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On the Growth of Perturbations of the Exponential Series

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We consider perturbations of the simple exponential series in the form

$$S(x)=\sum_{n=0}^{\infty}A_n(-x)^n/n!,$$

where the perturbation coefficients A_n satisfy the condition $A_n\to 1$ as $n\to\infty$. When $|\arg x|\leq\frac{1}{2}\pi$, the presence of the A_n can destroy the cancellation that takes place in the negative exponential series, thereby yielding a behaviour that is significantly different from that of e^{-x}. We determine the behaviour of $S(x)$ as $|x|\to\infty$ for some particular forms of A_n.

1. Introduction

We shall be concerned with the determination of the dominant behaviour for large $|x|$ of perturbations of the exponential series for e^{-x} in the form

$$S(x)=\sum_{n=0}^{\infty}A_n\frac{(-x)^n}{n!},\tag{1.1}$$

where the perturbing coefficients A_n possess the property that $A_n\to 1$ as $n\to\infty$. For large negative x, the growth of the terms when $A_n\equiv 1$ is characterised by a peaked distribution, with the maximum term occurring when $n=[-x]$ and of value approximately $e^{|x|}/(2\pi|x|)^{1/2}$. When x is positive, the individual terms in the negative exponential series have the same absolute values, but now an extreme cancellation takes place to yield the sum of the series as e^{-x}. A similar cancellation of terms – although not as extreme – occurs when x is pure imaginary to yield the sum of the series with unit modulus. The presence of the perturbing coefficients A_n in (1.1) can destroy the cancellation that occurs when $|\arg x| \le \frac{1}{2}\pi$ to produce a behaviour of $S(x)$ that is significantly different from that of the unperturbed function e^{-x}.

Such a significant cancellation of terms does not take place in the sector $|\arg(-x)| < \frac{1}{2}\pi$ and so we would expect the perturbed exponential series in (1.1) to behave essentially like the unperturbed series for large $|x|$ in this sector. A problem of this nature has been considered by Ramanujan [3, pp. 58–65] for the function defined by $f(x) = e^{-x}\sum_{n=0}^{\infty} x^n\varphi(n)/n!$, where the function $\varphi(n)$ possesses at most polynomial growth in n. Ramanujan showed that[1] [3, p. 58]

$$f(x) \sim \varphi(x) + \sum_{k=2}^{\infty} x^{1-k}F_k(x) \qquad (x \to +\infty), \tag{1.2}$$

where

$$F_k(x) = \sum_{j=k}^{2k-2} b_{kj}\frac{x^j\varphi^{(j)}(x)}{j!}$$

with the b_{kj} being computable integer coefficients satisfying $b_{22} = b_{33} = 1$, $b_{34} = 3, \ldots$. The result in (1.2) shows that in the case of $S(x)$ we have the leading behaviour $S(-x) \sim A_x e^x$ as $x \to +\infty$, thereby confirming the above statement.

A particularly simple example of (1.1) is furnished by $A_n = 1 + (\beta e^{i\phi})^n$, where β and ϕ are constants with $0 < \beta < 1$ and ϕ real, so that

$$S(x) = e^{-x} + \exp(-\beta x e^{i\phi}).$$

When $x > 0$, the dominant behaviour of $S(x)$ for large x is determined by the second term $\exp(-\beta x e^{i\phi})$, which is of slower exponential decay when $|\phi| < \frac{1}{2}\pi$; we note that when $\frac{1}{2}\pi < |\phi| \le \pi$, this term becomes exponentially large as $x \to +\infty$. Another example, although of a much more recondite character, is given by $A_n = \exp\{1/(n+a)\}$, where $a > 0$ is a constant. The function (1.1) in this case is one of the many integral functions studied by Barnes [2] in 1906; from page 281 of this reference, we consequently obtain the leading behaviour given by

$$S(x) \sim \frac{\Gamma(a)x^{-a}}{2\sqrt{\pi}(\log x)^{3/2}}\exp(2\sqrt{\log x})$$

[1] The proof given in [3] only deals with $x \to +\infty$; it is probable that this expansion holds in the sector $\mathrm{Re}(x) > 0$.

as $x \to \infty$ in $|\arg x| < \frac{1}{2}\pi$. With this form of perturbing coefficients, it is seen that $S(x)$ possesses a much slower algebraic-logarithmic decay as $|x| \to \infty$ in the right-half plane.

Another interesting example of a sum of type (1.1) is given by the Riesz function $S_{m,p}(x)$ defined by

$$S_{m,p}(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!\zeta(p+mn)} \qquad (m > 0,\ p > 0)$$

which has the coefficients $A_n = 1/\zeta(p+mn)$, where ζ denotes the Riemann zeta function. The case $m = 2$, $p = 2$ was discussed by Riesz [13], who showed that a necessary and sufficient condition for the truth of the Riemann hypothesis is that $S_{2,2}(x) = O(x^{-\frac{3}{4}+\epsilon})$ as $x \to +\infty$, where ϵ is an arbitrarily small positive quantity. An analogous result when $m = 2$, $p = 1$ was given by Hardy and Littlewood [5] in their famous memoir on the distribution of prime numbers; see also [15, p. 382]. In this case the Riemann hypothesis follows if $S_{2,1}(x) = O(x^{-\frac{1}{4}+\epsilon})$ as $x \to +\infty$. A numerical investigation of the Riesz function has been given in [9]. However, all attempts to establish these growth estimates have failed since knowledge of the singularities of the above form of A_n depends, among other things, on the location of the complex zeros of the zeta function itself.

