Mathematica
Balkanica

New Series Vol. 21, 2007, Fasc. 3-4

A Remark on a Generalization of Eneström-Kakeya Theorem

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Let $P(z) = \sum_{j=0}^{n} (\alpha_j + i\beta_j)z^j$ be a polynomial of degree n such that for some $k \geq 1$, $k\alpha_n \geq ... \geq \alpha \geq \alpha_0$. It has been claimed by Govil and McTume[10] that they have proved that all the zeros of $P(z)$ lie in

$$|z + k - 1| \leq (k\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|)/|\alpha_n|.$$

But, unfortunately , the proof of this result given by Govil and McTume is not correct. In this paper we consider a problem analogous to the above and obtain certain generalizations of the well-known result of Enestrom and Kakeya concerning the bounds for the moduli of the zeros of polynomial with complex coefficients, which generalize some known results in this direction.

AMS Subj. Classification: 26C10.

Key Words: Polynomials, Zeros.

1. Introduction and Statement of Results

The following result is well-known in the theory of the distribution of zeros of polynomials(see [13,14]).

Theorem A. (Eneström - Kakeya)
If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leq 1$.

In the literature, there exist several generalizations of Theorem A (see [1], [2], [3], [4], [5], [6], [7], [8], [12]). A.Joyal, G. Labelle and Q.I.Rahman [11] extended Theorem A to the polynomials whose coefficients are monotonic but not necessary non-negative by proving the following:

Theorem B. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 \,,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Recently, A.Aziz and B.A. Zarger [3] considered the class of polynomials $P(z)$ not satisfying the condition $a_{n-1} \leq a_n$ and proved the following result:

Theorem C. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that for some* $k \geq 1$,

$$ka_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + k - 1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

It may be remarked here that, although not explicitly mentioned in the statement of theorem C, excepting the case when $k = 1$, the theorem C makes sense only when both a_{n-1} and a_n are positive, for if a_{n-1} is negative and $a_n < a_{n-1}$, then it is not possible to find $k \geq 1$ such that $a_{n-1} \leq ka_n$.

More recently, Govil and McTume [10] considered the class of polynomials whose coefficients are not necessarily real and claimed that they have proved the following generalization of Theorem C:

Theorem D. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, ..., n$. *If for some* $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + k - 1| \leq \frac{k\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}.$$

Unfortunately, the proof of this result given by Govil and McTume [10] is not correct. The reason being that on page 251 between the lines 3-7, the authors of [10], use the inequality $|\beta_n||z| < |\beta_n|$ for $|z| > 1$, which is clearly absurd. Thus, the proof of the main result in [10, Theorem 1] is incorrect.

In this paper we consider an analagous problem and present some generalization of Theorem B and C. We first prove:

Theorem 1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$ for $j = 0, 1, 2, ..., n$. If for some $k \geq 1$,*

$$k\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$

then all the zeros of $P(z)$ lie in

$$|z + (k-1)\frac{\alpha_n}{a_n}| \leq \frac{k\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}. \tag{1}$$

Remark 1. It may be remarked here, that unless k=1, Theorem 1 makes sense only when α_{n-1} and α_n are both positive because otherwise it is not possible to find $k > 1$ that would satisfy the hypothesis of Theorem 1.

If all the coefficients of polynomial $P(z)$ in Theorem 1 are real, that is, if $\beta_j = 0$ for all j, then it reduces to theorem C. Theorem B is also a special case of Theorem 1 when $k = 1$ and all the coefficients a_j are real.

The following result can be easily deduced from Theorem 1.

