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Open Images of Metrizable Families [1]

Ekaterina P. Mihaylova

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First countable families and complete A-families are defined. It is shown that they are the open images of metrizable families (defined by M. Choban) and complete metrizable families, respectively. Selection theorems for set-valued mappings $\theta : Y \to 2^X$ are derived, for the case when the family of all images of the points of Y by θ constitute a complete A-family in X.

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1. Metrizable families

Let X be a topological space and ρ be a pseudometric on X. Denote by X/ρ the set of all equivalence classes of X of the type $H(x, \rho) = \{y \in X : \rho(x, y) = 0\}$ and by $\widetilde{\rho}$ a metric on X/ρ such that $\widetilde{\rho}(h_1, h_2) = \rho(\pi_\rho^{-1}(h_1), \pi_\rho^{-1}(h_2))$ for every $h_1, h_2 \in X/\rho$ where $\pi_\rho : X \to X/\rho$ is the natural projection. Notice that if the pseudometric ρ is continuous then π_ρ is a continuous mapping onto the metric space $(X/\rho, \widetilde{\rho})$.

Put $O_\rho(x, \varepsilon) = \{y \in X : \rho(y, x) < \varepsilon\}$ for $\varepsilon > 0$.

Definition 1.1. *Let $\mathcal{A}$ be a family of subsets of X. The triple $(X, \mathcal{A}, \rho)$ is called a metrizable family if the following conditions are valid:*

MF1) $H(x, \rho) \bigcap L = \{x\}$ for every $L \in \mathcal{A}$ and $x \in L$;

MF2) For every $L_0 \in \mathcal{A}$, $x_0 \in L_0$ and an open subset U of X such that $x_0 \in U$, there exist $\varepsilon > 0$ and an open subset V of X such that $x_0 \in V$ and $O_\rho(x_0, \varepsilon) \bigcap L \subseteq U$ provided $L \bigcap V \neq \emptyset$, $L \in \mathcal{A}$.

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Proposition 1.2. *Let the triple $(X, \mathcal{A}, \rho)$ satisfies condition MF1 of Definition 1.1. Then it is a metrizable family if and only if it satisfies the following condition:*

MF2) For every $L_0 \in \mathcal{A},\ x_0 \in L_0$ and an open subset U of X such that $x_0 \in U$, there exist $\varepsilon > 0$ and an open subset V of X such that $x_0 \in V$ and $O_\rho(x, \varepsilon) \bigcap L \subseteq U$ provided $x \in L \bigcap V,\ L \in \mathcal{A}$.*

Proof. Suppose first, that the triple $(X, \mathcal{A}, \rho)$ is a metrizable family, $L_0 \in \mathcal{A},\ x_0 \in L_0$ and U is an open subset of X such that $x_0 \in U$. Thus, there exist $\varepsilon > 0$ and an open subset V of X such that $x_0 \in V$ and $O_\rho(x_0, \varepsilon) \bigcap L \subseteq U$ provided $L \bigcap V \neq \emptyset,\ L \in \mathcal{A}$. Put $\varepsilon_1 = \frac{\varepsilon}{3}$ and $V_1 = V \bigcap O_\rho(x_0, \varepsilon_1)$. Suppose $L' \in \mathcal{A}$ and $x' \in L' \bigcap V_1$. Take $x_1 \in O_\rho(x', \varepsilon_1) \bigcap L'$. Then $\rho(x', x_1) < \varepsilon_1$ and $\rho(x', x_0) < \varepsilon_1$, hence $\rho(x_1, x_0) < 2\varepsilon_1 < \varepsilon$. From $x' \in L' \bigcap V_1$ and $V_1 \subseteq V$ follows that $L' \bigcap V \neq \emptyset$. MF2 implies $x_1 \in O_\rho(x_0, \varepsilon) \bigcap L' \subseteq U$. Thus $O_\rho(x', \varepsilon_1) \bigcap L' \subseteq U$. Therefore, MF2* holds for the triple $(X, \mathcal{A}, \rho)$.

Suppose now, that the triple $(X, \mathcal{A}, \rho)$ satisfies MF2*, $L_0 \in \mathcal{A},\ x_0 \in L_0$ and U is an open subset of X such that $x_0 \in U$. Thus, there exist $\varepsilon > 0$ and an open subset V of X such that $x_0 \in V$ and $O_\rho(x, \varepsilon) \bigcap L \subseteq U$ provided $x \in L \bigcap V,\ L \in \mathcal{A}$. Put $\varepsilon_1 = \frac{\varepsilon}{3}$ and $V_1 = V \bigcap O_\rho(x_0, \varepsilon_1)$. Then ε_1 and V_1 are the satisfying condition MF2 ones. ∎

Remark 1.3. Proposition 1.2 shows that Definition 1.1 for metrizable family is equivalent to the original one, given by M. M. Choban in [1], which is a generalization of the one given by Geiler in [5] for a uniform space X.