The problem we treat in this paper is the investigation of the dominant behaviour of $S(x)$ in (1.1) as $|x| \to \infty$ in $|\arg x| < \frac{1}{2}\pi$ for a given choice of the perturbing coefficients A_n. We shall be primarily concerned, however, with the asymptotic behaviour of the associated sums $S(x)$ as $x \to +\infty$, as it is along the positive real axis, where the cancellation between terms in the negative exponential series is greatest, that the most significant change in the behaviour of $S(x)$ can be expected to occur. It will be shown that the behaviour of $S(x)$ depends on the singularity structure of the perturbing coefficients $A_n \equiv A(n)$ considered as a function of a complex variable. In particular, it will be found that different types of perturbation result in different degrees of cancellation between the terms of the series, and hence different growth properties.

2. An integral representation for $S(x)$

An integral representation for $S(x)$ is given by the Mellin-Barnes integral

$$S(x) = \frac{1}{2\pi i} \int_{-c-\infty i}^{-c+\infty i} x^s \Gamma(-s) A(s)\, ds, \tag{2.1}$$

where $c > 0$ and the coefficients $A_n \equiv A(n)$ are expressed of a function of the complex variable s. The integration path lies to the left of the poles of $\Gamma(-s)$

situated at $s = 0, 1, 2, \ldots$ but may be indented (if necessary) to lie to the right of all the singularities of $A(s)$. The function $A(s)$ is bounded as $|s| \to \infty$ so that, since $|\Gamma(s)|$ behaves like $\exp(-\frac{1}{2}\pi|\mathrm{Im}\,(s)|)$ as Im $(s) \to \pm\infty$, the integral (2.1) defines $S(x)$ in the sector $|\arg x| < \frac{1}{2}\pi$. The result in (1.1) follows from (2.1) by straightforward deformation of the path into a loop surrounding the positive real axis and application of Cauchy's theorem, using the fact that the residue of $\Gamma(-s)$ at $s = k$ $(k = 0, 1, 2, \ldots)$ is $(-)^{k-1}/k!$.

The integration path may be displaced to the left to coincide with the line $s = -\ell$, where ℓ is an arbitrary positive quantity, so as to pass over the poles and branch points of $A(s)$. Let us suppose that $A(s)$ possesses branch points at $s = a_r$ $(r = 1, 2, \ldots, M)$ and poles at $s = b_r$ $(r = 1, 2, \ldots, N)$. Then we find

$$S(x) = \sum_{r=1}^{M} \frac{1}{2\pi i} \int_{-\infty}^{(a_r+)} x^s \Gamma(-s) A(s)\, ds + \sum_{r=1}^{N} \mathrm{Res}\,[x^s \Gamma(-s) A(s)]_{s=b_r} + O(|x|^{-\ell}), \tag{2.2}$$

where the term $O(|x|^{-\ell})$ can be made as small as we please when $|x|$ is large in $|\arg x| < \frac{1}{2}\pi$. In the case of branch points, the integration path can be deformed into a series of loops surrounding the points a_r in the positive sense and passing to infinity parallel[2] to the real s-axis; see Fig. 1.

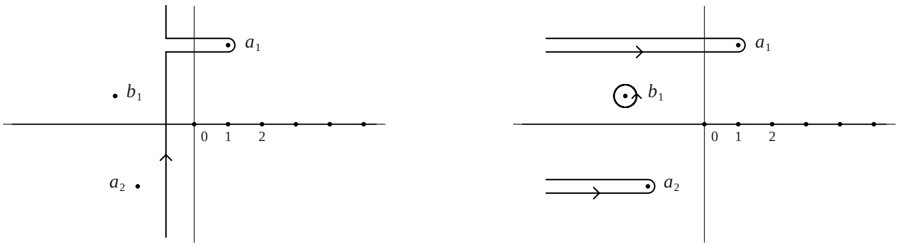

Figure 1: The integration paths in the complex s-plane. The heavy dots denote poles and branch points.

To evaluate the contribution from a branch point $s = a_r$, we suppose that $A(s)$ has an expansion of the form

$$A(s) = \sum_{k=0}^{\infty} \frac{b_k}{k!} (s - a_r)^{k-\beta_r}$$

[2] It is also possible to select branch cuts that are inclined at an acute angle to the negative real axis.

valid in some circle about the point $s = a_r$, where the coefficients $b_k \equiv b_k(a_r)$. Then the expansion of $A(s)\Gamma(-s)$ can be written in the form

$$A(s)\Gamma(-s) = \sum_{k=0}^{\infty} (-)^k \frac{c_k}{k!} (s - a_r)^{k-\beta_r}, \tag{2.3}$$

where the coefficients $c_k \equiv c_k(a_r)$ are specified by

$$c_k = \sum_{n=0}^{k} (-)^n \binom{k}{n} b_n \Gamma^{(k-n)}(-a_r). \tag{2.4}$$

The radius of convergence of the expansion (2.4) is determined by the nearest singularity of either $A(s)$ or $\Gamma(-s)$ to $s = a_r$.

