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Inverse Matrix Updating in One Inhomogeneous Network Flow Programming Problem

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We obtain the recurrence relations for updating the elements of the inverse matrix of determinants in one inhomogeneous network flow programming problem of special structure.

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1. Introduction

There is a well-known problem of the inverse matrix updating when the initial matrix takes low-rank increment. According to the famous Woodbury formula

$$(A + B^T C)^{-1} = A^{-1} - A^{-1} B^T (I_k + C A^{-1} B^T)^{-1} C A^{-1}, \tag{1}$$

where $A \in \mathbb{R}_{n\times n}; B, C \in \mathbb{R}_{k\times n}; I_k$ is the identity matrix of order k. The formula (1) is worth using when $k \ll n$, because of $O(kn^2)$ computation complexity instead of usual $O(n^3)$. In the current work we apply this speed-up technique to updating the inverse matrix of determinants, appearing in one inhomogeneous network flow programming problem.

However, instead of applying the general formula we build special problem-related recurrences for elements of the inverse matrix of determinants.

2. Problem statement

Let $S = \{I, U\}$ be a finite oriented connected network, where I is the set of nodes and U is the set of arcs, $U \subseteq I \times I, |I| < \infty, |U| < \infty$. We also introduce the connected networks $S^k = \{I^k, \widetilde{U}^k\}, I^k \subseteq I, \widetilde{U}^k \subseteq U$, each corresponding to some flow type (product) $k \in K, |K| < \infty$. Let $K(i) = \{k \in K : i \in I^k\}, i \in I$, $K(i,j) = \{k \in K : (i,j) \in \widetilde{U}^k\}, (i,j) \in U$.

The elements of every network S^k have the following characteristics: a_i^k − the intensity of the node $i \in I^k$ for the product k; d_{ij}^k − the capacity of the arc $(i,j) \in \widetilde{U}_1^k \subseteq \widetilde{U}^k$ for the product k; x_{ij}^k − the flow of the product k through the arc $(i,j) \in \widetilde{U}^k$; c_{ij}^k − the cost of transportation of one unit of product k through the arc $(i,j) \in \widetilde{U}^k, k \in K$; $a_i = (a_i^k, k \in K(i))$ is the vector of intensities of the node $i \in I$; $d_{ij} = (d_{ij}^k, k \in K_1(i,j))$ is the vector of capacities of the arc $(i,j) \in U, K_1(i,j) = \{k \in K(i,j) : (i,j) \in \widetilde{U}_1^k\}$; d_{ij}^0 is the overall capacity of the arc $(i,j) \in U_0, U_0 \subset U$, U_0 is a given set, $\sum_{k \in K_0(i,j)} x_{ij}^k \leq d_{ij}^0, (i,j) \in U_0, K_0(i,j) = K(i,j) \setminus K_1(i,j), |K_0(i,j)| > 1$; $x_{ij} = (x_{ij}^k, k \in K(i,j))$ is the flow through the arc $(i,j) \in U$; $c_{ij} = (c_{ij}^k, k \in K(i,j))$ is the vector of costs of unitary flow transportation, $(i,j) \in U$. Thus, every arc $(i,j) \in U$ corresponds to $|K(i,j)|$ arcs $(i,j)^k$ having the flow x_{ij}^k, the cost c_{ij}^k, the capacity d_{ij}^k of the arc $(i,j) \in \widetilde{U}_1^k$. The given network $S = (I, U)$ is an union of $|K|$ networks $S^k = \{I^k, U^k\}, U^k = \{(i,j)^k : (i,j) \in \widetilde{U}^k\}, k \in K$. These networks are interrelated by overall capacity constraints for the arcs $(i,j)^k, k \in K_0(i,j), (i,j) \in U_0$ and by the additional constraints $\sum\limits_{(i,j) \in U} \sum\limits_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, p = \overline{1,l}$.

