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# Involutions in $\mathbb{Q}$-Groups

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A finite group whose irreducible complex characters are rational is called a $\mathbb{Q}$-group. In this paper we will find the structure of $\mathbb{Q}$-groups $G$ which contain a CC-subgroup. We also study the Brauer-Fowler distance in relation with the set of involutions in $G$.

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## 1. Introduction

Let $G$ be a finite group and $\chi$ be a complex character of $G$. then $\chi$ is called a rational character if for every $x \in G$, $\chi(x) \in \mathbb{Q}$. If every irreducible complex character of a finite group $G$ is rational then $G$ is called a rational group or a $\mathbb{Q}$-group.

The symmetric group $S_n$ and the Weyl groups of the complex Lie algebras are examples of $\mathbb{Q}$-groups, see [11]. Some important properties of $\mathbb{Q}$-groups can be found in [6] and [5]. In [4] we found the structure of all Frobenius $\mathbb{Q}$-groups. Berkovich and Zhmud in ([2], p. 59) give a nice description of $\mathbb{Q}$-groups.

Let $K_1, \ldots, K_t$ be the conjugacy classes of $G$ containing involutions. We set $M = K_1 \cup \ldots \cup K_t$ which is the set of all involutions in $G$ and it is also denoted by $Inv(G)$. In this paper we consider properties of $G$ related to $Inv(G)$. Specially we consider the Brauer-Fowler's distance and will prove that the distance of two involutions in a $\mathbb{Q}$-group $G$ is greater than 1 if and only if a Sylow 2-subgroup of $G$ is isomorphic either to $\mathbb{Z}_2$ or $Q_8$. We also prove that a Sylow 2-subgroup of $G$ is isomorphic to $\mathbb{Z}_2$ or $Q_8$ if and only if $G$ contains a strongly embedded subgroup or equivalently $G$ contains a $CC$-subgroup. The notation used throughout this paper is standard, in particular the semi-direct product of groups $H$ and $K$ is denoted by $H : K$ also if $p$ is a prime number, $E(p^n)$ denotes the elementary abelian $p$-group of order $p^n$, and we write $F(G)$ for the Fitting subgroup of $G$.

## 2. Some Properties of $\mathbb{Q}$-groups

Let $G$ be a group. We define the distance $d$ on the set $G^{\#} = G - \{1\}$ as follow. We define $d(x,x) = 0$ for all $x \in G^{\#}$. For $x, y \in G^{\#}$, $x \neq y$, then define $d(x,y)$ to be the shortest length of a chain $x = a_1, a_2, \ldots, a_n = y$ such that $a_i \in G^{\#}$ for all $1 \leq i \leq n$ and the consecutive terms of the chain commute. If there is no such a chain we set $d(x,y) = \infty$. This distance is called the Brauer-Fowler distance. It satisfies the triangle inequality. Two elements $x, y$ of $G^{\#}$ are said to be connected if $d(x,y) < \infty$ otherwise they are called disconnected.

Before stating the results of this section, we mention some well-known properties about $\mathbb{Q}$-group. These can be found in ([9], p. 537, satz 13.7).

**Definition 1.** *An element $g \in G$ is said to be rational if it is conjugate to all generators of the cyclic group $< g >$.*

**Property 1.** A group $G$ is a $\mathbb{Q}$-group if and only if every $g \in G$ is rational. Equivalently, $N_G(A)/C_G(A) \cong Aut(A)$ for every cyclic subgroup $A$ of $G$.

**Property 2.** Every quotient of a $\mathbb{Q}$-group is a $\mathbb{Q}$-group

**Property 3.** All non-trivial $\mathbb{Q}$-groups are of even order, therefore the set of involutions in a $\mathbb{Q}$-group is not empty.

Now let $G$ be a $\mathbb{Q}$-group and set $M = Inv(G)$. We consider the Brauer-Fowler distance $d$ on $M$. If for every two elements $i$ and $j$ in $M$ we have $d(i,j) \leq 1$, then this means that the Sylow 2-subgroups of $G$ are abelian and hence by [11] are isomorphic to an elementary abelian 2-group. In the following Lemma we consider the case $d(i,j) > 1$.

