(n) = \frac{n(n+1)}{2}.$$

The only values of  $N(n)$  are known for  $n \leq 4$ : Dubnov [21] obtained in 1935

$$N(1) = 1, \quad N(2) = 3, \quad N(3) = 6.$$

In 1993, Vaughan-Lee [68] proved that

$$N(4) = 10.$$

In this way the conjecture of Kuzmin is confirmed for  $n \leq 4$ . Recently, Shestakov and Zhukavets [56] have proved that the class of nilpotency of the two-generated algebras satisfying the identity  $x^5 = 0$  is equal to 15, which agrees with the conjecture of Kuzmin for  $n = 5$ . They have obtained the same result also in the more general setup of 2-generated superalgebras. Their proof is based on computer calculations with the GAP package.

Since  $N(2) = 3$ , the algebra  $\Omega_{2d}^{GL_2}$  is generated by products of traces of degree  $\leq 3$ . The following result was established by Sibirskii [57].

**Theorem 2.10.** *The elements*

$$\begin{aligned} \operatorname{tr}(X_i), \operatorname{tr}(X_i^2), \quad i = 1, \dots, d, \quad \operatorname{tr}(X_i X_j), \quad 1 \leq i < j \leq d, \\ \operatorname{tr}(X_i X_j X_k), \quad 1 \leq i < j < k \leq d, \end{aligned}$$

*constitute a minimal set of generators of the algebra of  $2 \times 2$  matrix invariants  $\Omega_{2d}^{GL_2}$ .*

Now we give an idea about the Razmyslov-Procesi theory which is related with the second fundamental theorem of the matrix invariants, see Razmyslov [52] and Procesi [48], as well as the book by Razmyslov [53] for other applications of his method. For simplicity we consider the case  $n = 2$  only. The Cayley-Hamilton theorem for  $2 \times 2$  matrices implies that

$$X^2 - \operatorname{tr}(X)X + \det(X) = 0.$$

The Newton formulas give that

$$\det(X) = \frac{1}{2}(\operatorname{tr}^2(X) - \operatorname{tr}(X^2)).$$

This can be seen also directly. If  $\xi_1, \xi_2$  are the eigenvalues of  $X$ , then

$$\operatorname{tr}(X) = \xi_1 + \xi_2, \quad \operatorname{tr}(X^2) = \xi_1^2 + \xi_2^2,$$

$$\det(X) = \xi_1 \xi_2 = \frac{1}{2}((\xi_1 + \xi_2)^2 - (\xi_1^2 + \xi_2^2)) = \frac{1}{2}(\operatorname{tr}^2(X) - \operatorname{tr}(X^2)).$$

In this way we obtain the *mixed trace identity*

$$c(X) = X^2 - \operatorname{tr}(X)X + \frac{1}{2}(\operatorname{tr}^2(X) - \operatorname{tr}(X^2)) = 0.$$

Now we consider the identity  $c(X_1 + X_2) - c(X_1) - c(X_2) = 0$ , i.e. we *linearize* the identity  $c(X) = 0$ . In this way we obtain the *mixed Cayley-Hamilton identity*

$$\begin{aligned} \Psi_2(X_1, X_2) &= X_1 X_2 + X_2 X_1 - \operatorname{tr}(X_1)X_2 - \operatorname{tr}(X_2)X_1 \\ &\quad + \operatorname{tr}(X_1)\operatorname{tr}(X_2) - \operatorname{tr}(X_1 X_2) = 0. \end{aligned}$$

Since the trace is a nondegenerate bilinear form on  $M_2(K)$ , the vanishing of the polynomial  $\Psi_2(X_1, X_2)$  on  $M_2(K)$  is equivalent to the vanishing of the *pure Cayley-Hamilton identity*

$$\Phi_2(X_1, X_2, X_3) = \operatorname{tr}(\Psi_2(X_1, X_2)X_3) = 0$$

on all  $2 \times 2$  matrices. Direct calculations show that

$$0 = \Phi_2(X_1, X_2, X_3) = \text{tr}(\Psi_2(X_1, X_2)X_3) = \text{tr}(X_1X_2X_3) + \text{tr}(X_2X_1X_3) \\ - \text{tr}(X_1)\text{tr}(X_2X_3) - \text{tr}(X_2)\text{tr}(X_1X_3) + \text{tr}(X_1)\text{tr}(X_2)\text{tr}(X_3) - \text{tr}(X_1X_2)\text{tr}(X_3).$$

If we delete the symbols of traces and the  $X$ 's in the above expression, we shall obtain the following linear combination of permutations

$$(123) + (213) - (1)(23) - (2)(13) + (1)(2)(3) - (12)(3) = \sum_{\sigma \in S_3} \text{sign}(\sigma)\sigma.$$

This suggests the following construction. We write the permutations in the symmetric group  $S_m$  as products of disjoint cycles, including the cycles of length 1,

$$\sigma = (i_1 \dots i_p)(j_1 \dots j_q) \cdots (k_1 \dots k_r).$$

We define the *associated trace function*

$$\text{tr}_\sigma(x_1, \dots, x_m) = \text{tr}(x_{i_1} \cdots x_{i_p}) \text{tr}(x_{j_1} \cdots x_{j_q}) \cdots \text{tr}(x_{k_1} \cdots x_{k_r}).$$

For every element

$$\sum_{\sigma \in S_m} \alpha_\sigma \sigma \in KS_m, \quad \alpha_\sigma \in K, \sigma \in S_m,$$

where  $KS_m$  is the group algebra of  $S_m$ , we define the trace polynomial

$$f(x_1, \dots, x_m) = \sum_{\sigma \in S_m} \alpha_\sigma \text{tr}_\sigma(x_1, \dots, x_m).$$

We also assume that for  $m \leq k$  the symmetric group  $S_m$  acts on  $1, \dots, m$  and leaves invariant  $m+1, \dots, k$ , i.e.  $S_m$  is canonically embedded into  $S_k$ .

**Theorem 2.11.** (Second Fundamental Theorem of Matrix Invariants, Razmyslov [52], Procesi [48]) *Let*

$$f(x_1, \dots, x_m) = \sum_{\sigma \in S_m} \alpha_\sigma \text{tr}_\sigma(x_1, \dots, x_m), \quad \alpha_\sigma \in K,$$

*be a multilinear trace polynomial of degree  $m$ . Then  $f = 0$  is a trace identity for the  $n \times n$  matrix algebra, i. e.  $f(a_1, \dots, a_m) = 0$  for all  $a_1, \dots, a_m \in M_n(K)$ , if and only if*

$$\sum_{\sigma \in S_m} \alpha_\sigma \sigma$$

belongs to the two-sided ideal  $J(n, m)$  of the group algebra  $KS_m$  generated by the element

$$\sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \sigma.$$

As in the case of  $2 \times 2$  matrices, the *fundamental trace identity*

$$\sum_{\sigma \in S_{n+1}} (\text{sign} \sigma) \text{tr}_{\sigma}(x_1, \dots, x_{n+1}) = 0$$

is actually the linearization of the Cayley-Hamilton polynomial.

There are several important objects related with invariant theory of matrices. As above,  $n, d \geq 2$  are fixed integers and  $X_1, \dots, X_d$  are  $d$  generic  $n \times n$  matrices:

The algebra  $R_{nd}$  generated by  $X_1, \dots, X_d$ ;

The *pure* (or *commutative*) *trace algebra*  $C_{nd} = \Omega_{nd}^{GL_n}$  generated by the traces of all products  $\text{tr}(X_{i_1} \cdots X_{i_k})$ ;

The *mixed* (or *noncommutative*) *trace algebra*  $T_{nd}$  generated by  $R_{nd}$  and  $C_{nd}$  regarding the elements of  $C_{nd}$  as scalar matrices;

The field of fractions  $Q(C_{nd})$  of the algebra  $C_{nd}$ ;

The algebra  $Q(C_{nd})R_{nd}$ .