Definition 1.4. *(see [1]) Let $(X, \mathcal{A}, \rho)$ be a metrizable family of subsets of a space X. The set $L \in \mathcal{A}$ is complete if the metric space (L, ρ) is complete. The metrizable family $(X, \mathcal{A}, \rho)$ is called complete if every $L \in \mathcal{A}$ is complete.*

Proposition 1.5. *Let $(X_n, \mathcal{A}_n, \rho_n)$ be metrizable families, where the pseudometrics ρ_n are bounded by 1, for $n \in \mathbf{N}$, $X = \prod\{X_n : n \in \mathbf{N}\}$, $\rho : X \times X \to \mathbf{R}^+$ be a pseudometric on X such that $\rho(x, y) = \sum\{\frac{1}{2^n} \cdot \rho_n(x_n, y_n) : n \in \mathbf{N}\}$ for every $x = \{x_n : n \in \mathbf{N}\} \in X$, $y = \{y_n : n \in \mathbf{N}\} \in X$ and $\mathcal{A} = \{\prod\{L_n : n \in \mathbf{N}\} : L_n \in \mathcal{A}_n$ for every $n \in \mathbf{N}\}$. Then $(X, \mathcal{A}, \rho)$ is a metrizable family. Moreover, the family $(X, \mathcal{A}, \rho)$ is complete provided all families $(X_n, \mathcal{A}_n, \rho_n)$ are complete.*

Proof. To prove that MF1 holds take $L = \prod\{L_n : n \in \mathbf{N}\} \in \mathcal{A},\ x = \{x_n : n \in \mathbf{N}\} \in L$ and $y = \{y_n : n \in \mathbf{N}\} \in H(x, \rho) \bigcap L$. Hence $\rho_n(x_n, y_n) = 0$ for every $n \in \mathbf{N}$. Thus $y_n \in H(x_n, \rho_n) \bigcap L_n = \{x_n\}$ for every $n \in \mathbf{N}$.

To prove that MF2 holds take $L = \prod\{L_n : n \in \mathbf{N}\} \in \mathcal{A},\ x = \{x_n : n \in \mathbf{N}\} \in L$ and $U = \prod\{U_n : n \in \mathbf{N}\}$ an open subset of X where $x_{j_i} \in U_{j_i}$ and U_{j_i} is an open subset of X_{j_i} for $i \in \{1, ..., r\}$ and $U_n = X_n$ for $n \in \mathbf{N} \backslash \{j_1, ..., j_r\}$. Thus there exist $\varepsilon_i > 0$ and V_{j_i} an open subset of X_{j_i} such that $x_{j_i} \in V_{j_i}$ for every $i \in \{1, ..., r\}$ as in MF2. Put $\varepsilon = min\{\frac{1}{2^{j_1}} \cdot \varepsilon_1, ..., \frac{1}{2^{j_r}} \cdot \varepsilon_r\}$ and $V = \prod\{V_n : n \in \mathbf{N}\}$ where $V_n = X_n$ for $n \in \mathbf{N} \setminus \{j_1, ..., j_r\}$. Then V is an open subset of X and $x \in V$. Take $L' = \prod\{L'_n : n \in \mathbf{N}\} \in \mathcal{A}$ such that $L' \bigcap V \neq \emptyset$ and $y \in O_\rho(x, \varepsilon) \bigcap L'$. Then $V_{j_i} \bigcap L'_{j_i} \neq \emptyset$ for $i \in \{1, ..., r\}$ and $y_{j_i} \in O_{\rho_{j_i}}(x_{j_i}, \varepsilon_i) \bigcap L'_{j_i} \subseteq U_{j_i}$ for $i \in \{1, ..., r\}$ therefore $y \in U$.

Since the Cartezian product of complete metric spaces is a complete metric space, the proof is complete. ■

2. Special families of subsets

2.1. Definitions and notations

Let $\mathcal{A}$ be a family of subsets of the topological space X such that $\bigcup \mathcal{A} = X$. Let $(\gamma, p) = \{\gamma_n = \{U_\alpha : \alpha \in A_n\},\ p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ be an wA-sieve of X (i.e. γ_n is an open cover of X and $U_\alpha = \bigcup\{U_\beta : \beta \in p_n^{-1}(\alpha)\}$ for every $\alpha \in A_n$ and $n \in \mathbf{N}$).