Corollary 1. *If $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$ such that*

$$k\alpha_n \geq \alpha_{n-1} \geq .... \geq \alpha_1 \geq \alpha_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$|z + (k-1)\frac{\alpha_n}{a_n}| \leq k + \frac{2}{\alpha_n}\sum_{j=0}^{n} |\beta_j|.$$

For $k = 1$, Corollary 1 reduces to a result due to Govil and Rahaman [9, Theorem 4]. Taking $k = \dfrac{\alpha_{n-1}}{\alpha_n} \geq 1$ in Theorem 1, we immediately obtain the following result:

Corollary 2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, ..., n$. *such that*

$$\alpha_n \leq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + \frac{\alpha_{n-1} - \alpha_n}{a_n}| \leq \frac{\alpha_{n-1} - \alpha_0 + |\alpha_0| + 2\sum\limits_{j=0}^{n} |\beta_j|}{a_n}.$$

Applying Theorem 1 to the polynomial $\{-ip(z)\}$, we easily get the following:

Theorem 2. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, ..., n$. *If for some* $k \geq 1$,

$$k\beta_n \geq \beta_{n-1} \geq ... \geq \beta_1 \geq \beta_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + (k-1)\frac{\beta_n}{a_n}| \leq \frac{k\beta_n - \beta_0 + |\beta_0| + 2\sum_{j=0}^{n} |\alpha_j|}{|a_n|}.$$

(unless $k = 1, \beta_{n-1}$ *and* β_n *are positive)*

By applying Theorem 1 and Theorem 2 to the polynomial $P(tz)$, we get

Corollary 3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, ..., n$. *If for some* $k \geq 1$ *and* $t > 0$

$$kt^n\alpha_n \geq t^{n-1}\alpha_{n-1} \geq ... \geq t\alpha_1 \geq \alpha_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + t(k-1)\frac{\alpha_n}{a_n}| \leq \frac{kt^n\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|t^j}{t^{n-1}|a_n|}.$$

Corollary 4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, ..., n$. *If for some* $k \geq 1$ *and* $t > 0$

$$kt^n\beta_n \geq t^{n-1}\beta_{n-1} \geq ... \geq t\beta_1 \geq \beta_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + t(k-1)\frac{\beta_n}{a_n}| \leq \frac{kt^n\beta_n - \beta_0 + |\beta_0| + 2\sum_{j=0}^{n} |\alpha_j|t^j}{t^{n-1}|a_n|}.$$

Next we present the following result:

Theorem 3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, .., n$. *If for some* $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0$$

and

$$k\beta_n \geq \beta_{n-1} \geq ... \geq \beta_1 \geq \beta_0 \,,$$

then all the zeros of $P(z)$ *lie in*

$$|z + (k-1)| \leq \frac{k(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|}{|a_n|}. \tag{2}$$

Theorem 3 is a generalization of a result due to Aziz and Mohammad [4] For $k = 1$ and $\beta_j = 0$,$j = 0, 1, ..., n$, Theorem 3 reduces to Theorem B.

The following result is an immediate consequence of Theorem 3.

Corollary 5. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, ..., n$.*If for some* $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0 > 0$$

and

$$k\beta_n \geq \beta_{n-1} \geq ... \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + (k-1) \leq \sqrt{2}k.$$

Finally, in this paper we state the following generalization of Theorem A. As its proof is almost similar to the proof of Theorem 1, we omit the details.

Theorem 4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $Re(a_j) = \alpha_j$ *and* $Im(a_j) = \beta_j$ *for* $j = 0, 1, 2, .., n$. *If for some* $k \geq 1$,

$$k\alpha_n \geq \alpha_{n-1} \geq ... \geq \alpha_1 \geq \alpha_0$$

and

$$\beta_n \geq \beta_{n-1} \geq ... \geq \beta_1 \geq \beta_0,$$

then all the zeros of $P(z)$ lie in

$$|z+\frac{\alpha_n}{a_n}(k-1)|\leq\frac{k\alpha_n-(\alpha_0+\beta_0)+|a_0|+\beta_n}{|a_n|}.$$

Setting $k=\dfrac{\alpha_{n-1}}{\alpha_n}\geq 1$ in Theorem 4, we obtain:

Corollary 6. *Let $P(z)=\sum\limits_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$ for $j=0,1,2,...,n$ such that*

$$\alpha_n\leq\alpha_{n-1}\geq...\geq\alpha_1\geq\alpha_0$$

and

$$\beta_n\geq\beta_{n-1}\geq...\geq\beta_1\geq\beta_0,$$

then all the zeros of $P(z)$ lie in

$$|z+\frac{\alpha_{n-1}-\alpha_n}{a_n}|\leq\frac{\alpha_{n-1}-(\alpha_0+\beta_0)+|a_0|+\beta_n}{|a_n|}.$$

2. Proofs of the Theorems

Proof. **Theorem 1.** Consider the polynomial (use $a_{-1}=0$)

$$\begin{aligned}F(z) &= (1-z)P(z)=-a_nz^{n+1}+(a_n-a_{n-1})z^n+\sum_{j=0}^{n-1}(a_j-a_{j-1})z^j\\ &= -a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+i(\beta_n-\beta_{n-1})z^n\\ &\quad+\sum_{j=0}^{n-1}\{(\alpha_j-\alpha_{j-1})+i(\beta_j-\beta_{j-1})\}z^j\\ &= -a_nz^{n+1}-(k\alpha_n-\alpha_n)z^n+(k\alpha_n-\alpha_{n-1})z^n\\ &\quad+i(\beta_n-\beta_{n-1})z^n+\sum_{j=0}^{n-1}(\alpha_j-\alpha_{j-1})z^j+\sum_{j=0}^{n-1}(\beta_j-\beta_{j-1})z^j.\end{aligned}$$

Let $|z|>1$, then

$$\begin{aligned}|F(z)| &= |-a_nz^{n+1}-(k\alpha_n-\alpha_n)z^n+(k\alpha_n-\alpha_{n-1})z^n\\ &\quad+i(\beta_n-\beta_{n-1})z^n+\sum_{j=0}^{n-1}(\alpha_j-\alpha_{j-1})z^j+\sum_{j=0}^{n-1}(\beta_j-\beta_{j-1})z^j|\end{aligned}$$

$$\begin{aligned}
&\geq \; |a_n z+(k-1)\alpha_n||z|^n-\{|k\alpha_n-\alpha_{n-1}||z^n|+(|\beta_n|+|\beta_{n-1}|)|z|^n \\
&\quad +\sum_{j=0}^{n-1}|\alpha_j-\alpha_{j-1}||z|^j+\sum_{j=0}^{n-1}(|\beta_j|+|\beta_{j-1}|)|z|^j\} \\
&= \; |z|^n\Bigg[|a_n z+(k-1)\alpha_n|-\Bigg\{|k\alpha_n-a_{n-1}|+|\beta_n|+|\beta_{n-1}| \\
&\quad +\sum_{j=0}^{n-1}\frac{|\alpha_j-\alpha_{j-1}|}{|z|^{n-j}}+\sum_{j=0}^{n-1}\frac{(|\beta_j|+|\beta_{j-1}|)}{|z|^{n-j}}\Bigg\}\Bigg] \\
&> \; |z|^n\Bigg[|a_n z+(k-1)\alpha_n|-\Bigg\{|k\alpha_n-a_{n-1}|+\sum_{j=0}^{n-1}|\alpha_j-\alpha_{j-1}| \\
&\quad +|\beta_n|+|\beta_{n-1}|+\sum_{j=0}^{n-1}(|\beta_j|+|\beta_{j-1}|)\Bigg\}\Bigg] \\
&= \; |z|^n\Bigg[|a_n z+(k-1)\alpha_n|-\Bigg\{(k\alpha_n-a_{n-1})+|\alpha_0| \\
&\quad +\sum_{j=1}^{n-1}(\alpha_j-\alpha_{j-1})+2\sum_{j=0}^{n}|\beta_j|-|\beta_n|\Bigg\}\Bigg] \\
&> \; |z|^n\Bigg[|a_n z+(k-1)\alpha_n|-(k\alpha_n-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|)\Bigg] \\
&> \; 0\,,
\end{aligned}$$

if

$$|z+(k-1)\frac{\alpha_n}{a_n}|>\frac{k\alpha_n-\alpha_0+|\alpha_0|+2\sum\limits_{j=0}^{n}|\beta_j|}{|a_n|}\,.$$