Consider the mathematical model of an extremal inhomogeneous network flow programming problem (2)-(7):

(2)
$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} c_{ij}^k x_{ij}^k \longrightarrow \min,$$

(3)
$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} x_{ji}^k = a_i^k, \quad \text{for } i \in I^k, k \in K;$$

(4)
$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, \quad \text{for } p = \overline{1,l};$$

(5)
$$\sum_{k \in K_0(i,j)} x_{ij}^k \leq d_{ij}^0, x_{ij}^k \geq 0, \quad \text{for } k \in K_0(i,j), (i,j) \in U_0;$$

(6) $$0 \le x_{ij}^k \le d_{ij}^k, \quad \text{for } k \in K_1(i,j), (i,j) \in U;$$

(7) $$x_{ij}^k \ge 0, \quad \text{for } k \in K(i,j) \setminus K_1(i,j), (i,j) \in U \setminus U_0,$$

$$I_i^+(U^k) = \{j \in I^k : (i,j)^k \in U^k\}, I_i^-(U^k) = \{j \in I^k : (j,i)^k \in U^k\}.$$

The vector $x = (x_{ij}, (i,j) \in U)$ — is a plan (flow), if it satisfies the constrains (3)–(7). The plan $x^0 \in X$ (X is the set of plans) is optimal if $c^T x^0 = \min c^T x, \forall x \in X, c = (c_{ij}, (i,j) \in U)$.

Definition 1. The plan (flow) $x^\varepsilon = (x_{ij}^\varepsilon, (i,j) \in U), x_{ij}^\varepsilon = (x_{ij}^{\varepsilon k}, k \in K(i,j))$ is suboptimal (ε-optimal) if

$$\sum_{k \in K(i,j)} \sum_{(i,j)^k \in U^k} c_{ij}^k x_{ij}^{\varepsilon k} - \sum_{k \in K(i,j)} \sum_{(i,j)^k \in U^k} c_{ij}^k x_{ij}^{0k} \le \varepsilon$$

3. Network support. Matrix of determinants

Consider the structure of support $U_{on} = \{U_{on}^k, k \in K; U^*\}, U_{on}^k \subset U^k, k \in K; U^* \subset \overline{U}_0, \overline{U}_0 = \{(i,j) \in U_0 : |K_{on}^0| > 1\}, K_{on}(i,j) = \{k \in K(i,j) : (i,j)^k \in U_{on}^k\}, (i,j) \in U, K_{on}^0(i,j) = K_{on}(i,j) \bigcap K_0(i,j), (i,j) \in U_0$. Let the new set U_{on}^k contain l_k arcs $(i,j)^k$ such that removing them from the set U_{on}^k gives the set U_D^k with no cycles while every set $U_D^k \bigcup (i,j)^k$ contains a cycle. Let us denote $U_a^k = U_{on}^k \setminus U_D^k, k \in K$.

Definition 2. The elements of the set $U_a = \bigcup\limits_{k \in K} U_a^k$ are cycle arcs.

Let K be the set $\{1, 2, \ldots, |K|\}$. Let U_{on}^k contain $l(k)$ independent cycles, $k \in K$. The cycles $J_1, J_2, \ldots, J_n$ of the network S are called independent if every cycle has at least one edge that does not belong to other cycles and the unitary circulations [3,5] form a linearly-independent system of vectors.

Every arc $(i,j) \in U_a^k$ belongs to some cycle L_{ij}^k from the set U_{on}^k. If the set U_{on}^k is connected then the cycles $Z = \{L_{ij}^k, (i,j)^k \in U_a^k, k \in K\}$ introduced the same way form the fundamental set of cycles. Let us mark every arc $(i,j)^k \in U_a^k$ with the number $t = t(i,j)^k$, $\sum\limits_{s=1}^{k-1} l(s) + 1 \le t \le \sum\limits_{s=1}^{k} l(s)$. We denote $\widetilde{t} = \sum\limits_{s=1}^{|K|} l(s)$. Let us consider an arbitrary cycle $L_t^k, t = t(i,j)^k$. We choose the direction of cycle detour such that the arc $(i,j)^k$ is a forward one. Let L_t^{k+}, L_t^{k-} be sets of forward and backward arcs of the cycle L_t^k correspondingly.