All these algebras have no zero divisors and play important roles in mathematics. See the books [14, 26, 34, 47] for different aspects of the theory of algebras of matrix invariants and their applications to combinatorial and structure theory of PI-algebras, central division algebras, etc.

The algebra  $R_{nd}$  is a well known object in the theory of *PI-algebras*, or *algebras with polynomial identities*. Let  $K\langle x_1, \dots, x_d \rangle$  be the free associative algebra (i.e. the algebra of polynomials in noncommuting variables). A polynomial  $f(x_1, \dots, x_d) \in K\langle x_1, \dots, x_d \rangle$  is a *polynomial identity* in  $d$  variables for  $M_n(K)$  if  $f(a_1, \dots, a_d) = 0$  for all  $a_1, \dots, a_d \in M_n(K)$ . The set  $I(M_n(K))$  of all polynomial identities in  $d$  variables is a two-sided ideal of  $K\langle x_1, \dots, x_d \rangle$  and  $R_{nd}$  is isomorphic to the factor algebra  $K\langle x_1, \dots, x_d \rangle / I(M_n(K))$ .

Clearly, the algebra  $C_{nd}$  is the algebra of matrix invariants.

The algebra  $T_{nd}$  is also the algebra of invariant polynomial functions under a suitable action of  $GL_n$ . It is called the algebra of *matrix concomitants*. It is a finitely generated  $C_{nd}$ -module and, as a  $C_{nd}$ -module, has a generating set consisting of products  $X_{j_1} \cdots X_{j_k}$ , where  $k < N(n)$ , the class of nilpotency in the Nagata-Higman theorem.

The field of fractions  $Q(C_{nd})$  appears naturally in field theory. One of the main problems related with  $Q(C_{nd})$  is whether it is a purely transcendental extension of  $K$ .

Finally,  $Q(C_{nd})R_{nd}$  is a central division algebra of dimension  $n^2$  over its centre  $Q(C_{nd})$  and serves as a source of counterexamples to the theory of central division algebras.

General invariant theory gives that  $C_{nd}$  and  $T_{nd}$  have nice algebraic properties.

**Theorem 2.12** (Van den Bergh, [66]) *The algebra  $C_{nd}$  is a Cohen-Macaulay and even Gorenstein unique factorization domain. The algebra  $T_{nd}$  is a Cohen-Macaulay module over  $C_{nd}$ .*

Recall that the Noether normalization theorem gives that  $C_{nd}$  contains a homogeneous set of algebraically independent elements  $\{a_1, \dots, a_k\}$ , where  $k = (d-1)n^2 + 1$  is the transcendence degree of the quotient field  $Q(C_{nd})$  ( $k$  is also equal to the Krull dimension of  $C_{nd}$ ), such that  $C_{nd}$  is integral over the polynomial algebra  $K[a_1, \dots, a_k]$ . Such a set  $\{a_1, \dots, a_k\}$  is called a *homogeneous system of parameters* for  $C_{nd}$ . By a result of Stanley [59], *a graded  $C_{nd}$ -module is Cohen-Macaulay if and only if it is a free module with respect to some homogeneous system of parameters  $\{a_1, \dots, a_k\}$  of  $C_{nd}$ .*

The algebras  $R_{nd}$ ,  $C_{nd}$ , and  $T_{nd}$  are multigraded and their homogeneous components of degree  $(k_1, \dots, k_d)$  consist of all polynomials which are homogeneous of degree  $k_i$  in the generic matrix  $X_i$ . Hence we may consider their Hilbert series in  $d$  variables. For example,

$$H(C_{nd}, t_1, \dots, t_d) = \sum \dim \left( C_{nd}^{(k_1, \dots, k_d)} \right) t_1^{k_1} \dots t_d^{k_d},$$

where  $C_{nd}^{(k_1, \dots, k_d)}$  is the homogeneous component of degree  $(k_1, \dots, k_d)$ . Le Bruyn [38] for the case  $n = 2$ , and Formanek [25] and Teranishi [62, 64] in the general case proved:

**Theorem 2.13.** *Let  $d \geq 2$  for  $n \geq 3$  and  $d > 2$  for  $n = 2$ . Then  $H(C_{nd}, t_1, \dots, t_d)$  and  $H(T_{nd}, t_1, \dots, t_d)$  satisfy the functional equation*

$$H(C_{nd}, t_1^{-1}, \dots, t_d^{-1}) = (-1)^k (t_1 \dots t_d)^{n^2} H(t_1, \dots, t_d),$$

where  $k = (d-1)n^2 + 1$  is the Krull dimension of  $C_{nd}$ , and similarly for the Hilbert series of  $T_{nd}$ .

The proofs of this theorem given by Formanek and Teranishi are quite different and use, respectively, representation theory of general linear groups and the Molien-Weyl integral formula. Later, Van den Bergh paid attention that the proof can be considerably simplified using results of Stanley on Hilbert series of Cohen-Macaulay algebras.

We need some background on symmetric polynomials and representation theory of the general linear group, see e.g. the books by Weyl [70] and Macdonald [40]. As in the case of polynomial algebras, the general linear group  $GL_d$  acts canonically on the vector space with basis  $\{X_1, \dots, X_d\}$ . If  $g = (g_{ij})$ ,  $g_{ij} \in K$ , then the action of  $g$  on  $X_j$  is defined by

$$g(X_j) = g_{1j}X_1 + \dots + g_{dj}X_d.$$

This action is extended diagonally on  $R_{nd}, C_{nd}, T_{nd}$ . If  $f(X_1, \dots, X_d)$  is any polynomial expression depending on  $X_1, \dots, X_d$  (maybe including also traces), then

$$g(f(X_1, \dots, X_d)) = f(g(X_1), \dots, g(X_d)), \quad g \in GL_d.$$

Representation theory of  $GL_d$  says that every submodule of the  $GL_d$ -modules  $R_{nd}, C_{nd}, T_{nd}$  is a direct sum of irreducible (or simple) submodules. The irreducible  $GL_d$ -submodules which appear in the decomposition are *polynomial modules* and are described in terms of *partitions* of integers. If

$$\lambda = (\lambda_1, \dots, \lambda_d), \quad \lambda_1 \geq \dots \geq \lambda_d \geq 0,$$

is a partition of  $k$  (notation  $\lambda \vdash k$ ) in not more than  $d$  parts, then we denote the related  $GL_d$ -module by  $W(\lambda) = W(\lambda_1, \dots, \lambda_d)$ . To be explicit, below we consider the case of  $C_{nd}$  only. If

$$C_{nd} = \bigoplus m(\lambda)W(\lambda), \quad m(\lambda) \geq 0,$$

i.e. there are  $m(\lambda)$  direct summands isomorphic to  $W(\lambda)$ , then we say that  $W(\lambda)$  appears with *multiplicity*  $m(\lambda)$ . The multiplicities  $m(\lambda)$  for  $R_{nd}, C_{nd}$ , and  $T_{nd}$  play important role in the quantitative study of polynomial identities of matrices. (See the survey by Regev [54] and the book of the author [12] for applications of representation theory of  $S_n$  and  $GL_d$  to the theory of PI-algebras.) The Hilbert series of  $C_{nd}$  has the form

$$H(C_{nd}, t_1, \dots, t_d) = \sum m(\lambda)S_\lambda(t_1, \dots, t_d),$$

where  $S_\lambda(t_1, \dots, t_d)$  is the *Schur function* associated to  $\lambda$ . Schur functions are important combinatorial objects and appear in many places in mathematics. For example, *they form a basis of the vector space of all symmetric polynomials in  $d$  variables*. One of the possible ways to define Schur functions is via Vandermonde-like determinants. For a partition  $\mu = (\mu_1, \dots, \mu_d)$ , define the determinant

$$V(\mu_1, \dots, \mu_d) = \begin{vmatrix} t_1^{\mu_1} & t_2^{\mu_1} & \dots & t_d^{\mu_1} \\ t_1^{\mu_2} & t_2^{\mu_2} & \dots & t_d^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\mu_d} & t_2^{\mu_d} & \dots & t_d^{\mu_d} \end{vmatrix}.$$



Then the Schur function is

$$S_\lambda(t_1, \dots, t_d) = \frac{V(\lambda_1 + d - 1, \lambda_2 + d - 2, \dots, \lambda_{d-1} + 1, \lambda_d)}{V(d - 1, d - 2, \dots, 1, 0)}.$$

The Schur functions play the role of characters of the corresponding representation of  $GL_d$ . If we know the Hilbert series  $H(C_{nd}, t_1, \dots, t_d)$ , then we can uniquely determine the  $GL_d$ -module structure of  $C_{nd}$ .