This shows that all the zeros of $F(z)$ whose modulus is greater than 1 lie in the circle

$$|z+(k-1)\frac{\alpha_n}{a_n}|\leq\frac{k\alpha_n-\alpha_0+|\alpha_0|+2\sum\limits_{j=0}^{n}|\beta_j|}{|a_n|}.$$

But it can be easily verified that those zeros of $F(z)$ whose modulus are less than or equal to 1, also lie in the circle defined by (1). Since all the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that all zeros of $P(z)$ lie in the circle defined (1) and this compeletes the proof of Theorem 1. ∎

P r o o f. **Theorem 3**. Consider the polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \sum_{j=1}^{n-1}(a_j - a_{j-1})z^j + a_0, \\ &= -a_n z^{n+1} - (ka_n - a_n)z^n + (ka_n - a_{n-1})z^n + \\ &\quad + \sum_{j=0}^{n-1}(\alpha_j - \alpha_{j-1})z^j + i\sum_{j=1}^{n-1}(\beta_j - \beta_{j-1})z^j + a_0. \end{aligned}$$

Let $|z| > 1$, then

$$\begin{aligned} |F(z)| &\geq \;, |a_n||z+(k-1)||z|^n - \Big\{|k\alpha_n - \alpha_{n-1}||z|^n + |k\beta_n - \beta_{n-1}||z|^n \\ &\quad + \sum_{j=1}^{n-1}|\alpha_j - \alpha_{j-1}||z|^j + \sum_{j=1}^{n-1}|\beta_j - \beta_{j-1}||z|^j + |a_0|\Big\} \\ &= |z|^n\Bigg[|a_n||z+k-1| - \Bigg\{\frac{|k\alpha_n - \alpha_{n-1}|}{|z|^{n-j}} + \frac{|k\beta_n - \beta_{n-1}|}{|z|^{n-j}} \\ &\quad + \sum_{j=1}^{n-1}\frac{|\alpha_j - \alpha_{j-1}|}{|z|^{n-j}} + \frac{|\beta_j - \beta_{j-1}|}{|z|^{n-j}} + \frac{|a_0|}{|z|^n}\Bigg\}\Bigg] \\ &> |z|^n\Bigg[|a_n||z+k-1| - \Bigg\{(k\alpha_n - \alpha_{n-1}) + (k\beta_n - \beta_{n-1}) \\ &\quad + \sum_{j=1}^{n-1}(\alpha_j - \alpha_{j-1}) + \sum_{j=1}^{n-1}(\beta_j - \beta_{j-1}) + |a_0|\Bigg\}\Bigg] \\ &= |z|^n\Bigg[|a_n||z+k-1| - \Bigg\{k(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|\Bigg\}\Bigg] \\ &> 0, \end{aligned}$$

if

$$|z+(k-1)| > \frac{k(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|}{|a_n|}.$$

Thus, $|F(z)| > 0$ if

$$|z+k-1| > \frac{k(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|}{|a_n|}.$$

This shows that all the zeros of $F(z)$ whose modulus is greater than 1 lie in the circle defined (2). Since it can be easily verified that those zeros of $F(z)$ whose modulus is less than or equal to 1 also lie in circle defined by (2), it follows that all the zeros of $F(z)$ and, hence, that of $P(z)$ lie in the circle

$$|z+(k-1)| \leq \frac{k(\alpha_n+\beta_n)-(\alpha_0+\beta_0)+|a_0|}{|a_n|}.$$

This completes the proof of Theorem 3. ∎

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Received 07.02.2006

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