Definition 2.1. *A sequence $\mathbf{a} = \{\alpha_n \in A_n : n \in \mathbf{N}\}$ is called an s-sequence if $p_n(\alpha_{n+1}) = \alpha_n$ for every $n \in \mathbf{N}$.*

Definition 2.2. *An s-sequence $\mathbf{a} = \{\alpha_n \in A_n : n \in \mathbf{N}\}$ is called a b-sequence of X at the couple (L_0, x_0) where $L_0 \in \mathcal{A}$ and $x_0 \in L_0$ if the following hold:*

BS1) $\{x_0\} = \bigcap\{L_0 \bigcap U_{\alpha_n} : n \in \mathbf{N}\}$;

BS2) for every open subset V of X such that $x_0 \in V$ there exist $m \in \mathbf{N}$ and an open subset W of X such that $x_0 \in W$ and $L \bigcap U_{\alpha_m} \subseteq V$ provided $W \bigcap L \neq \emptyset,\ L \in \mathcal{A}$.

Condition 2.3. *(For a quartet $(X, \mathcal{A}, \gamma, p)$.) For every $L \in \mathcal{A}$, $m \in \mathbf{N}$, $\beta \in A_m$ and $x \in L \bigcap U_\beta$ there exists a b-sequence $\mathbf{a} = \{\alpha_n : n \in \mathbf{N}\}$ of X at (L, x) such that $\alpha_m = \beta$.*

Condition 2.4. *(For a quartet $(X, \mathcal{A}, \gamma, p)$.) For every $L \in \mathcal{A}$, $x \in L$ and an s-sequence $\mathbf{a} = \{\alpha_n : n \in \mathbf{N}\}$ such that $x \in \bigcap\{U_{\alpha_n} : n \in \mathbf{N}\}$ follows that $\mathbf{a}$ is a b-sequence of X at (L, x).*

Definition 2.5. *A subset $L \in \mathcal{A}$ is called complete relatively to the wA-sieve (γ, p) if for every s-sequence $\mathbf{a} = \{\alpha_n : n \in \mathbf{N}\}$ such that $L \bigcap U_{\alpha_n} \neq \emptyset$ for every $n \in \mathbf{N}$ follows that $\bigcap\{L \bigcap U_{\alpha_n} : n \in \mathbf{N}\} \neq \emptyset$.*

Condition 2.6. *(For a quartet $(X, \mathcal{A}, \gamma, p)$.) Every $L \in \mathcal{A}$ is complete relatively to the wA-sieve (γ, p).*

Definition 2.7. *The quartet $(X, \mathcal{A}, \gamma, p)$ is called first countable family if it satisfies Condition 2.3.*

Definition 2.8. *The quartet $(X, \mathcal{A}, \gamma, p)$ is called A-family if it satisfies Condition 2.4.*

Definition 2.9. *The quartet $(X, \mathcal{A}, \gamma, p)$ is called complete A-family if it satisfies Conditions 2.4 and 2.6.*

Remark 2.10. If $(X, \mathcal{A}, \gamma, p)$ is an A-family then it is a first countable family.

Remark 2.11. For a quartet $(X, \mathcal{A}, \gamma, p)$ put $\mathcal{A}(x) = \bigcup\{L \in \mathcal{A} : x \in L\}$ for $x \in X$. If $\mathbf{a} = \{\alpha_n : n \in \mathbf{N}\}$ is a b-sequence of X at (L_0, x_0) for some $L_0 \in \mathcal{A},\ x_0 \in L_0$ then $\{U_{\alpha_n} : n \in \mathbf{N}\}$ is a base of $\mathcal{A}(x_0)$ at x_0.

Remark 2.12. For a quartet $(X, \mathcal{A}, \gamma, p)$ and a b-sequence $\mathbf{a} = \{\alpha_n : n \in \mathbf{N}\}$ of X at (L_0, x_0) for some $L_0 \in \mathcal{A},\ x_0 \in L_0$ follows that $\mathbf{a}$ is a b-sequence of X at (L_1, x_0) for every $L_1 \in \mathcal{A}$ such that $x_0 \in L_1$ and L_1 is a T_1-subspace of X.