### 3. Concrete computations

We give a survey of some concrete results about generators, defining relations and Hilbert series of the algebras of matrix invariants. We start with  $2 \times 2$  matrices.

The first important reduction is the following. We take the generic  $2 \times 2$  matrix  $X$  and present it in the form

$$\begin{aligned} X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} &= \begin{pmatrix} \frac{x_{11}+x_{22}}{2} & 0 \\ 0 & \frac{x_{11}-x_{22}}{2} \end{pmatrix} + \begin{pmatrix} \frac{x_{11}-x_{22}}{2} & x_{12} \\ x_{21} & -\frac{x_{11}-x_{22}}{2} \end{pmatrix} \\ &= \frac{\text{tr}(X)}{2}E + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & -y_{11} \end{pmatrix} = \frac{\text{tr}(X)}{2}E + Y, \end{aligned}$$

where  $E$  is the identity matrix and

$$Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & -y_{11} \end{pmatrix}$$

is a generic  $2 \times 2$  *traceless* matrix. This reduction implies that  $C_{2d}$  is generated by  $\text{tr}(X_i)$ ,  $i = 1, \dots, d$ , and  $\text{tr}(Y_{i_1} \cdots Y_{i_k})$ , where  $Y_1, \dots, Y_d$  are generic traceless matrices and  $k \geq 2$ . Since the class of nilpotency in the Nagata-Higman theorem is  $N(2) = 3$  for  $n = 2$ , we obtain that it is sufficient to consider the cases  $k = 2, 3$  only. Similarly,  $T_{2d}$  is generated by  $C_{2d}$  and  $Y_1, \dots, Y_d$ . It is also easy to see:

**Proposition 3.1.** *The algebra  $C_{2d}$  has the presentation*

$$C_{2d} = (K[\text{tr}(X_1), \dots, \text{tr}(X_d)]) \otimes_K C_0,$$

where

$$C_0 = K[\text{tr}(Y_i Y_j), \text{tr}(Y_p Y_q Y_r) \mid 1 \leq i \leq j \leq d, 1 \leq p < q < r \leq d]/I$$

is the algebra generated by products of generic traceless  $2 \times 2$  matrices. Hence we may choose all defining relations as linear combinations of the traces of products of traceless matrices.

The description of  $T_{2d}$  is easier than that of  $C_{2d}$ .

**Theorem 3.2.**

- (i) (Procesi [50]) *The noncommutative trace algebra  $T_{2d}$  is isomorphic to the tensor product of  $K$ -algebras*

$$K[\operatorname{tr}(X_1), \dots, \operatorname{tr}(X_m)] \otimes_K W_d,$$

where  $W_d$  is the associative algebra generated by the generic traceless matrices  $Y_1, \dots, Y_m$ .

- (ii) (Razmyslov [51]) *The algebra of the generic traceless matrices  $W_d$  is isomorphic to the factor algebra  $K\langle y_1, \dots, y_d \rangle / I(M_2(K), sl_2(K))$  of the free associative algebra  $K\langle y_1, \dots, y_d \rangle$ , where  $sl_2(K)$  is the Lie algebra of all traceless  $2 \times 2$  matrices and  $I(M_2(K), sl_2(K))$  is the ideal of all polynomials in  $K\langle y_1, \dots, y_d \rangle$  which vanish under the substitutions of  $y_i$  by elements in  $sl_2(K)$ . (Such polynomials are called weak polynomial identities for the pair  $(M_2(K), sl_2(K))$ .) The ideal  $I(M_2(K), sl_2(K))$  is generated as a weak  $T$ -ideal by the weak polynomial identity  $[x_1^2, x_2] = 0$ , i.e.  $I(M_2(K), sl_2(K))$  is the minimal ideal of weak polynomial identities containing the element  $[x_1^2, x_2]$ . An equivalent description is that as an ideal of the free algebra,  $I(M_2(K), sl_2(K))$  is generated by all elements  $[uv + vu, w]$ , where  $u, v, w$  are all possible commutators  $[y_{j_1}, \dots, y_{j_k}]$  in the variables  $y_1, \dots, y_d$ .*
- (iii) (Drensky and Koshlukov [19]) *The ideal  $I(M_2(K), sl_2(K))$  is the minimal ideal of the free associative algebra  $K\langle y_1, \dots, y_d \rangle$  which is invariant under the diagonal action of  $GL_d$  and contains the elements  $[y_1^2, y_2]$  and*

$$s_4(y_1, y_2, y_3, y_4) = \sum_{\sigma \in S_4} \operatorname{sign}(\sigma) y_{\sigma(1)} y_{\sigma(2)} y_{\sigma(3)} y_{\sigma(4)},$$

the second polynomial appears for  $d \geq 4$  only. Hence the algebra of  $2 \times 2$  generic traceless matrices has a uniform set of defining relations for any  $d \geq 2$ .

- (iv) (Procesi [50]) *As a  $GL_d$ -module  $W_d$  has the description*

$$W_d \cong \bigoplus W(\lambda_1, \lambda_2, \lambda_3),$$

where the sum is on all partitions  $\lambda$  in at most three parts.

- (v) (Formanek [23]) *The  $GL_d$ -module  $T_{2d}$  has the decomposition*

$$T_{2d} \cong \bigoplus (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1) W(\lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

- (vi) (See [11, 12, 23, 50, 38, 39] for other descriptions of the Hilbert series.)  
*The Hilbert series of  $T_{2d}$  is*

$$H(T_{2d}, t_1, \dots, t_d) = \prod_{i=1}^d \frac{1}{1-t_i} \sum S_{(\lambda_1, \lambda_2, \lambda_3)}(t_1, \dots, t_d).$$

Now we give some results on  $C_{2d}$ . We follow the way we used in the previous theorem. For more details, especially for the Hilbert series and the  $GL_d$ -module decomposition of  $C_{2d}$ , see [23, 39, 50] or [14].