The contribution from the branch points can be evaluated by making the change of variable $s = a_r - u$ and appealing to Watson's lemma [7, p. 112]. We obtain

$$\frac{1}{2\pi i} \int_{-\infty}^{(a_r+)} x^s \Gamma(-s) A(s)\, ds \sim x^{a_r} \sum_{k=0}^{\infty} \frac{(-)^k c_k}{k!} \frac{i}{2\pi} \int_{\infty}^{(0+)} (-u)^{k-\beta_r} e^{-u \log x} du$$

$$= x^{a_r} (\log x)^{\beta_r - 1} \sum_{k=0}^{\infty} \frac{(-)^k c_k}{k! (\log x)^k} \frac{i}{2\pi} \int_{\infty}^{(0+)} (-w)^{k-\beta_r} e^{-w} dw$$

$$= \frac{x^{a_r} (\log x)^{\beta_r - 1}}{\Gamma(\beta_r)} \sum_{k=0}^{\infty} \frac{c_k (1-\beta_r)_k}{k! (\log x)^k}, \tag{2.5}$$

where use has been made of Hankel's representation for $1/\Gamma(z)$ [16, p. 103; 18, p. 245] and $(a)_k = \Gamma(a+k)/\Gamma(a)$ is Pochhammer's symbol. The expansion of $S(x)$ for $|x| \to \infty$ in $|\arg x| < \frac{1}{2}\pi$ then follows from (2.2) and (2.5).

We consider three different cases of the perturbation coefficients A_n given by

$$(a) \quad 1 + (n+a)^{-\beta}, \qquad (b) \quad \frac{n^{p/2}}{\sqrt{1+n^p}}, \qquad (c) \quad \tanh \lambda(n+a), \tag{2.6}$$

where, for simplicity in presentation, we take the parameters a, β and λ to be fixed positive constants and p to be a positive integer. When viewed as functions of a complex variable, these coefficients consist of a single branch point, multiple branch points and an infinite number of simple poles, respectively.

As stated in Section 1, we shall be primarily concerned with the asymptotic behaviour of the associated sums $S(x)$ as $x \to +\infty$, as it is along the positive real axis that the most significant change in the behaviour of $S(x)$ will take place. A deficiency of the Mellin-Barnes integral approach in (2.1) is its restriction to the sector $|\arg x| < \frac{1}{2}\pi$, which precludes the case of purely imaginary values of x. Although it is possible in certain cases to use the Mellin-Barnes integral representation to extract the asymptotic behaviour of a function outside its sector of definition [11, §5.4; 12, pp. 367-371], it is generally necessary to resort to other integral definitions. We carry out such a procedure in the case of (2.6a) to obtain the full asymptotic expansion in the sector $|\arg x| < \pi$, which thereby enables us to confirm the fact that the greatest change in $S(x)$ in this case does indeed occur for positive x.

3. The coefficients A_n in Case (a)

For the coefficients A_n specified in (2.6a), we have[3]) has been investigated by Barnes in [2, p. 259 *et seq.*], where it is called $G_\beta(-x;a)$, and also in [4]. from (1.1) and (2.1)

$$S(x) - e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(n+a)^\beta} = \frac{1}{2\pi i}\int_{-c-\infty i}^{-c+\infty i} \frac{x^s\Gamma(-s)}{(s+a)^\beta}\,ds \quad (0 < c < a) \tag{3.1}$$

valid in the sector $|\arg x| < \frac{1}{2}\pi$. The integrand has a single branch point at $s = -a$, so that from (2.4) the coefficients $c_k = \Gamma^{(k)}(a)$. Then, from (2.2) and (2.5) and neglecting the exponentially small term on the left-hand side of (3.1), we obtain the expansion

$$S(x) \sim \frac{x^{-a}(\log x)^{\beta-1}}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{(1-\beta)_k\Gamma^{(k)}(a)}{k!(\log x)^k} \tag{3.2}$$

as $|x| \to \infty$ in $|\arg x| < \frac{1}{2}\pi$. The derivatives of $\Gamma(a)$ can be expressed in terms of the more familiar digamma function $\psi(a)$ as

$$\Gamma'(a) = \psi(a)\Gamma(a), \quad \Gamma''(a) = \{\psi'(a) + \psi^2(a)\}\Gamma(a),$$
$$\Gamma'''(a) = \{\psi''(a) + 3\psi(a)\psi'(a) + \psi^3(a)\}\Gamma(a), \ldots .$$

To obtain an expansion valid in a wider sector, we employ the standard Laplace transform

$$(n+a)^{-\beta} = \frac{1}{\Gamma(\beta)}\int_0^\infty e^{-(n+a)t}t^{\beta-1}dt \qquad (a > 0,\ \beta > 0)$$

[3]The sum on the right-hand side of (3.1

for $n \geq 0$. Substitution of this result in the sum in (3.1) then yields

$$\begin{aligned} S(x) - e^{-x} &= \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} \exp(-xe^{-t} - at)\, dt \\ &= \frac{x^{-a}}{\Gamma(\beta)} \int_0^x e^{-u} u^{a-1} (\log\, x/u)^{\beta-1} du, \end{aligned} \tag{3.3}$$

where the change of variable $u = xe^{-t}$ has been made. If this last integral is now written as the difference between the integrals over the intervals $[0, \infty)$ and $[x, \infty)$, we obtain

$$S(x) - e^{-x} = \frac{x^{-a}(\log\, x)^{\beta-1}}{\Gamma(\beta)} \int_0^\infty e^{-u} u^{a-1} \left(1 - \frac{\log\, u}{\log\, x}\right)^{\beta-1} du$$