**Lemma 1.** *Let $G$ be a $\mathbb{Q}$-group and $M = Inv(G)$. Then for every two distinct elements $i$ and $j$ of $M$ we have $d(i,j) > 1$, if and only if a Sylow 2-subgroup of $G$ is isomorphic to either $\mathbb{Z}_2$ or $Q_8$.*

Proof. Let $i$ be an involution in $G$ and Let $P$ be a Sylow 2-subgroup of $G$ containing $i$. Since for every pair of involutions $i \neq j$ we have assumed $d(i,j) > 1$, hence the only involution in $C_G(i)$ is $i$. Now $Z(P)$ is an elementary abelian 2-group and $Z(P) \neq C_G(i)$ implying that $Z(P) =< i >$. By ([9], satz 8.2) $P$ is isomorphic either to a cyclic group or the generalized quaternion group. If $P$ is cyclic then $P \cong \mathbb{Z}_2$. If $P$ is the generalized quaternion group, then $P =< x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}}, y^{-1}xy = x^{-1} >$ and $P$ has order $2^{n+1}$ where $n \geq 2$. Let $A =< x >$ be the cyclic subgroup of $P$ with order $2^n$. Then we must have $\frac{N_G(A)}{C_G(A)} \cong Aut(A)$, hence $|N_G(A)|_2 = |C_G(A)|_2 \times 2^{n-1}$, where $|n|_2$ denotes the 2-part of the positive integer $n$. Therefore $2^{n+1} \geq 2^n \times 2^{n-1}$ implying $n \leq 2$. Hence $n = 2$ and $P \cong Q_8$ proving the first part of the Lemma. Conversely assume a Sylow 2-subgroup of $G$ is isomorphic either to $\mathbb{Z}_2$ or $Q_8$. Let $i \in M$. If there is a $j \in C_G(i)$ different from $i$, then $< i, j >$ is a fourgroup contained in a Sylow 2-subgroup of $G$ which contradicts the structure of the Sylow 2-subgroups of $G$. Therefore $C_G(i) =< i >$ and if there is $j \in M$ different from $i$, then $d(i, j) > 1$ proving the Lemma. ■

Next we consider $\mathbb{Q}$-groups containing a strongly embedded subgroup.

**Definition 2.** *A subgroup $H$ of a finite group $G$ is said to be strongly embedded in $G$ if the following two conditions are satisfied:*

*1) $H$ is a proper subgroup of even order,*

*2) for any element $x \in G - H$, the order of $H \cap H^x$ is odd.*

For Example, a Frobenius group has such a subgroup. In the following, we will find the structure of a Sylow 2-subgroup of a $\mathbb{Q}$-group having a strongly embedded subgroup.

**Theorem 1.** *Let $G$ be a $\mathbb{Q}$-group having a strongly embedded subgroup. Then $G$ is isomorphic to one of the following groups:*

*1) $G \cong G' : \mathbb{Z}_2$,*

*2) $G \cong E(p^n) : Q_8$.*

*Where $E(p^n)$ is an elementary abelian $p$-group of odd order $p^n$ and $G'$ is the commutator subgroup of $G$.*

Proof. By ([12], p. 391) every Sylow 2-subgroup of $G$ contains exactly one element of order 2 or there exists a proper normal subgroup $L$ such that $G/L$ is of odd order. But the second case is impossible because of Property 2. Now if every Sylow 2-subgroup of $G$ contains exactly one involution then by using the same argument as used in the proof of Lemma 1 it can be shown that a Sylow 2-subgroup $P$ of $G$ is isomorphic to $\mathbb{Z}_2$ or $Q_8$. If $P \cong \mathbb{Z}_2$ then by [8] case (1) holds. If $P \cong Q_8$ then by ([11], p. 35) case (2) holds. ■

## 3. $\mathbb{Q}$-groups with a CC-subgroup

First we characterize CC-subgroups of a $\mathbb{Q}$-group and show that groups having a CC-subgroup are exactly Frobenius $\mathbb{Q}$-groups.