Definition 3. The number $R_p(L_t^k) = \sum\limits_{(i,j)\in L_t^k} \lambda_{ij}^{kp} \mathrm{sign}(i,j)^{L_t^k}$ is called the determinant of the cycle L_t^k with respect to the additional restriction (3) with the number p, where

$$\mathrm{sign}(i,j)^{L_t^k} = \begin{cases} 1, & (i,j)^k \in L_t^{k+}; \\ -1, & (i,j)^k \in L_t^{k-}; \\ 0, & (i,j)^k \notin L_t^k. \end{cases}$$

Let us put the arcs of the set U^* in an arbitrary order. Let $\tau = \tau(i,j)$ be the serial number of the arc (i,j) in the set $U^*, 1 \le \tau \le m, m = |U^*|$.

We build the matrix $D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$, where $D_1 = (R_p(L_t^k), p = \overline{1,l}, t = \overline{1,\widetilde{t}})$ is an $l \times \widetilde{t}$-matrix consisting of arc determinants $R_p(L_t^k)$ with respect to the restrictions (3), $t \in R(k), R(k) = \{t : \sum\limits_{s=1}^{k-1} l(s) + 1 \le t \le \sum\limits_{s=1}^{k} l(s)\}, k \in K$; $D_2 = (\delta_{\tau(i,j)}(L_t^k), \tau(i,j) = \overline{1,m}, t = \overline{1,\widetilde{t}})$ is an $m \times \widetilde{t}$-matrix that consists of the following elements:

$$\delta_{\tau(i,j)}(L_t^k) = \begin{cases} 1, & (i,j) \bigcap L_t^k \ne \emptyset, (i,j)^k \in L_t^{k+}; \\ -1, & (i,j) \bigcap L_t^k \ne \emptyset, (i,j)^k \in L_t^{k-}; \\ 0, & (i,j) \bigcap L_t^k \ne \emptyset, k \in K_{on}^0(i,j), (i,j) \in U^*; \end{cases}$$

$$\tau = \tau(i,j), (i,j) \in U^*, t \in R(k), k \in K.$$

If $\widetilde{t} \ne l + m, m = |U^*|$ we supply the matrix D with zeros to make it a square matrix of order $\max\{\widetilde{t}, l + |U^*|\}$. Let $R(U_{on}) = \det D$.

The set of arcs $U_{on} = \{U_{on}^k, k \in K, U^*\}, U_{on}^k \subset U^k, k \in K; U^* \subset \overline{U}_0, \overline{U}_0 = \{(i,j) \in U_0 : |K_{on}^0| > 1\}, K_{on}(i,j) = \{k \in K(i,j) : (i,j)^k \in U_{on}^k\}, (i,j) \in U, K_{on}^0(i,j) = K_{on}(i,j) \bigcap K_0(i,j), (i,j) \in U_0$ is a support of the network S iff the following conditions are met:

1. $I(U_{on}^k) = I^k, k \in K$;

2. U_{on}^k is a connected set, $k \in K$;

3. $R(U_{on}) \ne 0$.

Let $D = D(U_{on})$ be the matrix of determinants that corresponds to the support $U_{on} = \{U^k_{on}, k \in K, U^*\}$. Let $\overline{D} = \overline{D}(\overline{U}_{on})$ be the matrix of determinants corresponding to the support $\overline{U}_{on} = \{\overline{U}^k_{on}, k \in K, \overline{U}^*\}$. Let us denote

$$D^{-1} = [D(U_{on})]^{-1} = (\upsilon_{pq}, p = \overline{1, l + |U^*|}, q = \overline{1, \widetilde{t}});$$

$$\overline{D}^{\,-1} = [D(\overline{U}_{on})]^{-1} = (\overline{\upsilon}_{pq}, p = \overline{1, l + |U^*|}, q = \overline{1, \widetilde{t}}).$$

Let $\widetilde{D}^{-1} = [D(\widetilde{U}_{on})]^{-1} = (\widetilde{\upsilon}_{pq}, p = \overline{1, l + |U^*|}, q = \overline{1, \widetilde{t}})$ be the inverse support matrix, that can be obtained through elementary transformations (the definition of elementary transformations is given further).