$$- \frac{e^{-x}}{x\Gamma(\beta)} \int_0^\infty e^{-u} \left(1 + \frac{u}{x}\right)^{a-1} \left(\log\, \frac{1}{1+t/x}\right)^{\beta-1} du,$$

where in the second integral the variable u has been replaced by $u + x$. In the first integral we employ the binomial expansion of the factor involving the logarithms and in the second integral the expansion [12, p. 239]

$$(1-\tau)^{a-1} \left(\log\, \frac{1}{1-\tau}\right)^{\lambda} = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \tau^{k+\lambda} B_k^{(k+\lambda+1)}(a) \qquad (|\tau| < 1),$$

where $B_k^{(\nu)}(a)$ denotes the generalised Bernoulli function [14, p. 4]. Then, upon application of Watson's lemma, we obtain [2; 12, p. 417]

$$S(x) \sim \frac{x^{-a}(\log\, x)^{\beta-1}}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{(1-\beta)_k \Gamma^{(k)}(a)}{k!(\log\, x)^k} + e^{-x}\left\{1 + (-x)^{-\beta} \sum_{k=0}^{\infty} \frac{(\beta)_k B_k^{(k+\beta)}(a)}{k!\, x^k}\right\} \tag{3.4}$$

valid for $|x| \to \infty$ in $0 < |\arg\, x| < \pi$.

From (3.2) and (3.4), the leading behaviour of $S(x)$ when the coefficients A_n are defined by (2.6a) is given by

$$S(x) = O(x^{-a}(\log\, x)^{\beta-1}) + e^{-x}(1 + O(x^{-\beta}))$$

as $|x| \to \infty$ in $|\arg\, x| \leq \frac{1}{2}\pi$. As remarked in Section 1, the greatest change in the behaviour of $S(x)$ occurs for large positive values of x. Results of computations with $x > 0$ are presented in Table 1 where the leading two terms in (3.2) are compared with the exact values of the sum $S(x)$ obtained from (1.1). In Table

x	$\beta = 2 \quad a = 1$		$\beta = 3 \quad a = 2$	
	$S(x)$	Asymptotic	$S(x)$	Asymptotic
5	0.4442983216	0.4373307155	0.0477467259	0.0410616156
10	0.2880258914	0.2879800758	0.0209382796	0.0208929248
15	0.2190180316	0.2190177244	0.0130389150	0.0130386092
20	0.1786473990	0.1786473969	0.0090812460	0.0090812440
25	0.1518436596	0.1518436596	0.0067704494	0.0067704493
x	$\beta = 0.5 \quad a = 1$		$\beta = 1.5 \quad a = 2$	
	$S(x)$	Asymptotic	$S(x)$	Asymptotic
5	0.0962804747	0.0984659199	0.0525771199	0.0474632582
10	0.0359236131	0.0377223878	0.0151534298	0.0152178844
15	0.0219260052	0.0227323167	0.0074659891	0.0074927075
20	0.0156190834	0.0160753317	0.0044711325	0.0044819986
25	0.0120542753	0.0123513537	0.0029886848	0.0029942116

Table 1: Values of $S(x)$ and its asymptotic value in (3.2) for different x, β and a when the coefficients A_n have the form (2.6a).

x	$S(x)$	Asymptotic
$5i$	$+0.59364843 + 0.48358761i$	$+0.59414636 + 0.48096729i$
$10i$	$-0.67323677 + 0.25149539i$	$-0.67305716 + 0.25143990i$
$20i$	$+0.48549415 - 1.08937166i$	$+0.48548755 - 1.08936129i$
$30i$	$+0.20647667 + 0.85431678i$	$+0.20647653 + 0.85431433i$
$40i$	$-0.62726343 - 0.85129004i$	$-0.62726296 - 0.85128942i$
$50i$	$+0.99599837 + 0.17247751i$	$+0.99599807 + 0.17247741i$

Table 2: Values of $S(x)$ and its asymptotic value in (3.5) for different imaginary values of x when the coefficients A_n have the form (2.6a) with $\beta = 2$, $a = 1$.

2 we show results when x is pure imaginary for the particular case $\beta = 2$, where, from (3.4), we obtain the estimate[4]

$$(3.5)\quad S(x) = \frac{\Gamma(a)}{x^a}\biggl(\log x - \psi(a) + O(1/\log x)\biggr) + e^{-x}\biggl(1 + \frac{1}{x^2} + \frac{2a-3}{x^3} + O(x^{-4})\biggr).$$

[4]In deriving (3.5) we have employed the values of the generalised Bernoulli functions $B_0^{(2)}(a) = 1$, $B_1^{(3)}(a) = a - \frac{3}{2}$.

It may be remarked that the case $\beta = 1$ is particularly simple, since the sum in (3.1) can be evaluated in terms of the incomplete gamma function $\gamma(a,x)$ to yield

$$S(x) = e^{-x} + x^{-a}\gamma(a,x) \qquad (\beta = 1).$$

Use of the identity $\gamma(a,x) = \Gamma(a) - \Gamma(a,x)$, where $\Gamma(a,x)$ denotes the complementary incomplete gamma function, together with the standard asymptotics of $\Gamma(a,x)$ for large x [1, p. 263] readily shows that the expansion of $S(x)$ in this case reduces to that given in (3.4). Extension to the perturbing coefficients $A_n = n^2/(n^2+a^2)$, for example, then follows from this result by a straightforward partial-fraction decomposition.