**Definition 3.** *A proper subgroup $H$ of $G$ is called a CC-subgroup of $G$ if $C_G(x) \leq H$ for any $x \in H^{\#}$.*

For finding the structure of $\mathbb{Q}$-groups with a CC-subgroup of even order, we use strongly embedded subgroups rather than using the classification of CC-subgroups.

**Theorem 2.** *([12], p. 407) The following conditions on a group $G$ are equivalent:*

*1) The group $G$ contains a strongly embedded subgroup.*

*2) The group $G$ contains a subgroup $K$ of even order such that $Inv(G) \nsubseteq K$ and $C_G(t) \subseteq K$ for any $t \in Inv(K)$.*

**Theorem 3.** *Let $H$ be a CC-subgroup of even order of a $\mathbb{Q}$-group $G$. Then a Sylow 2-subgroup of $H$ is isomorphic to either $\mathbb{Z}_2$ or $Q_8$.*

P r o o f. If $Inv(G) \subseteq H$, then we show $H \trianglelefteq G$. In this case the subgroup $K$ of $G$ generated by $Inv(G)$ is normal in $G$ and hence $K \trianglelefteq H$. If we let $C = \bigcap_{x \in G} H^x$, then $K \leq C \underset{\neq}{\lhd} G$, hence $C$ is a proper non-trivial subgroup of $G$. But since $C$ is also a CC-subgroup of $G$, hence by [12] page 280, $G$ would be a Frobenius group with kernel $C$. Since $C$ has even order and $(\left|\frac{G}{C}\right|, |C|) = 1$ we deduce that $\frac{G}{C}$ is a $\mathbb{Q}$-group of odd order. Therefore $G = C$ implying that $G \leq H$ a contradiction to the fact that $H$ is a proper subgroup of $G$.

Hence we must have $Inv(G) \nsubseteq H$. Now by Theorem 2, $G$ contains a strongly embedded subgroup, hence by Theorem 1 every Sylow 2-subgroup of $G$ contains exactly one involution. If $t \in H$ is an involution, then we may assume $t$ is in the center of a Sylow 2-subgroup $P$ of $G$. Since $H$ is a CC-subgroup of $G$, hence $P \subseteq C_G(t) \subseteq H$ which implies that $H$ contains a Sylow 2-subgroup of $G$ and the Theorem is proved. ■

We use the classification of CC-subgroups of odd order for finding the structure of $\mathbb{Q}$-groups with such a CC-subgroup.

According to [7] a 2-Frobenius group $G$ contains a normal series $K < F < G$ such that $F$ and $G/K$ are Frobenius groups having kernels $K$ and $F/K$ respectively.

**Theorem 4.** *(Theorem A in [1]) Let $G$ be a finite group containing a non-normal CC-subgroup $A$ of odd order. Then either $G$ is a Frobenius group with complement $A$ or we have one of the following cases:*

*1) If $A$ is non-nilpotent, then $G \cong PSL(2, q)$, $q \equiv 3 (mod\ 4)$ and $A$ is solvable of odd order $|A| = \frac{q(q-1)}{2}$.*

*2) If $A$ is nilpotent then*

*(i) $G$ is a simple non-abelian group which is classified in ([1], p. 9-13, Tables 1-6),*

*(ii) $G$ is not simple and if $H$ is the normal closure of $A$ in $G$, then $F(H) = F(G)$ and $H/F(G)$ is a simple group of type (2) (i);*

*(iii) $G$ is a 2-Frobenius group.*

**Theorem 5.** *Let $H$ be a CC-subgroup of odd order in a $\mathbb{Q}$-group $G$. Then either all the composition factors of $G$ are of order 2 or; $H \cong E(3^n)$ or $E(5^n)$.*

Proof. We consider two cases.