**Theorem 3.3.**

- (i) *The commutative trace algebra  $C_{2d}$  is isomorphic to the tensor product of  $K$ -algebras*

$$K[\operatorname{tr}(X_1), \dots, \operatorname{tr}(X_m)] \otimes_K C(W_d),$$

*where  $C(W_d)$  is the centre of the algebra  $W_d$  defined above.*

- (ii) *As a subalgebra of  $W_d$ , its centre  $C(W_d)$  is generated by*

$$Y_i^2, \quad i = 1, \dots, d, \quad Y_i Y_j + Y_j Y_i, \quad 1 \leq i < j \leq d,$$

$$s_3(Y_i, Y_j, Y_k) = \sum_{\sigma \in S_3} \operatorname{sign}(\sigma) Y_{\sigma(1)} Y_{\sigma(2)} Y_{\sigma(3)}, \quad 1 \leq i < j < k \leq d,$$

*where the symmetric group  $S_3$  acts on  $\{i, j, k\}$ .*

- (iii) *As a  $GL_d$ -module  $W_d$  has the decomposition*

$$W_d \cong \bigoplus W(\lambda_1, \lambda_2, \lambda_3),$$

*where the sum is on all partitions  $\lambda$  in at most three parts such that both  $\lambda_1 - \lambda_2$  and  $\lambda_2 - \lambda_3$  are even.*

Concerning the defining relations of  $C_{2d}$ , the case  $d = 2$  is trivial. Formanek, Halpin and Li [28] showed that  $C_{22}$  is generated by the algebraically independent elements

$$\operatorname{tr}(X_1), \operatorname{tr}(X_2), \det(X_1), \det(X_2), \operatorname{tr}(X_1 X_2).$$

For  $d = 3$  Sibirskii [57] found one relation between the generators of  $C_{23}$  and, using the Hilbert series of  $C_{23}$ , Formanek [23] proved that there are no more relations. In the general case, the description of the defining relations of  $C_{2d}$

is reduced to a similar description of the defining relations of the subalgebra of  $C_{2d}$  generated by

$$\mathrm{tr}(Y_i^2), \quad i = 1, \dots, d, \quad \mathrm{tr}(Y_i Y_j), \quad 1 \leq i < j \leq d,$$

$$\mathrm{tr}(Y_i Y_j Y_k), \quad 1 \leq i < j < k \leq d,$$

where  $Y_1, \dots, Y_d$  are generic traceless  $2 \times 2$  matrices. Since  $GL_2$  acts on the generic matrices by conjugation, we may replace its action with the action of  $SL_2$  and even with the action of  $PSL_2$ . Since  $PSL_2(\mathbb{C}) \cong SO_3(\mathbb{C})$ , the special orthogonal group, we may apply invariant theory of special orthogonal groups. (The restriction  $K = \mathbb{C}$  is not essential in the final version of the result.)

We consider the action of the special orthogonal group  $SO_3 = SO_3(K)$ , i.e. the group of orthogonal  $3 \times 3$  matrices with determinant 1, on the polynomial algebra in  $3d$  variables

$$K[u_i^{(1)}, u_i^{(2)}, u_i^{(3)} \mid i = 1, \dots, d],$$

induced by the action of  $SO_3$  on the 3-dimensional vectors  $u_i = (u_i^{(1)}, u_i^{(2)}, u_i^{(3)})$ . It is a classical result that the algebra of invariants  $K[u_i^{(j)}]^{SO_3}$  is generated by all scalar products

$$\langle u_i, u_j \rangle = u_i^{(1)} u_j^{(1)} + u_i^{(2)} u_j^{(2)} + u_i^{(3)} u_j^{(3)}, \quad 1 \leq i \leq j \leq d,$$

and all  $3 \times 3$  determinants of the coordinates

$$\Delta(u_i, u_j, u_k) = \begin{vmatrix} u_i^{(1)} & u_j^{(1)} & u_k^{(1)} \\ u_i^{(2)} & u_j^{(2)} & u_k^{(2)} \\ u_i^{(3)} & u_j^{(3)} & u_k^{(3)} \end{vmatrix}, \quad 1 \leq i < j < k \leq d.$$

The defining relations express the fact that the underlying vector space is three-dimensional and every four vectors are linearly dependent. In particular, they use the properties of the Gram determinant:

$$\Gamma_4(u_i, u_j, u_k, u_l; u_p, u_q, u_r, u_s) = \begin{vmatrix} \langle u_i, u_p \rangle & \langle u_i, u_q \rangle & \langle u_i, u_r \rangle & \langle u_i, u_s \rangle \\ \langle u_j, u_p \rangle & \langle u_j, u_q \rangle & \langle u_j, u_r \rangle & \langle u_j, u_s \rangle \\ \langle u_k, u_p \rangle & \langle u_k, u_q \rangle & \langle u_k, u_r \rangle & \langle u_k, u_s \rangle \\ \langle u_l, u_p \rangle & \langle u_l, u_q \rangle & \langle u_l, u_r \rangle & \langle u_l, u_s \rangle \end{vmatrix} = 0,$$

$$1 \leq i < j < k < l \leq d, \quad 1 \leq p < q < r < s \leq d,$$

$$\Delta(u_i, u_j, u_k) \Delta(u_p, u_q, u_r) - \Gamma_3(u_i, u_j, u_k; u_p, u_q, u_r) = 0,$$

$$\begin{aligned} & \langle u_p, u_i \rangle \Delta(u_j, u_k, u_l) - \langle u_p, u_j \rangle \Delta(u_i, u_k, u_l) \\ & + \langle u_p, u_k \rangle \Delta(u_i, u_j, u_l) - \langle u_p, u_l \rangle \Delta(u_i, u_j, u_k) = 0. \end{aligned}$$

In order to apply invariant theory of  $SO_3$  we need a scalar product (i.e. nondegenerate symmetric bilinear form) on  $sl_2(K)$ . We use the trace and define

$$\langle u, v \rangle = \text{tr}(uv), \quad u, v \in sl_2(K).$$

The following result gives the generators and the defining relations of  $C_{2d}$  for  $d \geq 2$ . It is a translation of the description of the invariants of  $SO_3$ .

**Theorem 3.4.**

(i) *The algebra  $C_{2d}$  is generated by*

$$\text{tr}(X_i), \quad \text{tr}(Y_i^2), \quad \text{tr}(Y_i Y_j), \quad \text{tr}(s_3(Y_i, Y_j, Y_k)),$$

where  $i, j, k = 1, \dots, d$ , and in the traces involving two or three traceless matrices we require  $i < j$  or  $i < j < k$ , respectively.

(ii) Drensky [13]) *The defining relations of  $C_{2d}$  with respect to the above generators are*

$$\text{tr}(s_3(Y_i, Y_j, Y_k)) \text{tr}(s_3(Y_p, Y_q, Y_r)) + 18 \begin{vmatrix} \text{tr}(Y_i Y_p) & \text{tr}(Y_i Y_q) & \text{tr}(Y_i Y_r) \\ \text{tr}(Y_j Y_p) & \text{tr}(Y_j Y_q) & \text{tr}(Y_j Y_r) \\ \text{tr}(Y_k Y_p) & \text{tr}(Y_k Y_q) & \text{tr}(Y_k Y_r) \end{vmatrix} = 0,$$

$$\begin{aligned} & \text{tr}(Y_p Y_i) \text{tr}(s_3(Y_j, Y_k, Y_l)) - \text{tr}(Y_p Y_j) \text{tr}(s_3(Y_i, Y_k, Y_l)) \\ & + \text{tr}(Y_p Y_k) \text{tr}(s_3(Y_i, Y_j, Y_l)) - \text{tr}(Y_p Y_l) \text{tr}(s_3(Y_i, Y_j, Y_k)) = 0, \end{aligned}$$

where, again,  $i, j, k, p, q, r = 1, \dots, d$ , and, where necessary, we require  $i < j < k < l$  and  $p < q < r$ .

In order to work efficiently with an algebra  $R = K[x_1, \dots, x_p]/I$ , it is not sufficient to know the generators of the ideal  $I$ . For computational purposes one needs also the *Gröbner basis* of  $I$  with respect to some ordering on the monomials of  $K[x_1, \dots, x_p]$ , see e.g. the book by Adams and Loustaunau [2]. The Gröbner basis of  $C_{2d}$  is given by Domokos and Drensky [10], see their paper for more details.