We note that by binomial expansion of the logarithmic term in the integrand in (3.3) when $\beta = m$, a positive integer, we find the exact result

$$S(x) = e^{-x} + x^{-a}(\log x)^{m-1}\sum_{k=0}^{m-1}\frac{(-)^k\gamma^{(k)}(a,x)}{k!\Gamma(m-k)(\log x)^k} \qquad (|\arg x| \leq \pi),$$

where the derivatives of the incomplete gamma function are with respect to the parameter a. However, extraction of the asymptotic form of the exponential contribution from this expansion to produce the second series on the right-hand side of (3.4) is not simple and involves considerable cancellation of terms.

4. The coefficients A_n in Case (b)

When the perturbing coefficients are defined by $A_n = n^{p/2}/\sqrt{1+n^p}$ for positive integer p, we have upon rearrangement of (2.1)

$$S(x) = -\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^s\Gamma(1-s)\frac{s^{p/2-1}}{\sqrt{1+s^p}}ds \qquad (\cos\pi/p < c < 1). \tag{4.1}$$

When $p = 2m$ is an even integer, the integrand has branch points at $s = \omega_r = \exp\{(2r-1)\pi i/p\}$ $(1 \leq r \leq m)$ in the upper half-plane, together with a conjugate set of points in the lower half-plane. When $p = 2m+1$ is an odd integer, the integrand has a similar array of branch points together with two additional branch points at $s = 0$ and $s = -1$.

The expansion of $A(s)/s$ about $s = \omega_r$ takes the form

$$\frac{s^{p/2-1}}{\sqrt{1+s^p}} = \sum_{k=0}^{\infty}\frac{b_k}{k!}(s-\omega_r)^{k-\frac{1}{2}},$$

where the first few coefficients $b_k \equiv b_k(\omega_r)$ are given by

$$b_0 = (\omega_r p)^{-1/2}, \quad b_1 = -(\omega_r p)^{-1/2}\frac{(3-p)}{4\omega_r}, \quad b_2 = (\omega_r p)^{-1/2}\frac{(p^2-30p+65)}{48\omega_r^2},$$

$$b_3 = -(\omega_r p)^{-1/2}\,\frac{(p^3+7p^2-133p+245)}{64\omega_r^3}, \ldots .$$

Then, from (2.2) and a slight modification of the argument leading to (2.5) (with $\beta_r = \frac{1}{2}$), the contribution to $S(x)$ from the branch point $s = \omega_r$ has the expansion

$$-\frac{x^{\omega_r}}{(\pi \log x)^{1/2}} \sum_{k=0}^{\infty} \frac{c_k(\frac{1}{2})_k}{k!(\log x)^k} \qquad (1 \le r \le m) \tag{4.2}$$

as $|x| \to \infty$ in $|\arg x| < \frac{1}{2}\pi$. The coefficients $c_k \equiv c_k(\omega_r)$ are defined as in (2.4), but with the argument of the gamma function replaced by $1-\omega_r$; that is

$$c_k = (\omega_r p)^{-1/2}\Gamma(1-\omega_r)C_k, \tag{4.3}$$

where

$$\begin{aligned} C_0 &= 1, \quad C_1 = g^{(1)}(\xi) + \frac{3-p}{4\omega_r}, \quad C_2 = g^{(2)}(\xi) + \frac{(3-p)}{2\omega_r}\,g^{(1)}(\xi) + \frac{p^2-30p+65}{48\omega_r^2}, \\ C_3 &= g^{(3)}(\xi) + \frac{3(3-p)}{4\omega_r}\,g^{(2)}(\xi) + \frac{(p^2-30p+65)}{16\omega_r^2}\,g^{(1)}(\xi) + \frac{p^3+7p^2-133p+245}{64\omega_r^3}, \ldots \end{aligned}$$

and

$$g^{(n)}(\xi) \equiv \frac{\Gamma^{(n)}(\xi)}{\Gamma(\xi)}, \qquad \xi = 1-\omega_r.$$

When $p = 2m+1$ there is an additional contribution $I(x)$ from the branch points at $s = 0$ and $s = -1$ given by

$$I(x) = -\frac{1}{2\pi i}\int_{-\infty}^{(-1+,0+)} x^s\Gamma(1-s)\frac{s^{p/2-1}}{\sqrt{1+s^p}}ds,$$

where the phase of $s^{p/2-1}/\sqrt{(1+s^p)}$ at the point where s crosses the positive real axis is zero. By taking the loop to be along the negative real axis with vanishingly small circular indentations around the branch points (see, for example, [17, p. 171]), we find that

$$\begin{aligned} I(x) &= -\frac{\sin(\frac{1}{2}\pi p)}{\pi}\int_0^1 \Gamma(1+u)\frac{u^{p/2-1}}{\sqrt{1-u^p}}e^{-u\log x}du \\ &= -\frac{\sin(\frac{1}{2}\pi p)}{\pi}\sum_{k=0}^{\infty}\frac{d_k}{k!}\int_0^1 u^{k+p/2-1}e^{-u\log x}du \\ &= (-)^{(p+1)/2}\frac{\Gamma(\frac{1}{2}p)}{\pi}\sum_{k=0}^{\infty}\frac{d_k(\frac{1}{2}p)_k}{k!(\log x)^{k+p/2}}P(k+\tfrac{1}{2}p, \log x) \end{aligned} \tag{4.4}$$

and that the contributions from $[-1, -\infty)$ on either side of the cut cancel[5]. The coefficients d_k appear in the Maclaurin expansion of $\Gamma(1+u)/\sqrt{1-u^p}$, valid in $|u| < 1$, and P denotes the normalised incomplete gamma function $P(a, x) = \gamma(a, x)/\Gamma(a)$. The series in (4.4) is an example of a Hadamard expansion; see [10] for further details on the computation of such series.