4. Updating the inverse matrix

Definition 4. Let us define the elementary transformation of the matrix D^{-1} as the transfer from the matrix D^{-1} to some matrix $\widetilde{D}^{-1}$ that corresponds to such a modifications of the matrix D as:

1) rank-1 increment: rank$\widetilde{D}$ = rank$D$;
2) edging: rank$\widetilde{D}$ = rank$D + 1$;
3) truncation: rank$\widetilde{D}$ = rank$D - 1$;

We obtain the matrix $\overline{D}^{\,-1}$ from D^{-1} through elementary transformations performed in a suitable order.

Now we are going to get the recurrence relations connecting the matrices $[D(\widetilde{U}_{on})]^{-1}$ and $[D(U_{on})]^{-1}$ met on the adaptive method iterations [3] when searching for an optimal inhomogeneous flow. Let us suppose that while building the new flow $\overline{x} = x + \Delta x$ the maximal admissible step θ_0 is achieved either on the arc $(i_0, j_0)^k$ or (i_0, j_0). The step θ_0 is computed using standard rules [3]. The maximal dual step σ_0 is computed according to [3] and is achieved either on the arc $(i_*, j_*)^l$ or (i_*, j_*).

If one is searching for the arc (i_*, j_*) while building the co-flow increment $\delta = (\delta_{ij}, (i,j) \in U)$, $\delta_{ij} = (\delta^k_{ij}, k \in K(i,j))$, several cases are possible:

a) $\theta_0 = \theta^{k_0}_{i_0 j_0}$
b) $\theta_0 = \theta_{i_0 j_0}$

Let us consider the case a). Wlog $k_0 = 1$. Let $(i_0, j_0) \notin U^*$. The non-support components of δ, according to the formula

$$\sigma\delta_{ij}^k = -(\Delta u_i^k - \Delta u_j^k + \sum_{p=1}^{l} \lambda_{ij}^{kp} r_p), k \in K(i,j), (i,j) \in U,$$

depend on the potential increments $\Delta u_i^k, i \in I^k, k \in K$; $\Delta_p, p = \overline{1,l}, \gamma_{ij}, (i,j) \in U^*$, that are obtained from the system

(8)
$$\begin{gathered}
-(\Delta u_{i_0}^{k_0} - \Delta u_{j_0}^{k_0} + \sum_{p=1}^{l} \lambda_{i_0 j_0}^{k_0 p} \Delta r_p) = \alpha_0; \\
-(\Delta u_{i_0}^{k} - \Delta u_{j_0}^{k} + \sum_{p=1}^{l} \lambda_{ij}^{kp} \Delta r_p) = 0, \\
(i,j)^k \in U_{on}^k \backslash (U^* \bigcup (i_0, j_0)^{k_0}), k \in K; \\
-(\Delta u_{i_0}^{k} - \Delta u_{j_0}^{k} + \sum_{p=1}^{l} \lambda_{ij}^{kp} \Delta r_p) = -\Delta\gamma_{ij}, \\
(i,j) \in U^*, k \in K, \alpha_0 = 1.
\end{gathered}$$

Now let us suppose that $(i_0, j_0) \in U^*$. The potential increments are computed from the system