Remark 2.13. If $(X, \mathcal{A}, \gamma, p)$ is a first countable family and $L \in \mathcal{A}$ then L is a first countable (i.e. satisfies the first Axiom of countability) T_1-space. (It follows from condition BS1 for a b-sequence).

2.2 Constructions

Let $(X, \mathcal{A}, \gamma, p)$ be a first countable family where $(\gamma, p) = \{\gamma_n = \{U_\alpha : \alpha \in A_n\},\ p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$. Put $\mathbf{A} = \prod\{A_n : n \in \mathbf{N}\}$, $\mathbf{B_1} = \{a \in \mathbf{A} : a$ is an s-sequence $\}$ and $\mathbf{B} = \{a \in \mathbf{B_1} : a$ is a b-sequence of X at (L, x) for some $L \in \mathcal{A},\ x \in L\}$. For $a = \{\alpha_n \in A_n : n \in \mathbf{N}\} \in \mathbf{A}$ and $b = \{\beta_n \in A_n : n \in \mathbf{N}\} \in \mathbf{A}$ put $d(a, b) = \sum\{2^{-n} : n \in \mathbf{N}$ is such that $\alpha_n \neq \beta_n\}$ (The Baire metric). Then (A, d) is a complete metric space, $\mathbf{B} \subseteq \mathbf{B_1} \subseteq \mathbf{A}$ and $\mathbf{B_1}$ is a closed subset of $\mathbf{A}$. Moreover, the family $\mathcal{B} = \{s_m(\alpha) = \{a = \{\alpha_n : n \in \mathbf{N}\} \in \mathbf{B} : \alpha_m = \alpha\} \subseteq \mathbf{B} : m \in \mathbf{N},\ \alpha \in A_m\}$ is a base for the metric space $(\mathbf{B}, d)$. Note if $a = \{\alpha_n \in A_n : n \in \mathbf{N}\}$ is an s-sequence and $\alpha = \alpha_m$ then $\pi_{m-1}(\alpha_m) = \alpha_{m-1}, \dots, \pi_1(\alpha_2) = \alpha_1$.

Put $\mathbf{Z} = \{(x, a) : x \in X,\ a \in \mathbf{B}$ where a is a b-sequence of X at (L, x) for some $L \in \mathcal{A}$ such that $x \in L\}$. Thus $\mathbf{Z} \subseteq X \times \mathbf{B}$. Denote by π and q

the projections from $\mathbf{Z}$ onto X and onto $\mathbf{B}$, respectively (i.e. $\pi((x,a)) = x$ and $q((x,a)) = a$ for $(x,a) \in \mathbf{Z}$). Put $\rho((x,a),(y,b)) = d(a,b)$ for $(x,a),\ (y,b) \in \mathbf{Z}$. Finally denote by $\mathcal{A}'$ the family $\{\pi^{-1}(L) : L \in \mathcal{A}\}$.

2.3 Properties

Property 2.14. $(\mathbf{B}, d)$ *is a metric space.* ■

Property 2.15. ρ *is a continuous pseudometric on* $\mathbf{Z}$.

Proof. It follows from the fact that $q : \mathbf{Z} \to \mathbf{B}$ is continuous. ■

Property 2.16. $(\mathbf{Z}, \mathcal{A}', \rho)$ *is a metrizable family.*

Proof. Let $L_0 \in \mathcal{A},\ (x_0, a_0) \in \pi^{-1}(L_0)$ and $a_0 = \{\alpha_{0n} \in A_n : n \in \mathbf{N}\}$. Obviously $H((x_0,a_0),\rho) = \{(x,a_0) \in \mathbf{Z}\}$.

Assume $(x_1, a_0) \in H((x_0,a_0),\rho) \bigcap \pi^{-1}(L_0)$.

Then $x_1 \in L_0$ and from Remarks 2.13 and 2.12 follows that a_0 is a b-sequence of X at (L_0, x_1). Thus $\{x_1\} = \bigcap\{L_0 \bigcap U_{\alpha_{0n}} : n \in \mathbf{N}\} = \{x_0\}$. Therefore $H((x_0,a_0),\rho) \bigcap \pi^{-1}(L_0) = \{(x_0,a_0)\}$.