For large positive x, the contribution to $S(x)$ from the branch points $s = \omega_r$ and $s = \overline{\omega}_r$ is, from (4.2),

$$J(\omega_r) \sim -\frac{2x^{\cos(\pi/p)}}{(\pi \log x)^{1/2}} \sum_{k=0}^{\infty} \frac{|c_k|\,(\frac{1}{2})_k}{k!(\log x)^k} \cos\Big\{\sin(\pi/p)\log x + \psi_k\Big\} \qquad (1 \le r \le m) \tag{4.5}$$

where $\psi_k \equiv \psi_k(\omega_r) = \arg c_k$. It then follows that we have the expansion

$$S(x) \sim \sum_{r=1}^{m} J(\omega_r) + \delta_p\, I(x), \qquad \delta_p = \begin{cases} 0 & p = 2m \\ 1 & p = 2m+1 \end{cases} \tag{4.6}$$

valid for $x \to +\infty$. When $p \ge 2$, the dominant contribution to $S(x)$ arises from the branch points with the greatest real part corresponding to $r = 1$. If we let $\Gamma(-\overline{\omega}_1) = \Gamma(e^{\pi i(p-1)/p}) = M_p e^{i\phi_p}$, where M_p and ϕ_p denote the modulus and phase, then we find from (4.3) and (4.6) that $S(x)$ has the leading behaviour

$$S(x) \sim \frac{2M_p x^{\cos(\pi/p)}}{(\pi p \log x)^{1/2}} \cos\Big\{\sin(\pi/p)\log x + \frac{\pi}{2p} - \phi_p\Big\} \tag{4.7}$$

for $x \to +\infty$. It is then evident that the perturbation (2.6b) with $p \ge 3$ results in $|S(x)| \to \infty$ as $x \to +\infty$, where the growth of $S(x)$ is seen to consist of a slow oscillation with an amplitude that is controlled by the factor $x^{\cos(\pi/p)}/(\log x)^{1/2}$.

When $p = 2$, the sum $S(x)$ is associated with a pair of branch points in the integrand in (4.1) at $s = \pm i$, which results in a decay controlled by $(\log x)^{-1/2}$ with no algebraic dependence. However, if the perturbation coefficients are modified to

$$A_n = \frac{n}{\sqrt{1 + (n-\alpha)^2}}, \tag{4.8}$$

for example, where α is real, then this has the effect of displacing the branch points to $s = \alpha \pm i$. The expansion of $S(x)$ in this case is easily shown to be

$$S(x) \sim M(\alpha)\left(\frac{2}{\pi \log x}\right)^{1/2} x^{\alpha} \sum_{k=0}^{\infty} \frac{|C_k(\alpha)|(\frac{1}{2})_k}{k!(\log x)^k} \cos\Big\{\log x + \frac{\pi}{4} - \phi_k(\alpha)\Big\} \tag{4.9}$$

[5] The contour of integration is effectively replaced by a dumbbell contour surrounding the branch points.

as $x \to +\infty$, where

$$M(\alpha) = |\Gamma(-\alpha + i)|\sqrt{1 + \alpha^2}, \qquad \phi_k(\alpha) = \arg\{C_k(\alpha)\Gamma(-\alpha + i)\} + \arctan \alpha$$

and

$$C_0(\alpha) = 1, \quad C_1(\alpha) = g^{(1)}(\xi) + \tfrac{1}{4}i, \quad C_2(\alpha) = g^{(2)}(\xi) + \tfrac{1}{2}ig^{(1)}(\xi) - \tfrac{3}{16},$$
$$C_3(\alpha) = g^{(3)}(\xi) + \tfrac{3}{4}ig^{(2)}(\xi) - \tfrac{9}{16}g^{(1)}(\xi) - \tfrac{15}{64}i, \ldots ,$$

with $g^{(n)}(\xi)$ defined at (4.3) and $\xi = 1-\alpha+i$. For $\alpha > 0$, which corresponds to a non-monotonic growth of the A_n, (4.9) shows that $|S(x)| \to \infty$ as $x \to +\infty$. In Fig. 2 we show the approximate behaviour of $S(x)$ for two values of p compared with the exact behaviour computed from (1.1). In each case we have employed the first two terms in the asymptotic expansions (4.5) and (4.9); the behaviour consists of a slow oscillation with a growing or decaying amplitude.