Now we consider two generic  $3 \times 3$  matrices. Using the Molien-Weyl formula, Teranishi [62] calculated the Hilbert series of  $C_{32}$ , namely,

$$H(C_{32}, t_1, t_2) = \frac{1 + t_1^3 t_2^3}{(1 - t_1)(1 - t_2) q_2(t_1, t_2) q_3(t_1, t_2) (1 - t_1^2 t_2^2)},$$

where

$$q_2(t_1, t_2) = (1 - t_1^2)(1 - t_1 t_2)(1 - t_2^2),$$

$$q_3(t_1, t_2) = (1 - t_1^3)(1 - t_1^2 t_2)(1 - t_1 t_2^2)(1 - t_2^3).$$

He also found the following system of generators of  $C_{32}$ :

$$\begin{aligned} &\text{tr}(X_1), \text{tr}(X_2), \text{tr}(X_1^2), \text{tr}(X_1 X_2), \text{tr}(X_2^2), \\ &\text{tr}(X_1^3), \text{tr}(X_1^2 X_2), \text{tr}(X_1 X_2^2), \text{tr}(X_2^3), \text{tr}(X_1^2 X_2^2), \text{tr}(X_1^2 X_2^2 X_1 X_2), \end{aligned}$$

where  $X_1, X_2$  are generic  $3 \times 3$  matrices. He showed that the first ten of these generators form a homogeneous system of parameters of  $C_{32}$  and  $C_{32}$  is a free module with generators 1 and  $\text{tr}(X_1^2 X_2^2 X_1 X_2)$  over the polynomial algebra on these ten elements.

Abeasis and Pittaluga [1] found a system of generators of  $C_{3d}$  in terms of representation theory of the symmetric and general linear groups, in the spirit of its use in theory of PI-algebras. They showed that  $C_{3d}$  has a minimal system of generators which spans a  $GL_d$ -module isomorphic to

$$\begin{aligned} G = &W(1) \oplus W(2) \oplus W(3) \oplus W(1^3) \oplus W(2^2) \oplus W(2, 1^2) \\ &\oplus W(3, 1^2) \oplus W(2^2, 1) \oplus W(1^5) \oplus W(3^2) \oplus W(3, 1^3). \end{aligned}$$

(The partitions in [1] are given in “Francophone” way, i.e. transposed to ours.)

It follows from the description of the generators of  $C_{32}$  given by Teranishi [62], that  $\text{tr}(X_1^2 X_2^2 X_1 X_2)$  satisfies a quadratic equation with coefficients depending on the other ten generators. The explicit (but very complicated) form of the equation was found by Nakamoto [45], over  $\mathbb{Z}$ , with respect to a slightly different system of generators. A much simpler description of  $C_{32}$  was obtained by Aslaksen, Drensky, and Sadikova [3]. The following generators are in the spirit of the ideas of [1].

**Proposition 3.5.** *Let  $X_1, X_2$  and  $Y_1, Y_2$  be, respectively, two generic and two generic traceless  $3 \times 3$  matrices. The algebra  $C_{32}$  is generated by*

$$\begin{aligned} &\text{tr}(X_1), \text{tr}(X_2), \text{tr}(Y_1^2), \text{tr}(Y_1 Y_2), \text{tr}(Y_2^2), \\ &\text{tr}(Y_1^3), \text{tr}(Y_1^2 Y_2), \text{tr}(Y_1 Y_2^2), \text{tr}(Y_2^3), \end{aligned}$$

$$V = \text{tr}(Y_1^2 Y_2^2) - \text{tr}(Y_1 Y_2 Y_1 Y_2), \quad W = \text{tr}(Y_1^2 Y_2^2 Y_1 Y_2) - \text{tr}(Y_2^2 Y_1^2 Y_2 Y_1).$$

Now we define the following elements of  $C_{32}$ :

$$\begin{aligned}
 U &= \begin{vmatrix} \text{tr}(Y_1^2) & \text{tr}(Y_1 Y_2) \\ \text{tr}(Y_1 Y_2) & \text{tr}(Y_2^2) \end{vmatrix}, \\
 W_1 &= U^3, \quad W_2 = U^2 V, \quad W_4 = U V^2, \quad W_7 = V^3, \\
 W_5 &= V \begin{vmatrix} \text{tr}(Y_1^2) & \text{tr}(Y_1 Y_2) & \text{tr}(Y_2^2) \\ \text{tr}(Y_1^3) & \text{tr}(Y_1^2 Y_2) & \text{tr}(Y_1 Y_2^2) \\ \text{tr}(Y_1^2 Y_2) & \text{tr}(Y_1 Y_2^2) & \text{tr}(Y_2^3) \end{vmatrix}, \\
 W_6 &= \begin{vmatrix} \text{tr}(Y_1^3) & \text{tr}(Y_1 Y_2^2) \\ \text{tr}(Y_1^2 Y_2) & \text{tr}(Y_2^3) \end{vmatrix}^2 - 4 \begin{vmatrix} \text{tr}(Y_2^3) & \text{tr}(Y_1 Y_2^2) \\ \text{tr}(Y_1 Y_2^2) & \text{tr}(Y_1^2 Y_2) \end{vmatrix} \begin{vmatrix} \text{tr}(Y_1^3) & \text{tr}(Y_1^2 Y_2) \\ \text{tr}(Y_1^2 Y_2) & \text{tr}(Y_1 Y_2^2) \end{vmatrix}, \\
 W_3' &= U \begin{vmatrix} \text{tr}(Y_1^2) & \text{tr}(Y_1 Y_2) & \text{tr}(Y_2^2) \\ \text{tr}(Y_1^3) & \text{tr}(Y_1^2 Y_2) & \text{tr}(Y_1 Y_2^2) \\ \text{tr}(Y_1^2 Y_2) & \text{tr}(Y_1 Y_2^2) & \text{tr}(Y_2^3) \end{vmatrix},
 \end{aligned}$$

where  $U, V$  are defined above. Finally, we define one more element  $W_3''$  as follows. Recall that a linear mapping  $\delta$  of an algebra  $R$  is a *derivation* if  $\delta(rs) = \delta(r)s + r\delta(s)$  for all  $r, s \in R$ . We consider the derivation  $\delta$  of  $C_{32}$  which commutes with the trace and satisfies the conditions

$$\delta(X_1) = 0, \quad \delta(X_2) = X_1, \quad \delta(Y_1) = 0, \quad \delta(Y_2) = Y_1.$$

Then

$$W_3'' = \frac{1}{144} \sum_{i=0}^6 (-1)^i \delta^i(\text{tr}^3(Y_2^2)) \delta^{6-i}(\text{tr}^2(Y_2^3)).$$

The following theorem gives the defining relation of  $C_{32}$ . It uses representation theory of  $GL_2$ , combinatorics, computations by hand and easy computer calculations with standard functions of Maple.

**Theorem 3.6.** (Aslaksen, Drensky, Sadikova [3]) *The algebra of invariants  $C_{32}$  of two  $3 \times 3$  matrices is generated by the elements from the previous theorem, subject to the defining relation*

$$W^2 - \left( \frac{1}{27} W_1 - \frac{2}{9} W_2 + \frac{4}{15} W_3' + \frac{1}{90} W_3'' + \frac{1}{3} W_4 - \frac{2}{3} W_5 - \frac{1}{3} W_6 - \frac{4}{27} W_7 \right) = 0.$$

The calculation of the Hilbert series of  $C_{nd}$  and  $T_{nd}$  based on the Molien-Weyl formula is quite complicated because requires evaluations of multiple integrals. Van den Bergh [67] subjected a way which involves graph theory. As

a consequence, he established important properties of  $H(C_{2d}, t_1, \dots, t_d)$  and  $H(T_{2d}, t_1, \dots, t_d)$ .

Berele and Stembridge [7] applied the method of van den Bergh [67] and calculated the Hilbert series of  $T_{32}$ . Using the above results of Aslaksen, Drensky, and Sadikova on  $C_{32}$  and the explicit form of the Hilbert series of  $T_{32}$ , Benanti and Drensky [4] found a polynomial subalgebra  $S$  of  $C_{32}$  and a finite set of generators of the free  $S$ -module  $T_{32}$ . They gave also a set of defining relations of  $T_{32}$  as an algebra and a Gröbner basis of the corresponding ideal. (See the survey article by Ufnarovski [65] for a background on Gröbner bases in the noncommutative case, as well as the paper by Mikhalev and Zolotykh [41] which is closer to the situation in [4].)