**Case 1** $H$ is not a normal subgroup of $G$.

By Theorem 4 and Property 2, $G$ cannot be a Frobenius group with complement $H$. By [3], $PSL(2, 3)$ is not a $\mathbb{Q}$-group and in other case $PSL(2, q)$ is a simple group. But the only non-cyclic simple $\mathbb{Q}$-groups are $Sp_6(2)$ or $O_8^+(2)$ ([5] Theorem B.1). Therefore, case 1 and (2) (i) in Theorem 4 are impossible. Assume $G$ is not simple. By ([5] Theorem B), a non-cyclic finite simple group is a composition factor of a $\mathbb{Q}$-group if and only if it is isomorphic to an alternating group or one of the following groups: $PSP_4(3)$, $SP_6(2)$, $O_8^+(2)$, $PSL_3(4)$ or $PSP_4(3)$. But none of these groups occur in tables of simple groups in ([1] p. 9-13). Therefore all the composition factors of $G$ must be cyclic of prime order $p$, and since $G$ is assumed to be a $\mathbb{Q}$-group we must have $p = 2$. Thus case (2) (ii) in Theorem 4 is possible only if all the composition factors of $G$ are of order 2. By properties 2 and 3 case (2) (iii) in Theorem 3 is not possible.

**Case 2** $H$ is a normal subgroup of $G$.

By ([10] p. 121) $G$ is a Frobenius group with Frobenius kernel $H$ and complement $K$ which is a CC-subgroup. Because $G/H$ is of even order, thus $K$ must be a CC-subgroup of even order. By Theorem 3, $K \cong \mathbb{Z}_2$ or $Q_8$. Since $K$ is of even order, $H$ is abelian ([10] p. 114, Lemma (7.21)). Because $G/H$ is solvable then by [6] $\pi(G) \subseteq \{2, 3, 5\}$. Thus $\pi(H) \subseteq \{3, 5\}$. if $g \in H$, then we may put $o(g) = 3^r 5^s$ where $r$ and $s$ are non-negative integers. Since $H$ is a CC-subgroup of a $\mathbb{Q}$-group $G$ we must have $C_G(g) = K$ and $N_G(< g >)/C_G(< g >) \cong Aut(< g >)$. But $|Aut(< g >)| = \varphi(o(g))$ must be a power of 2 from which we obtain $r$, $s \leq 1$. Therefore the order of every non-identity element of $H$ is either 3, 5 or 15. If $H$ has an element of order 15, say $g$, then from $N_G(< g >)/C_G(< g >) \cong Aut(< g >)$ we deduce that $N_G(< g >)/K$ has order $\varphi(15) = 8$, hence it must be isomorphic to $Q_8$. But this is a contradiction because $Aut(< g >)$ is an abelian group. Therefore $H$ is elementary abelian 3 or 5-group and the Theorem is proved. ∎

**Theorem 6.** *Suppose $d(i,j) > 6$ for some $i, j \in M$. Then $G$ contains an abelian CC-subgroup of odd order.*

Proof. Let $i, j \in M$ and put $x = ij$. It is clear that $x^i = x^{-1}$. We will show that either $d(x,i) \geq 4$ or $d(x,j) \geq 4$. Otherwise, $d(x,i) \leq 3$ and $d(x,j) \leq 3$ and by the triangle inequality $6 < d(x,i) + d(x,j) = 3 + 3$, which is impossible. We may assume that $d(x,i) \geq 4$. Let $H = C_G(x)$, then

$H^i = C_G(x)^i = C_G(x^i) = C_G(x^{-1}) = C_G(x) = H$

If $y \in H^{\#}$ then clearly $d(x,y) \leq 1$. By assumption and triangle inequality $4 \leq d(x,i) \leq d(x,y) + d(y,i) \leq 1 + d(y,i)$ implying $d(y,i) \geq 3$, that is $y^i \neq y$. Therefore, we may consider $i$ as a fixed point free automorphism of order 2 of $H$. Thus $H$ is an abelian subgroup of $G$ of odd order (Burnside). On the other hand, for any $y \in H^{\#}$, $C_G(y)$ is an abelian subgroup of odd order such that $H \leq C_G(y)$. Since $H$ is a maximal abelian subgroup of $G$, hence $H = C_G(x) = C_G(y)$ which shows that $H$ must be a CC-subgroup of $G$. ∎

**Corollary 1.** *Suppose $G$ is a $\mathbb{Q}$-group with a CC-subgroup, then $G$ is a Frobenius $\mathbb{Q}$-group.*

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