Now let U be an open subset of $\mathbf{Z}$ such that $(x_0, a_0) \in U$. Since $\mathbf{Z} \subseteq X \times \mathbf{B}$, it can be assumed that $U = (V \times s_m(\alpha)) \bigcap \mathbf{Z}$ for some open subset V of X and $m \in \mathbf{N}$ where $\alpha = \alpha_{0m}$. Thus $x_0 \in V \bigcap L_0 \bigcap U_\alpha$. From the fact that a_0 is a b-sequence of X at (L_0, x_0) follows that there exist $l \geq m$ and an open subset W of X such that $x_0 \in W$ and $L \bigcap U_{\alpha_{0l}} \subseteq V$ provided $L \bigcap W \neq \emptyset,\ L \in \mathcal{A}$. Fix $0 < \varepsilon < 2^{-l}$ and let $W_1 = W \times s_l(\alpha_{ol})$. Take $L' \in \mathcal{A}$ such that $\pi^{-1}(L') \bigcap W_1 \neq \emptyset$ and $(x,a) \in \pi^{-1}(L') \bigcap O_\rho((x_0,a_0),\varepsilon),\ a = \{\alpha_n : n \in \mathbf{N}\}$. From $d(a,a_0) < \varepsilon$ it follows that $\alpha_l = \alpha_{0l}$. From $(x,a) \in \mathbf{Z}$ it follows that $x \in U_{\alpha_l} = U_{\alpha_{0l}}$. Since $L' \bigcap W \neq \emptyset,\ L' \bigcap U_{\alpha_{0l}} \subseteq V$. Therefore $x \in V$ and $a \in s_l(\alpha_{ol}) \subseteq s_m(\alpha)$ (i.e. $(x,a) \in U$). Thus ε and $W_1 \bigcap \mathbf{Z}$ are the required ones from condition MF2 for metrizable family. ■

Property 2.17. $\pi : \mathbf{Z} \to X$ *is an open continuous mapping.*

Proof. It suffices to show that $\pi((V \times s_m(\alpha)) \bigcap \mathbf{Z}) = V \bigcap U_\alpha$ for every $V \subseteq X,\ \alpha \in A_m$ and $m \in \mathbf{N}$. Take $x_0 \in V \bigcap U_\alpha$. Then $x_0 \in L_0$ for some $L_0 \in \mathcal{A}$. There exists a b-sequence $a = \{\alpha_n \in A_n : n \in \mathbf{N}\}$ of X at (L_0, x_0) such that $\alpha_m = \alpha$. Thus $a \in s_m(\alpha)$ and $(x_0, a_0) \in \mathbf{Z}$. Hence $x_0 = \pi((x_0,a_0)) \in \pi((V \times s_m(\alpha)) \bigcap \mathbf{Z}))$. ■

Definition 2.18. *A family* $(X, \mathcal{A}, \gamma, p)$ *is called open continuous image of a metrizable family* $(Y, \mathcal{B}, \rho)$ *if there exists an open continuous mapping* $\varphi : Y \to X$ *such that* $\mathcal{B} = \{\varphi^{-1}(L) : L \in \mathcal{A}\}$.

Theorem 2.19. *Let $(X, \mathcal{A}, \gamma, p)$ be a first countable family. Then it is an open continuous image of a metrizable family $(\mathbf{Z}, \mathcal{A}', \rho)$.*

Proof. It follows from Properties 2.16 and 2.17. ■

Property 2.20. *Let $(X, \mathcal{A}, \gamma, p)$ be an A-family and $L \in \mathcal{A}$ be such that for every s-sequence $\mathbf{a} = \{\alpha_n \in A_n : n \in \mathbf{N}\}$ with $L \bigcap U_{\alpha_n} \neq \emptyset$ for every $n \in \mathbf{N}$ follows that $\bigcap\{L \bigcap U_{\alpha_n} : n \in \mathbf{N}\} \neq \emptyset$. Then $(\pi^{-1}(L), \rho)$ is a complete metric space.*

Proof. Let $S_n = \{\beta \in A_n : L \bigcap U_\beta \neq \emptyset\}$ and $S = \mathbf{B} \bigcap (\prod\{S_n : n \in \mathbf{N}\})$. Then S is a complete metric space. It is sufficient to prove that S is closed in $\mathbf{B_1}$. Take $b \in \mathbf{B_1} \setminus S$ where $b = \{\beta_n : n \in \mathbf{N}\}$. Assume that $L \bigcap U_{\beta_n} \neq \emptyset$ for every $n \in \mathbf{N}$. But b is an s-sequence, hence $\bigcap\{L \bigcap U_{\beta_n} : n \in \mathbf{N}\} \neq \emptyset$. Now from Condition 2.4 follows that b is a b-sequence of X at (L, x) for some $x \in \bigcap\{L \bigcap U_{\beta_n} : n \in \mathbf{N}\}$. Thus $b \in \mathbf{B} \bigcap (\prod\{S_n : n \in \mathbf{N}\}) = S$ which is a contradiction. Hence $L \bigcap U_{\beta_m} = \emptyset$ for some $m \in \mathbf{N}$. Thus the open subset $s'_m(\beta_m) = \{a = \{\alpha_n : n \in \mathbf{N}\} \in \mathbf{B_1} : \alpha_m = \beta_m\}$ of $\mathbf{B_1}$ contains b and it does not meet S.