For two generic  $4 \times 4$  matrices, the Hilbert series of  $C_{42}$  was calculated (with some typos) by Teranishi [62, 63] and corrected by Berele and Stembridge [7]. Teranishi found also a homogeneous system of parameters and a system of generators, in the spirit of the  $3 \times 3$  case. Recently, Drensky and Sadikova [20] have found another system of generators of  $C_{42}$  which is minimal and seems to be more convenient for concrete calculations.

The Hilbert series of a graded vector space with  $GL_d$ -module structure determines uniquely its decomposition into irreducible submodules. Hence, in principle, one may calculate the multiplicities  $m(\lambda)$  if one knows the concrete form of the Hilbert series. Berele [5] used the Hilbert series of  $C_{32}$  found by Teranishi [62] and described the asymptotics of  $m(\lambda_1, \lambda_2)$ . (Due to a technical error (an omitted summand) some of the coefficients of the polynomials in the asymptotics of Berele are slightly different from the real ones.) Another approach to the problem was suggested by Drensky and Genov [15]. Let

$$f(t_1, t_2) = \sum_{i, j \geq 0} a_{ij} t_1^i t_2^j,$$

$a_{ij} \in K$ ,  $a_{ij} = a_{ji}$ , be a symmetric function in two variables which is a formal power series from  $K[[t_1, t_2]]$ . We present it in the form

$$f(t_1, t_2) = \sum_{\lambda_1 \geq \lambda_2} m(\lambda_1, \lambda_2) S_{(\lambda_1, \lambda_2)}(t_1, t_2)$$

and want to find the multiplicities  $m(\lambda_1, \lambda_2)$ . In most of the cases which we consider,  $f(t_1, t_2)$  is given explicitly as a rational function. So, it is natural to express  $m(\lambda_1, \lambda_2)$  not in terms of the coefficients  $a_{ij}$  but in a more direct way. We introduce the generating function of the multiplicities

$$M(f, t, u) = \sum_{\lambda_1 \geq \lambda_2} m(\lambda_1, \lambda_2) t^{\lambda_1} u^{\lambda_2} \in K[[t, u]]$$



and call it the *multiplicity series* of  $f(t_1, t_2)$ . It is more convenient to introduce a new variable  $v = tu$  and to consider the series

$$M'(f, t, v) = \sum_{\lambda_1 \geq \lambda_2} m(\lambda_1, \lambda_2) t^{\lambda_1 - \lambda_2} v^{\lambda_2} \in K[[t, v]],$$

because the mapping  $M' : K[[t_1, t_2]]^{S_2} \rightarrow K[[t, v]]$  is a bijective linear mapping which is continuous with respect to the *formal power series topology*. It is easy to see that  $f(t_1, t_2)$  and  $M'(f, t, v)$  are related by

$$f(t_1, t_2) = \frac{t_1 M'(f, t_1, t_1 t_2) - t_2 M'(f, t_2, t_1 t_2)}{t_1 - t_2}.$$

Hence, if we have a potential candidate  $h(t, v)$  for  $M'(f, t, v)$ , it is easy to verify whether  $h(t, v) = M'(f, t, v)$ .

Also, the elementary symmetric function  $e_2 = t_1 t_2$  behaves like a constant,

$$M'(g(t_1 t_2) f(t_1, t_2), t, v) = g(v) M'(f, t, v),$$

and this simplifies the calculations. Applying quite complicated (also technically) arguments, Drensky and Genov [15] found the multiplicity series of the Hilbert series of  $C_{32}$ . They also corrected the technical errors in [5].

**Theorem 3.7.**

- (i) [15] *The multiplicity series of the Hilbert series of the algebra  $C_{32}$  of invariants of two  $3 \times 3$  matrices is*

$$\begin{aligned} M'(H(C_{32}, t_1, t_2), t, v) &= \frac{1}{(1 - v^2)(1 - v^3)^2} \times \\ &\times \left( \frac{(1 + v^2 + v^4)((1 + v^2)(1 - t^2 v) + 2tv(1 - v))}{3(1 - v)(1 - v^2)^3(1 - t)^2(1 - t^2)} \right. \\ &+ \frac{(1 - v)(1 + tv)}{3(1 - v^2)(1 - t)(1 - t^2)} + \frac{(1 - v^2)(1 - tv)}{3(1 - v^3)(1 - t^3)} \\ &\left. - \frac{v^3((1 - v + v^2)(1 - t^2 v^2) + tv(1 - v^2))}{(1 - v)(1 - v^2)^2(1 - v^4)(1 - t)(1 - t^2)(1 - tv)} \right). \end{aligned}$$

- (ii) [5, 15] *Let  $\lambda = (p, q)$  and let  $m(p, q)$  be the multiplicity of  $S_{(p, q)}(t_1, t_2)$  in  $H(C_{32}, t_1, t_2)$ . Then for  $p > 2q \geq 0$*

$$m(p, q) = \frac{q^7}{7!2^5 \cdot 3^2} + \frac{(p - q)q^6}{6!2^4 \cdot 3^2} + \frac{(p - q)^2 q^5}{2!5!2^3 3^2} + \mathcal{O}((p + q)^6)$$

$$\begin{aligned}
&= \frac{p^2 q^5}{17280} - \frac{11pq^6}{103680} + \frac{71q^7}{1451520} + \mathcal{O}((p+q)^6); \\
&\text{for } 2q \geq p \geq q \geq 0 \\
m(p, q) &= \frac{q^7}{7!2^5 \cdot 3^2} + \frac{(p-q)q^6}{6!2^4 \cdot 3^2} + \frac{(p-q)^2 q^5}{2!5!2^3 3^2} - \frac{(2q-p)^7}{7!2^5 \cdot 3^2} + \mathcal{O}((p+q)^6) \\
&= \frac{p^7}{1451520} - \frac{p^6 q}{103680} + \frac{p^5 q^2}{17280} - \frac{p^4 q^3}{5184} + \frac{p^3 q^4}{2592} \\
&\quad - \frac{7p^2 q^5}{17280} + \frac{7pq^6}{34560} - \frac{19q^7}{483840} + \mathcal{O}((p+q)^6).
\end{aligned}$$

Later the methods for calculating the multiplicity series of symmetric functions of special kinds were significantly improved [16]. The Hilbert series of  $C_{42}$  calculated by Teranishi [62, 63] (with some typos corrected in [7]) and the Hilbert series of  $T_{32}$  and  $T_{42}$  calculated by Berele and Stembridge [7] allowed to express their multiplicity series and to determine the asymptotics of the multiplicities. We shall state simplified versions of the results:

**Theorem 3.8.**

- (i) (Drensky, Genov, Valenti [18]) *The multiplicities  $m_{(\lambda_1, \lambda_2)}(C_{32})$  and  $m_{(\lambda_1, \lambda_2)}(T_{32})$  of the Hilbert series of  $C_{32}$  and  $T_{32}$ , respectively, are related by*

$$m_{(\lambda_1, \lambda_2)}(T_{32}) \approx 9m_{(\lambda_1, \lambda_2)}(C_{32}).$$