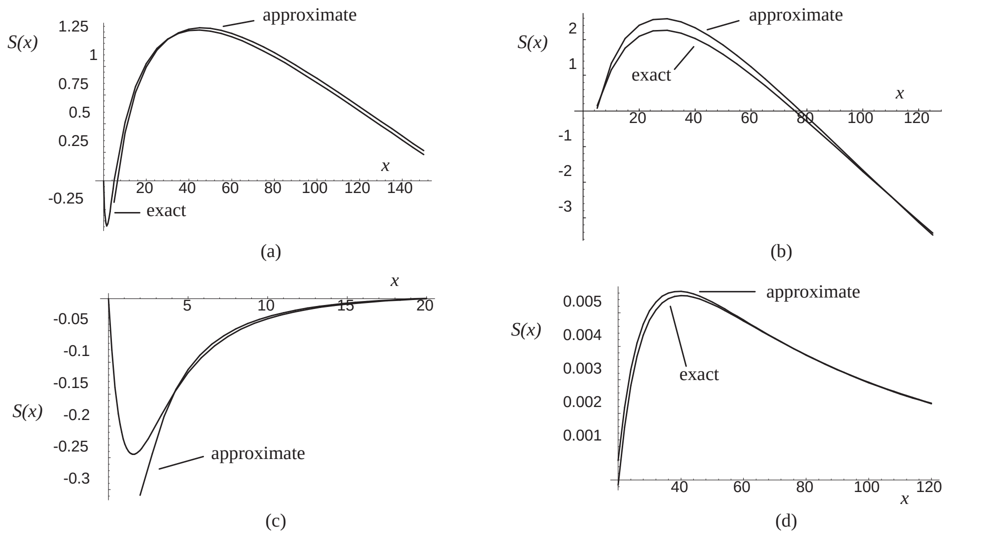

Figure 2: Comparison of the exact and asymptotic behaviour of $S(x)$: (a) when A_n are given by (2.6b) with $p = 3$, and when A_n are given by (4.8) with (b) $\alpha = 3/4$, (c) $\alpha = -1$, $0 \le x \le 20$ and (d) $\alpha = -1$, $20 \le x \le 120$.

To conclude this section, we note that the logarithmic dependence of the behaviour of $S(x)$ in (4.7) can be modified by allowing the perturbation coefficients to have the form

$$A_n = \frac{n^{p/q}}{(1 + n^p)^{1/q}},$$

where p and q are positive integers. By the same procedure as described above, the dominant behaviour as $x \to +\infty$ when $p \geq 2$ is then found to be

$$S(x) \sim \frac{2p^{-1/q} M_p \, x^{\cos(\pi/p)}}{\Gamma(1/q)(\log x)^{(q-1)/q}} \cos\left\{\sin(\pi/p) \log x + \frac{\pi}{pq} - \phi_p\right\},$$

where we recall that M_p and ϕ_p are defined at (4.7). This result shows that increasing the value of q strengthens the logarithmic dependence in $S(x)$.

5. The coefficients A_n in Case (c)

Our final example has the perturbing coefficients given by $A_n = \tanh \lambda(n+a)$, where $a > 0$ and $\lambda > 0$. From the Mellin-Barnes integral for $S(x)$ in (2.1) (with $0 < c < a$), the integrand possesses an infinite string of simple poles situated at $s_k = -a + (k + \frac{1}{2})\pi i/\lambda$, $k = 0, \pm 1, \pm 2, \ldots$. Evaluation of the residues in (2.2) then yields the expansion

$$S(x) \sim \frac{x^{-a}}{\lambda} \sum_{k=-\infty}^{\infty} x^{-iK} \Gamma(a + iK), \qquad K = (k + \tfrac{1}{2})\frac{\pi}{\lambda} \tag{5.1}$$

as $|x| \to \infty$ in $|\arg x| < \frac{1}{2}\pi$. The terms in this expansion decay rapidly when $\mathrm{Re}(x) > 0$ on account of the exponential decay of the Gamma function: if we let $\theta = \arg x$, then use of Stirling's formula shows that the late terms are controlled by $\exp\{\theta K - \frac{1}{2}\pi|K|\}$ as $k \to \pm\infty$. We then obtain

$$S(x) \sim \frac{2x^{-a}}{\lambda} \sum_{k=0}^{\infty} |\Gamma(a + iK)| \cos\left\{K \log x - \psi_k\right\} \tag{5.2}$$

as $x \to +\infty$, where $\psi_k = \arg \Gamma(a+iK)$. In Table 3 we show results of numerical calculations for $x > 0$ using (5.2) with $k \leq 5$.

A representation for $S(x)$ in $|\arg x| \leq \pi$ can be obtained by expressing the coefficients $A(n)$ in the form

$$A(n) = \frac{1}{2\pi i} \oint \frac{A(s)}{s - n} ds,$$

by Cauchy's integral theorem, where the path is a closed contour surrounding the point $s = n$ that does not enclose or lie on any singularity of $A(s)$. Substitution of this representation into (1.1) then produces

$$S(x) = \frac{1}{2\pi i} \oint_{\mathcal{S}} A(s) \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(s-n)} ds = \frac{i}{2\pi} \oint_{\mathcal{S}} x^s \gamma(-s, x) A(s) ds \tag{5.3}$$

x	$S(x)$	Asymptotic
10	$-8.8722453596\times10^{-3}$	$-8.8268454341\times10^{-3}$
20	$-1.9917003282\times10^{-3}$	$-1.9916982674\times10^{-3}$
30	$-2.5570991725\times10^{-4}$	$-2.5570991770\times10^{-4}$
40	$+1.4832260291\times10^{-4}$	$+1.4832260303\times10^{-4}$
50	$+2.1569016284\times10^{-4}$	$+2.1569016292\times10^{-4}$

Table 3: Values of $S(x)$ and its asymptotic value for different positive x when A_n are given by (2.6c) with $a=2$, $\lambda=1$.

valid in the sector $|\arg x|\leq\pi$. The infinite sum in the integrand has been expressed in terms of the incomplete gamma function $\gamma(\alpha,x)$ through use of the definitions [1, Eqs. (6.5.12) and (6.5.29)]

$$\gamma(-s,x)=x^{-s}\sum_{n=0}^\infty\frac{(-x)^n}{n!(n-s)}=-\frac{x^{-s}e^{-x}}{s}{}_1F_1(1;1-s;x) \tag{5.4}$$

and $\mathcal{S}$ denotes a 'sausage contour' surrounding the points $s=0,1,2,\ldots$ in the positive sense.