Clearly, to prove the property, it is enough to show that $q_L = q|\pi^{-1}(L) : (\pi^{-1}(L), \rho) \to (S, d)$ is an isometry. Fix $a \in S$, $a = \{\alpha_n \in A_n : n \in \mathbf{N}\}$. There exists $x \in \bigcap\{L \bigcap U_{\alpha_n} : n \in \mathbf{N}\} \neq \emptyset$. Therefore a is a b-sequence of X at (L, x). From $(x, a) \in \pi^{-1}(L)$ follows that $a = q_L((x, a))$. Thus q_L is onto S. Moreover q_L is a one-to-one mapping because $(\mathbf{Z}, \mathcal{A}', \rho)$ is a metrizable family and $\pi^{-1}(L) \in \mathcal{A}'$. The proof follows from the equality $\rho((x, a), (y, b)) = d(a, b)$ for $(x, a),\ (y, b) \in \pi^{-1}(L)$. ■

Corollary 2.21. *Let $(X, \mathcal{A}, \gamma, p)$ be a complete A-family. Then $(\pi^{-1}(L), \rho)$ is a complete metric space for every $L \in \mathcal{A}$.* ■

Property 2.22. *Let $(X, \mathcal{A}, \gamma, p)$ be a first countable family [A-family] and $\mathcal{A}_1 = \{H \bigcap L : L \in \mathcal{A},\ H \subseteq X\}$. Then $(X, \mathcal{A}_1, \gamma, p)$ is a first countable family [A-family], too.*

Proof. Let $(X, \mathcal{A}, \gamma, p)$ be a first countable family, $L_0 \in \mathcal{A}_1$, $x_0 \in L_0 \bigcap U_\beta$ and $\beta \in A_m$, $m \in \mathbf{N}$. Then $L_0 \subseteq L_1$ for some $L_1 \in \mathcal{A}$. From Condition 2.3 follows that there exists a b-sequence $a = \{\alpha_n \in A_n : n \in \mathbf{N}\}$ of X at (L_1, x_0) such that $\alpha_m = \beta$. Clearly, $\{x_0\} = \bigcap\{L_0 \bigcap U_{\alpha_n} : n \in \mathbf{N}\} = \bigcap\{L_1 \bigcap U_{\alpha_n} : n \in \mathbf{N}\}$. Let V be an open subset of X such that $x_0 \in V$. There exist $k \in \mathbf{N}$ and an open subset W of X such that $x_0 \in W$ and $L' \bigcap U_{\alpha_k} \subseteq V$ provided $W \bigcap L' \neq \emptyset$, $L' \in \mathcal{A}$. Now if $L'' \in \mathcal{A}_1$ and $W \bigcap L'' \neq \emptyset$. Then $L'' \subseteq L_2$ for some $L_2 \in \mathcal{A}$. Hence $L'' \bigcap U_{\alpha_k} \subseteq L_2 \bigcap U_{\alpha_k} \subseteq V$. Therefore a is a b-sequence of X at (L_0, x_0). Thus $(X, \mathcal{A}_1, \gamma, p)$ is a first countable family. One can easily prove

$(X, \mathcal{A}_1, \gamma, p)$ is an A-family provided $(X, \mathcal{A}, \gamma, p)$ is an A-family, in a similar way. ■

Corollary 2.23. *Let $(X, \mathcal{A}, \gamma, p)$ be an A-family. Then $(\pi^{-1}(x), \rho)$ is a complete metric space for every $x \in X$.*

P r o o f. Let $\mathcal{A}_1 = \{H \bigcap L : L \in \mathcal{A},\ H \subseteq X\}$ and $x \in X$. Then $\{x\} \in \mathcal{A}_1$. Take an s-sequence $a = \{\alpha_n \in A_n : n \in \mathbf{N}\}$ such that $x \in U_{\alpha_n}$ for every $n \in \mathbf{N}$. Clearly, $\{x\} = \bigcap\{\{x\} \bigcap U_{\alpha_n} : n \in \mathbf{N}\}$. From Property 2.20 follows that $(\pi^{-1}(x), \rho)$ is a complete metric space. ■