- (ii) (Drensky and Genov [17]) *Let  $\lambda = (\lambda_1, \lambda_2)$ . The multiplicities  $m_\lambda(C_{42})$  of the Hilbert series of  $C_{42}$  satisfy the condition*

$$m_\lambda(C_{42}) = \begin{cases} m_1 + \mathcal{O}((\lambda_1 + \lambda_2)^{13}), & \text{if } \lambda_1 > 3\lambda_2, \\ m_1 + m_2 + \mathcal{O}((\lambda_1 + \lambda_2)^{13}), & \text{if } 3\lambda_2 \geq \lambda_1 > 2\lambda_2, \\ m_1 + m_2 + m_3 + \mathcal{O}((\lambda_1 + \lambda_2)^{13}), & \text{if } 2\lambda_2 \geq \lambda_1, \end{cases}$$

where

$$\begin{aligned}
m_1 &= \frac{(\lambda_1 - \lambda_2)^3 \lambda_2^{11}}{11!3!2^8 3^2} - \frac{(\lambda_1 - \lambda_2)^2 \lambda_2^{12}}{12!2!2^8 3^3} + \frac{127(\lambda_1 - \lambda_2) \lambda_2^{13}}{13!2^{10} 3^4} - \frac{305 \lambda_2^{14}}{14!2^9 3^5}, \\
m_2 &= \frac{(3\lambda_2 - \lambda_1)^{14}}{14!2^{10} 3^5 5^2}, \\
m_3 &= -\frac{(\lambda_1 - \lambda_2)(2\lambda_2 - \lambda_1)^{13}}{13!2^{10} 3^2 5} - \frac{7(2\lambda_2 - \lambda_1)^{14}}{14!2^9 3 \cdot 5^2}.
\end{aligned}$$

The multiplicities  $m_\lambda(T_{42})$  satisfy

$$m_\lambda(T_{42}) = 16m_\lambda(C_{42}) + \mathcal{O}((\lambda_1 + \lambda_2)^{13}).$$

We want to mention that Berele and Stembridge [7] computed also the Hilbert series of  $C_{33}$  and  $T_{33}$  but the methods of [15, 16, 17, 18] do not work successfully for symmetric functions in three variables. One can introduce the multiplicity series of a symmetric function in any number of variables, generalizing in an obvious way the case of symmetric functions in two variables. A recent theorem of Berele [6] gives the rationality of the multiplicity series of a class of rational symmetric functions in any number of variables, including the Hilbert series of  $C_{nd}$  and  $T_{nd}$ . Unfortunately, it is not clear how to perform the concrete calculations, even for the Hilbert series of  $C_{33}$  and  $T_{33}$ .

## References

- [1] S. Abeasis, M. Pittaluga, On a minimal set of generators for the invariants of  $3 \times 3$  matrices, *Commun. Algebra*, **17** (1989), 487-499.
- [2] W.W. Adams, P. Lounstau, *An Introduction to Gröbner Bases*, Graduate Studies in Math. **3**, AMS, Providence, R.I., 1994.
- [3] H. Aslaksen, V. Drensky, L. Sadikova, Defining relations of invariants of two  $3 \times 3$  matrices, *J. Algebra*, **298** (2006), No. 1, 41-57. Announcement: C.R. Acad. Bulg. Sci. **58** (2005), No. 6, 617-622.
- [4] F. Benanti, V. Drensky, Defining relations of noncommutative trace algebra of two  $3 \times 3$  matrices, *Adv. Appl. Math.*, **37** (2006), No. 2, 162-182.
- [5] A. Berele, Approximate multiplicities in the trace cocharacter sequence of two three-by-three matrices, *Commun. Algebra*, **25** (1997), 1975-1983.
- [6] A. Berele, Applications of Belov's theorem to the cocharacter sequence of P.I. algebras, *J. Algebra*, **298** (2006), No. 1, 208-214.
- [7] A. Berele, J.R. Stembridge, Denominators for the Poincaré series of invariants of small matrices, *Israel J. Math.*, **114** (1999), 157-175.
- [8] C. Chevalley, Invariants of finite groups generated by reflections, *Amer. J. Math.*, **67** (1955), 778-782.
- [9] J.A. Dieudonné, J.B. Carrell, *Invariant Theory, Old and New*, Academic Press, New York-London, 1971.
- [10] M. Domokos, V. Drensky, Gröbner bases for the rings of invariants of special orthogonal and  $2 \times 2$  matrix invariants, *J. Algebra*, **243** (2001), 706-716.

- [11] V. Drensky, Codimensions of T-ideals and Hilbert series of relatively free algebras, *J. Algebra*, **91** (1984), 1-17.
- [12] V. Drensky, *Free Algebras and PI-Algebras*, Springer-Verlag, Singapore, 1999.
- [13] V. Drensky, Defining relations for the algebra of invariants of  $2 \times 2$  matrices, *Algebras and Representation Theory*, **6** (2003), 193-214.
- [14] V. Drensky, E. Formanek, *Polynomial Identity Rings*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, Basel-Boston, 2004.
- [15] V. Drensky, G.K. Genov, Multiplicities of Schur functions in invariants of two  $3 \times 3$  matrices, *J. Algebra*, **264** (2003), No. 2, 496-519.
- [16] V. Drensky, G.K. Genov, Multiplicities of Schur functions with applications to invariant theory and PI-algebras, *C.R. Acad. Bulg. Sci.* **57** (2004), No. 3, 5-10.
- [17] V. Drensky, G.K. Genov, Multiplicities in the trace cocharacter sequence of two  $4 \times 4$  matrices, *Mediterr. J. Math.* **2** (2005), 231-241.
- [18] V. Drensky, G.K. Genov, A. Valenti, Multiplicities in the mixed trace cocharacter sequence of two  $3 \times 3$  matrices, *International J. Algebra and Computations*, **16** (2006), No. 2, 275-285.
- [19] V. Drensky, P. Koshlukov, Weak polynomial identities for a vector space with a symmetric bilinear form, in "Math. and Education in Math. Proc. of 16-th Spring Conf. of the Union of Bulg. Mathematicians", Sofia, Publ. House of the Bulg. Acad. of Sci., 1987, 213-219.
- [20] V. Drensky, L. Sadikova, Generators of invariants of two  $4 \times 4$  matrices, *C.R. Acad. Bulg. Sci.* **59** (2006), No. 5, 477-484. Preprint, <http://xxx.lanl.gov/abs/math.RA/0503146>.
- [21] J. Dubnov, Sur une généralisation de l'équation de Hamilton-Cayley et sur les invariants simultanés de plusieurs affineurs, *Proc. Seminar on Vector and Tensor Analysis*, Mechanics Research Inst., Moscow State Univ. **2/3** (1935), 351-367 (see also *Zbl. für Math.* **12** (1935), p. 176).
- [22] J. Dubnov, V. Ivanov, Sur l'abaissement du degré des polynômes en affineurs, *C.R. (Doklady) Acad. Sci. USSR*, **41** (1943), 96-98. (See also MR **6** (1945), p. 113, *Zbl. für Math.* **60** (1957), p. 33).

- [23] E. Formanek, Invariants and the ring of generic matrices, *J. Algebra*, **89** (1984), 178-223.
- [24] E. Formanek, *Generating the Ring of Matrix Invariants*, Lect. Notes in Math. **1195**, Springer Verlag, Berlin-Heidelberg-New York, 1986, 73-82.
- [25] E. Formanek, Functional equations for character series associated with  $n \times n$  matrices, *Trans. Amer. Math. Soc.* **294** (1986), 647-663.
- [26] E. Formanek, The Polynomial Identities and Invariants of  $n \times n$  Matrices, *CBMS Regional Conf. Series in Math.* **78**, Published for the Confer. Board of the Math. Sci. Washington DC, AMS, Providence RI, 1991.
- [27] E. Formanek, The ring of generic matrices, *J. Algebra*, **258** (2002), 310-320.
- [28] E. Formanek, P. Halpin, W.-C.W. Li, The Poincaré series of the ring of  $2 \times 2$  generic matrices, *J. Algebra*, **69** (1981), 105-112.
- [29] G. Freudenburg, A survey of counterexamples to Hilbert's fourteenth problem, *Serdica Math. J.* **27** (2001), 171-192.
- [30] B. G. Gurevich, *Foundations of the Theory of Algebraic Invariants*, Noordhoff, Groningen, 1964.
- [31] G. Higman, On a conjecture of Nagata, *Proc. Camb. Philos. Soc.* **52** (1956), 1-4.
- [32] D. Hilbert, Über die Theorie der algebraischen Formen, *Math. Ann.* **36** (1890), 473-534. Reprinted in "*Gesammelte Abhandlungen, Band II, Algebra, Invariantentheorie, Geometrie*", Zweite Auflage, Springer-Verlag, Berlin-Heidelberg-New York, 1970, 199-257.
- [33] D. Hilbert, Mathematische Probleme, *Archiv f. Math. u. Phys.* **1** (1901), 44-63, 213-237. Reprinted in "*Gesammelte Abhandlungen, Band III, Analysis, Grundlagen der Mathematik, Physik, Verschiedenes, Lebensgeschichte*", Zweite Auflage, Springer-Verlag, Berlin-Heidelberg-New York, 1970, 290-329.
- [34] N. Jacobson, *PI-Algebras: An Introduction*, Lecture Notes in Math. **441**, Springer-Verlag, Berlin-New York, 1975.
- [35] A. Kanel-Belov, L.H. Rowen, *Computational Aspects of Polynomial Identities*, Research Notes in Mathematics **9**, A. K. Peters, Wellesley, MA, 2005.