(9)
$$\begin{gathered}
-(\Delta u_{i_0}^{k_0} - \Delta u_{j_0}^{k_0} + \sum_{p=1}^{l} \lambda_{i_0 j_0}^{k_0 p} \Delta r_p) = -\Delta\gamma_{i_0 j_0} + 1; \\
-(\Delta u_{i_0}^{k} - \Delta u_{j_0}^{k} + \sum_{p=1}^{l} \lambda_{ij}^{kp} \Delta r_p) = -\Delta_{i_0 j_0}, \\
k \in K_{on}^0(i_0, j_0); \\
-(\Delta u_{i}^{k} - \Delta u_{j}^{k} + \sum_{p=1}^{l} \lambda_{ij}^{kp} \Delta r_p) = 0, \\
k \in K(i,j), (i,j) \in U \backslash (U^* \backslash (i_0, j_0)); \\
-(\Delta u_{i}^{k} - \Delta u_{j}^{k} + \sum_{p=1}^{l} \lambda_{ij}^{kp} \Delta r_p) = -\Delta\gamma_{ij}, \\
k \in K(i,j), (i,j) \in U^* \backslash (i_0, j_0).
\end{gathered}$$

Similarly to (8), (9) we build a system to compute the potential increments for the case b) putting $\alpha_0 = -1$.

The effective algorithms for solving (8), (9), based on variable decomposition principles, are considered in [6].

Thus, $(i_0, j_0)^k$ is the arc excluded from the support, $(i_*, j_*)^l$ is the arc introduced into the support, $L^k_{i_0 j_0}$ is the cycle corresponding to the arc $(i_0, j_0)^{k_0}$, $(i_0, j_0)^{k_0} \in U_a$. Let us denote the cycle generated by the arc $(i_*, j_*)^k$ as $L^k_{i_* j_*}$.

Wlog we put $l = k = 1$.

Let us consider the case

1) $\mathrm{rank} D = \mathrm{rank} \widetilde{D}$. We consider the following subcases:

1. If $(i_0, j_0)^1 \in U^1_a$ then $\widetilde{U}_a = (U_a \setminus (i_0, j_0)^1) \bigcup (i_*, j_*)^1, \widetilde{U}^k_D = U^k_D, k \in K, \widetilde{U}^* = U^*$. Let μ be the serial number of a cycle that corresponds to the arc $(i_0, j_0)^1$. The elements $\widetilde{\upsilon}_{qp}$ of the matrix $[D(\widetilde{U}_{on})]^{-1}$ may be evaluated the following way

$$(10) \qquad \begin{aligned} &\widetilde{\upsilon}_{qp} = \upsilon_{qp} - \frac{\Delta r_p}{\delta^1_{i_*, j_*}} y_q, q = \overline{1, l + |U^*|}, q \neq \mu, p = \overline{1, \widetilde{t}}, \\ &y_q = \sum_{p=1}^{l} R_p(L^1_{i_*, j_*}) \upsilon_{qp} + \sum_{\tau=1}^{|U^*|} \delta_{l+\tau}(L^1_{i_*, j_*}) \upsilon_{q, l+\tau}, \\ &\tau = \tau(i, j), (i, j) \in U^*; \end{aligned}$$

$$(11) \qquad \widetilde{\upsilon}_{\mu p} = \Delta r_p / \delta^1_{i_*, j_*}, p = \overline{1, \widetilde{t}}.$$

2. If $(i_0, j_0)^1 \in U^1_D, (i_0, j_0)^1 \in L^1_{i_*, j_*}$ then $\widetilde{U}^1_D = (U^1_D \setminus (i_0, j_0)^1) \bigcup (i_*, j_*)^1$, $\widetilde{U}^k_D = U^k_D, k \in K \setminus \{1\}; \widetilde{U}^k_a = U^k_a, k \in K, \widetilde{U}^* = U^*$. If the arc $(i_0, j_0)^1$ does not belong to any support cycle then $[D(\widetilde{U}_{on})]^{-1} = [D(U_{on})]^{-1}$. Otherwise the elements $\widetilde{\upsilon}_{qp}$ are evaluated using the relations (10) for $q = \overline{1, l + |U^*|}$.