If we now expand the contour $\mathcal{S}$ into the infinite circle surrounding the origin (with the radius R chosen to pass between the poles parallel to the imaginary axis), we find that the contribution round the circle vanishes. This follows because as $R\to\infty$ the integrand in (5.3) behaves like[6] $-e^{-x}A(s)/s\sim\mp e^{-x}/s$ in the right- and left-hand half planes, respectively. Evaluation of the residues at the simple poles s_k then leads to

$$S(x)=\frac{x^{-a}}{\lambda}\sum_{k=-\infty}^\infty x^{-iK}\gamma(a+iK,x)\qquad(|\arg x|\leq\pi). \tag{5.5}$$

Since $\gamma(\alpha,z)=\Gamma(\alpha)-\Gamma(\alpha,z)$, it is seen that the expansions in (5.1) and (5.5) differ by the sum

$$\frac{x^{-a}}{\lambda}\sum_{k=-\infty}^\infty x^{-iK}\Gamma(a+iK,x). \tag{5.6}$$

To estimate this difference in the sector $|\arg x|<\frac{1}{2}\pi$, we appeal to the asymptotics of $\Gamma(\alpha,z)$ for $|z|\to\infty$ in $\mathrm{Re}(z)>0$ uniformly valid in the parameter α

[6] The behaviour of the confluent hypergeometric function in the second expression in (5.4) satisfies ${}_1F_1\sim1$ for $|s|\to\infty$ in $|\arg(-s)|<\pi$.

away from the transition point $z = \alpha$. The required uniform approximation is [8, Eq. (4.2)] $\Gamma(\alpha, z) \sim z^\alpha e^{-z}/(z - \alpha)$ when $\mathrm{Re}(z - \alpha) \geq 0$. This yields the leading estimate for the sum in (5.6) given by

$$\frac{e^{-x}}{\lambda} \sum_{k=-\infty}^{\infty} \frac{1}{x - a - iK} = \frac{2e^{-x}}{\lambda} \sum_{k=0}^{\infty} \frac{(x-a)}{(x-a)^2 + K^2} = e^{-x} \tanh \lambda(x - a);$$

see [6, p. 342, Ex. 30]. Since this expression decays exponentially for large $|x|$ in $\mathrm{Re}(x) > 0$, we see that (5.5) agrees with (5.1) up to an exponentially small contribution as $|x| \to \infty$ in $|\arg x| < \frac{1}{2}\pi$.

The behaviour of $S(x)$ when $\arg x = \pm\frac{1}{2}\pi$ is more complicated and would necessitate, in part, the use of the uniform asymptotics of $\gamma(\alpha, z)$ valid in the vicinity of the transition point $z = \alpha$; see [8, Eq. (3.4)]. We do not discuss this case any further here.

6. Concluding remarks

We have seen that the perturbed exponential series $S(x)$ in (1.1), with the three examples chosen for the coefficients A_n in (2.6), possesses a radically different behaviour as $x \to +\infty$ than that of the unperturbed function e^{-x}. The modification to $S(x)$ becomes less pronounced as $|\arg x|$ increases, with no change to leading order when $|\arg x| = \frac{1}{2}\pi$, as would be expected from the general considerations described in Section 1.

If we take the coefficients $A_n = 1 + (n + a)^{-\beta}$, we have from (3.2) the dominant behaviour

$$\frac{x^{-a}(\log x)^{\beta-1}}{\Gamma(\beta)} \qquad (x \to +\infty).$$

The coefficients $A_n \to 1$ more rapidly as n increases the larger the value of β. Naively, therefore, one might be led to expect that $S(x)$ would also decay more rapidly the larger the value of β. However, this is clearly seen not to be the case, since the above leading behaviour indicates that, for sufficiently large $x > 0$, $S(x)$ decays more slowly with increasing β. Manifestly, the principal mechanism in operation is the degree of cancellation between the terms of $S(x)$, not just simply the rate of decay of the perturbing coefficients A_n.

The forms of the coefficients chosen in (2.6a,c) with the parameter $a > 0$ result in $S(x)$ decaying as $x \to +\infty$, although much more slowly than the unperturbed function e^{-x}. However, for the coefficients in (2.6b) the interference between the cancellation of the terms when $p \geq 3$ is such that $S(x)$ *grows* with increasing x. In the particular case $p = 3$, for example, we have from (4.7) the result

$$S(x) \sim \left(\frac{4x}{3\pi \log x}\right)^{\frac{1}{2}} |\Gamma(e^{2\pi i/3})| \cos\left(\frac{\sqrt{3}}{2}\log x + \frac{\pi}{6} - \phi\right) \qquad (x \to +\infty),$$

where $\phi = \arg \Gamma(e^{2\pi i/3})$. In this case, the growth of $S(x)$ consists of a slow oscillation with an increasing amplitude that is controlled by the factor $(x/\log x)^{1/2}$. Even when $p = 2$, which is associated with a slow logarithmic decay of $S(x)$, a modification of the coefficients A_n as in (4.8) can result in $|S(x)| \to \infty$ as $x \to +\infty$.

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