- [36] A. A. Kirillov, Certain division algebras over a field of rational functions (Russian), *Funkts. Anal. Prilozh.* **1** (1967), No. 1, 101-102. Translation: *Funct. Anal. Appl.* **1** (1967), 87-88.
- [37] E.N. Kuzmin, On the Nagata-Higman theorem (Russian), in “*Mathematical Structures, Computational Mathematics, Mathematical Modelling*. Proc. Dedicated to the 60th Birthday of Acad. L. Iliev”, Sofia, 1975, 101-107.
- [38] L. Le Bruyn, The functional equation for Poincaré series of trace rings of generic  $2 \times 2$  matrices, *Israel J. Math.* **52** (1985), 355-360.
- [39] L. Le Bruyn, *Trace Rings of Generic 2 by 2 Matrices*, Memoirs of AMS, **66**, No. 363, Providence, R.I., 1987.
- [40] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Univ. Press (Clarendon), Oxford, 1979. Second Edition, 1995.
- [41] A.A. Mikhalev, A.A. Zolotykh, Standard Gröbner-Shirshov bases of free algebras over rings. I: Free associative algebras, *Int. J. Algebra Comput.* **8** (1998), No. 6, 689-726.
- [42] T. Molien, Über die Invarianten der linearen Substitutionsgruppen, *Sitz. König Preuss. Akad. Wiss.* (1897), N 52, 1152-1156.
- [43] M. Nagata, On the nilpotency of nil algebras, *J. Math. Soc. Japan*, **4** (1953), 296-301.
- [44] M. Nagata, On the 14th problem of Hilbert, *Amer. J. Math.* **81** (1959), 766-772.
- [45] K. Nakamoto, The structure of the invariant ring of two matrices of degree 3, *J. Pure Appl. Algebra*, **166** (2002), No. 1-2, 125-148.
- [46] E. Noether, Der Endlichkeitssatz der Invarianten endlicher Gruppen, *Math. Ann.* **77** (1916), 89-92. Reprinted in “*Gesammelte Abhandlungen. Collected Papers*”, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983, 181-184.
- [47] C. Procesi, *Rings with Polynomial Identities*, Marcel Dekker, New York, 1973.
- [48] C. Procesi, The invariant theory of  $n \times n$  matrices, *Adv. Math.* **19** (1976), 306-381.

- [49] C. Procesi, Trace identities and standard diagrams, in “*Ring Theory*”, Lect. Notes in Math. **51**, Marcel Dekker, New York, 1979, 191-218.
- [50] C. Procesi, Computing with  $2 \times 2$  matrices, *J. Algebra*, **87** (1984), 342-359.
- [51] Yu. P. Razmyslov, Finite basing of the identities of a matrix algebra of second order over a field of characteristic 0 (Russian), *Algebra i Logika*, **12** (1973), 83-113. Translation: *Algebra and Logic*, **12** (1973), 43-63.
- [52] Yu. P. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero (Russian), *Izv. Akad. Nauk SSSR, Ser. Mat.* **38** (1974), 723-756. Translation: *Math. USSR, Izv.* **8** (1974), 727-760.
- [53] Yu. P. Razmyslov, *Identities of Algebras and Their Representations* (Russian), “Sovremennaya Algebra”, “Nauka”, Moscow, 1989. Translation: *Translations of Math. Monographs*, **138**, AMS, Providence, R.I., 1994.
- [54] A. Regev, On the codimensions of matrix algebras, in “*Algebra – Some Current Trends (Varna, 1986)*”, Lect. Notes in Math. **1352**, Springer-Verlag, Berlin-New York, 1988, 162-172.
- [55] G.C. Shephard, J.A. Todd, Finite unitary reflection groups, *Canad. J. Math.* **6** (1954), 274-304.
- [56] I.P. Shestakov, N. Zhukavets, On associative algebras satisfying the identity  $x^5 = 0$ , *Algebra Discrete Math.* (2004) No. 1, 112-120.
- [57] K. S. Sibirskii, Algebraic invariants for a set of matrices (Russian), *Sib. Mat. Zh.* **9** (1968), No. 1, 152-164. Translation: *Siber. Math. J.* **9** (1968), 115-124.
- [58] T.A. Springer, *Invariant Theory*, Lect. Notes in Math. **585**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
- [59] R. P. Stanley, Combinatorics and invariant theory, in “*Relations between Combinatorics and Other Parts of Mathematics*”, Ed. D. K. Ray-Chaudhuri, Proc. Sympos. Pure Math., **34**, AMS, Providence, RI, 1979, 345-355.
- [60] R.P. Stanley, *Combinatorics and Commutative Algebra*, Progress in Math. **41**, Birkhäuser Boston, Boston, MA, 1983, Second Edition, 1996.

- [61] B. Sturmfels, *Algorithms in Invariant Theory, Texts and Monographs in Symbolic Computation*, Springer-Verlag, Wien, 1993.
- [62] Y. Teranishi, The ring of invariants of matrices, *Nagoya Math. J.* **104** (1986), 149-161.
- [63] Y. Teranishi, Linear diophantine equations and invariant theory of matrices, in “*Commut. Algebra and Combinatorics (Kyoto, 1985)*”, Adv. Stud. Pure Math. **11**, North-Holland, Amsterdam-New York, 1987, 259-275.
- [64] Y. Teranishi, The Hilbert series of matrix concominants, *Nagoya Math. J.*, **111** (1988), 143-156.
- [65] V.A. Ufnarovski, Combinatorial and asymptotic methods in algebra, in A.I. Kostrikin, I.R. Shafarevich (Eds.), “*Algebra VI*”, Encyclopedia of Math. Sciences **57**, Springer-Verlag, 1995, 1-196.
- [66] M. Van den Bergh, Trace rings are Cohen-Macaulay, *J. Amer. Math. Soc.*, **2** (1989), 775-799.
- [67] M. Van den Bergh, Explicit rational forms for the Poincaré series of the trace rings of generic matrices, *Isr. J. Math.* **73** (1991), 17-31.
- [68] M. R. Vaughan-Lee, An algorithm for computing graded algebras, *J. Symbolic Comput.* **16** (1993), 345-354.
- [69] H. Weyl, Zur Darstellungstheorie und Invariantenabzählung der projektiven, der Komplex- und der Drehungsgruppe, *Acta Math* **48** (1926), 255-278. Reprinted in “*Gesammelte Abhandlungen*”, *Band III*, Springer-Verlag, Berlin-Heidelberg-New York, 1968, 1-25.
- [70] H. Weyl, *The Classical Groups, Their Invariants and Representations*, Princeton Univ. Press, Princeton, N.J., 1946, New Edition, 1997.

*Institute of Mathematics and Informatics,  
Bulgarian Academy of Sciences,  
1113 Sofia, Bulgaria  
E-mail: drensky@math.bas.bg*

*Received 21.02.2006*