Property 2.24. *Let $(X, \mathcal{A}, \gamma, p)$ be a complete A-family and $\mathcal{A}_2 = \{H \bigcap L : L \in \mathcal{A},\ H$ is a closed subset of $X\}$. Then $(X, \mathcal{A}_2, \gamma, p)$ is a complete A-family, too.*

P r o o f. That $(X, \mathcal{A}_2, \gamma, p)$ satisfies Condition 2.4 can be seen in a similar way as in the proof of the previous property. To prove that Condition 2.6 holds for $(X, \mathcal{A}_2, \gamma, p)$, take $L \in \mathcal{A}_2$ and an s-sequence $a = \{\alpha_n : n \in \mathbf{N}\}$ such that $L \bigcap U_{\alpha_n} \neq \emptyset$ for every $n \in \mathbf{N}$. Then $L = L_0 \bigcap H$ for some $L_0 \in \mathcal{A}$ and a closed subset H of X. Take $x \in \bigcap\{L_0 \bigcap U_{\alpha_n} : n \in \mathbf{N}\} \neq \emptyset$. Then a is a b-sequence of X at (L_0, x). Assume that $\bigcap\{H \bigcap L_0 \bigcap U_{\alpha_n} : n \in \mathbf{N}\} = \emptyset$. Thus $X \setminus H$ is an open subset of X containing x. There exist $m \in \mathbf{N}$ and an open subset W of X such that $x \in W$ and $L' \bigcap U_{\alpha_m} \subseteq X \setminus H$ provided $L' \bigcap W \neq \emptyset$ and $L' \in \mathcal{A}$. Thus $L_0 \bigcap U_{\alpha_m} \subseteq X \setminus H$. Therefore $L \bigcap U_{\alpha_m} = \emptyset$ which is a contradiction. Hence $\bigcap\{L \bigcap U_{\alpha_n} : n \in \mathbf{N}\} \neq \emptyset$. ■

3. Examples

E x a m p l e 3.1. Let X be a first countable T_1-space and $\mathcal{A} = \{L \subseteq X\}$. Then there exists an wA-sieve $(\gamma, p) = \{\gamma_n = \{U_\alpha : \alpha \in A_n\},\ p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ of X such that $(X, \mathcal{A}, \gamma, p)$ is a first countable family.

P r o o f. Take as $\gamma_1 = \{U_\alpha : \alpha \in A_1\}$ an arbitrary base of X. Put $A_n = \{(\alpha_1, \alpha_2, ..., \alpha_n) \in A_1^n : U_{\alpha_n} \subseteq U_{\alpha_{n-1}} \subseteq ... \subseteq U_{\alpha_1}\}$, $\gamma_n = \{U_{(\alpha_1, \alpha_2, ..., \alpha_n)} = U_{\alpha_n} : (\alpha_1, \alpha_2, ..., \alpha_n) \in A_n\}$ and $p_n((\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1})) = (\alpha_1, \alpha_2, ..., \alpha_n)$ for every $(\alpha_1, \alpha_2, ..., \alpha_{n+1}) \in A_{n+1}$ and every $n \in \mathbf{N}$. Clearly, $(\gamma,\ p) = \{\gamma_n = \{U_\alpha : \alpha \in A_n\},\ p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ is an wA-sieve of X. Let $L_0 \in \mathcal{A},\ x_0 \in L_0$, $m \in \mathbf{N}$ and $x_0 \in U_\alpha = U_{\alpha_m}$ for some $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m) \in A_m$. There exists a countable subset $\{U_{\beta_i} : i \in \mathbf{N}\}$ of γ_1 which is a base of X at x_0. Put $\alpha_{m+1} = \beta_{j_1}$ for some $j_1 \in \mathbf{N}$ such that $U_{\beta_{j_1}} \subseteq U_{\alpha_m} \bigcap U_{\beta_1}$. Next put $\alpha_{m+2} = \beta_{j_2}$ for some $j_2 \in \mathbf{N}$ such that $U_{\beta_{j_2}} \subseteq U_{\alpha_{m+1}} \bigcap U_{\beta_2}$. And so on an s-sequence $a = \{a_n = (\alpha_1, \alpha_2, ..., \alpha_n) : n \in \mathbf{N}\}$ can be constructed. Moreover, a is a b-sequence of X at (L_0, x_0) such that $\alpha = a_m = (\alpha_1, \alpha_2, ..., \alpha_m)$. ■