3. If $(i_0, j_0)^1 \in U^1_D, (i_0, j_0)^1 \notin L^1_{i_* j_*}$ then $\widetilde{U}^1_a = (U^1_a \setminus (\xi_1, \eta_1)^1) \bigcup (i_*, j_*)^1$, $\widetilde{U}^1_D = (U^1_D \setminus (i_0, j_0)^1) \bigcup (\xi_1, \eta_1)^1$, where $(\xi_1, \eta_1)^1$ is the cycle arc of the support cycle that includes the arc $(i_0, j_0)^1$ (it is easy to prove the existence of such a cycle). The process of excluding the arc $(i_0, j_0)^1$ from the set U^1_D has 2 stages. At the first stage we perform the following transformations of the support sets:

$$\widetilde{\widetilde{U}}^1_a = (U^1_a \setminus (\xi_1, \eta_1)^1) \bigcup (i_0, j_0)^1, \widetilde{\widetilde{U}}^2_a = U^2_a, \widetilde{\widetilde{U}}^1_D = (U^1_D \setminus (i_0, j_0)^1) \bigcup (\xi_1, \eta_1)^1.$$

At the second stage we get:

$\widetilde{U}_a^1 = (\widetilde{\widetilde{U}}_a^1 \setminus (i_0,j_0)^1) \bigcup (i_*,j_*)^1, \widetilde{U}_a^k = U_a^k, k \in K \setminus \{1\}, \widetilde{U}_D^k = \widetilde{\widetilde{U}}_D^k, k \in K, \widetilde{U}^* = U^*$. We use the relations (10), (11) to evaluate the elements of $[D(\widetilde{U}_{on})]^{-1}$.

4. If $(i_0,j_0) \in U^*, (i_*,j_*) \in U^*$ then $\widetilde{U}_a^k = U_a^k, \widetilde{U}_D^k = U_D^k, k \in K; \widetilde{U}^* = (U^* \setminus (i_0,j_0)) \bigcup (i_*,j_*)$. To transform the matrix $D(U_{on})$ into $D(\widetilde{U}_{on})$ we replace the row that corresponds to the arc (i_0,j_0) (and to the restriction $\sum\limits_{k\in K_0(i_0,j_0)} x_{i_0j_0}^k = d_{i_0j_0}^0$) by the row that corresponds to the arc (i_*,j_*) (and to the restriction $\sum\limits_{k\in K_0(i_0,j_0)} \overline{x}_{i_0j_0}^k = d_{i_0j_0}^0$). It is easy to prove the following relations:

$$\widetilde{\upsilon}_{qp} = \upsilon_{qp} - z_{i_qj_q}/z_{i_\mu j_\mu}, q = \overline{1,\widetilde{t}}, q \neq \mu, p = \overline{1, l + |U^*|};$$
$$\widetilde{\upsilon}_{p\mu} = \upsilon_{p\mu}/z_{i_\mu j_\mu}, p = \overline{1, l + |U^*|}, z^T = b^T D^{-1}$$

where $b^T = (\delta_{\tau(i_*,j_*)}(L_1^k), \ldots, \delta_{\tau(i_*,j_*)}(L_{\widetilde{t}}^k))$, μ is the number of the replaced row.

Now let us consider the case

2) $\mathrm{rank}\widetilde{D} = 1 + \mathrm{rank}D$. Let us denote $\widetilde{U}^* = U^* \bigcup (i_0,j_0), \widetilde{U}_a^1 = U_a^1 \bigcup (i_*,j_*)^1$. To obtain the matrix $D(\widetilde{U}_{on})$, we add a column, that corresponds to the cycle $L_{i_*j_*}^1$, and a row, that corresponds to the restriction $\sum\limits_{k\in K_0(i_0,j_0)} x_{i_0j_0}^k = d_{i_0j_0}^0$, to the matrix $D(U_{on})$. In this case the recurrence relations that connect the elements of the matrices $[D(\widetilde{U}_{on})]^{-1}$ and $[D(U_{on})]^{-1}$ are the following:

$$\widetilde{\upsilon}_{qp} = \upsilon_{qp} - \frac{\Delta r_p}{\delta_{i_*j_*}^1} y_q, q = \overline{1, l + |U^*|}, p = \overline{1,\widetilde{t}};$$
$$\widetilde{\upsilon}_{l+|U^*|+1,p} = \frac{\Delta r_p}{\delta_{i_*j_*}^1}, p = \overline{1,\widetilde{t}};$$
$$\widetilde{\upsilon}_{q,\widetilde{t}+1} = -\frac{\Delta r_{p_0}}{\delta_{i_*j_*}^1} y_q, q = \overline{1, l + |U^*|};$$
$$\widetilde{\upsilon}_{\widetilde{t}+1,\widetilde{t}+1} = \frac{\Delta r_{p_0}}{\delta_{i_*j_*}^1},$$
$$y_q = \sum_{p=1}^{l} \upsilon_{qp} R_p(L_{i_*j_*}^1) + \sum_{\tau=1}^{|U^*|} \upsilon_{q,l+\tau} \delta_\tau(L_{i_*j_*}^1).$$

Here Δr_{p_0} is the increment of the potential r_{p_0} corresponding to the restriction $\sum\limits_{k\in K_0(i_0,j_0)} x_{i_0j_0}^k = d_{i_0j_0}^0$. The numbers $\Delta r_p, \delta_{i_*j_*}^1$ are evaluated using (8),(9).

Finally, we consider the case

3) $\mathrm{rank}\widetilde{D} = \mathrm{rank}D - 1$. There are the following subcases:

1. $(i_0, j_0)^1 \in U_a^1, (i_*, j_*) \in U^*$.
   $\widetilde{U}_a^1 = U_a^1 \setminus (i_0, j_0)^1, \widetilde{U}_a^k = U_a^k, k \in K \setminus \{1\}, \widetilde{U}_D^k = U_D^k, k \in K, \widetilde{U}^* = U^* \setminus (i_*, j_*)$. To obtain the matrix $D(\widetilde{U}_{on})$ we exclude a column with the serial number η, corresponding to the cycle $L^1_{i_0 j_0}$, and a row with the serial number $(l+\tau)$, $\tau = \tau(i,j), (i,j) \in U^*$, that corresponds to the restriction $\sum\limits_{k \in K_0(i_*, j_*)} x^k_{i_* j_*} = d^0_{i_* j_*}$. The elements of the matrices $[D(\widetilde{U}_{on})]^{-1}$ and $[D(U_{on})]^{-1}$ are connected with the following recurrence relations

$$(12)\quad \widetilde{v}_{qp} = v_{qp} - \frac{\Delta r_p}{\Delta r_{l+\tau}} v_{q,l+\tau}, \quad q = \overline{1, l + |U^*|}, q \neq \eta, p = \overline{1, \widetilde{t}}, p \neq l + \tau$$

2. In this case the arc $(i_0, j_0)^1$ is included into at least one support cycle. Let $(i_0, j_0)^1 \in L^1_{\xi\eta}, (\xi, \eta)^1 \in \check{U}_a^1$ where $\check{U}_a^1 \subseteq U_a^1$. Then $\widetilde{U}_a^1 = U_a^1 \setminus (\xi_1, \eta_1)^1, \widetilde{U}_a^k = U_a^k, k \in K \setminus \{1\}, \widetilde{U}_D^k = U_D^k, k \in K, \widetilde{U}_D^1 = (U_D^1 \setminus (i_0, j_0)^1) \bigcup (\xi_1, \eta_1)^1$. Here $(\xi_1, \eta_1)^1$ is an arbitrary arc from the set $\check{U}_a^1, \widetilde{U}^* = U^* \setminus (i_*, j_*)$. The elements of the matrix $[D(\widetilde{U}_{on})]^{-1}$ are evaluated using (12).

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