E x a m p l e 3.2. Let $(X_i, \mathcal{A}_i, \gamma^i, p^i), i = 1,\ 2$ be first countable families where $(\gamma^1, p^1) = \{\gamma_n^1 = \{U_\alpha : \alpha \in A_n^1\},\ p_n^1 : A_{n+1}^1 \to A_n^1 : n \in \mathbf{N}\}$ and $(\gamma^2, p^2) = \{\gamma_n^2 = \{V_\beta : \beta \in A_n^2\},\ p_n^2 : A_{n+1}^2 \to A_n^2 : n \in \mathbf{N}\}$. If $X = X_1 \times X_2$, $\mathcal{A} = \{L_1 \times L_2 : L_i \in \mathcal{A}_i,\ i = 1,\ 2\}$ and $(\gamma,\ p) = \{\gamma_n = \{U_\alpha \times V_\beta :\ U_\alpha \in \gamma_n^1,\ V_\beta \in \gamma_n^2\},\ p_n : A_{n+1}^1 \times A_{n+1}^2 \to A_n^1 \times A_n^2 = A_n$ where $p_n((\alpha,\ \beta)) = (p_n^1(\alpha),\ p_n^2(\beta)) : n \in \mathbf{N}\}$. Then $(X, \mathcal{A}, \gamma, p)$ is a first countable family. ■

E x a m p l e 3.3. Finite product of first countable families is a first countable family. ■

E x a m p l e 3.4. Let $(X_j, \mathcal{A}_j, \gamma^j, p^j),\ j \in \mathbf{N}$ be first countable families, $X = \prod\{X_j : j \in \mathbf{N}\}$ and $\mathcal{A} = \{\prod\{L_j : j \in \mathbf{N}\} : L_j \in \mathcal{A}_j,\ j \in \mathbf{N}\}$. Then $(X, \mathcal{A}, \gamma, p)$ is a first countable family for some wA-sieve (γ, p) of X.

P r o o f. Let $(\gamma^j, p^j) = \{\gamma_n^j = \{U_\alpha^j : \alpha \in A_n^j\},\ p_n^j : A_{n+1}^j \to A_n^j : n \in \mathbf{N}\}$ be an wA-sieve of $X_j,\ j \in \mathbf{N}$. Put $A_n = \{(\alpha_n^1,\ ...,\ \alpha_n^n) : \alpha_n^j \in A_n^j,\ j \in 1,\ 2,\ ..., n\}$, $\gamma_n = \{U_\alpha = \prod\{U_{\alpha_n^j}^j : j \in 1,\ 2,\ ..., n\} \times \prod\{X_j : j > n\} : \alpha = (\alpha_n^1,\ ...,\ \alpha_n^n) \in A_n\}$ and $p_n : A_{n+1} \to A_n$ such that if $\beta = (\beta_{n+1}^1,\ ...,\ \beta_{n+1}^{n+1}) \in A_{n+1}$ then $p_n(\beta) = (p_n^1(\beta_{n+1}^1),\ ...,\ p_n^n(\beta_{n+1}^n)) \in A_n$ for $n \in \mathbf{N}$. Then $(\gamma, p) = \{\gamma_n = \{U_\alpha : \alpha \in A_n\},\ p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ is an wA-sieve of X. To prove that $(X, \mathcal{A}, \gamma, p)$ is a first countable family take $L = \prod\{L_j : j \in \mathbf{N}\} \in \mathcal{A},\ x = \{x_j : j \in \mathbf{N}\} \in L,\ m \in \mathbf{N}$, U_α where $\alpha = (\alpha_m^1,\ ...,\ \alpha_m^m) \in A_m$ and $x \in U_\alpha = \prod\{U_{\alpha_m^j}^j : j \in 1,\ 2,\ ..., m\} \times \prod\{X_j : j > m\}$. Choose $\alpha_m^j \in A_m^j$ such that $x_j \in U_{\alpha_m^j}^j$ for every $j > m$. There exists a b-sequence $a_j = \{\alpha_n^j \in A_n^j : n \in \mathbf{N}\}$ of X_j at $(L_j,\ x_j)$ for every $j \in \mathbf{N}$. Put $\alpha_k = (\alpha_k^1,\ ...,\ \alpha_k^k)$ for every $k \in \mathbf{N}$. Obviously $a = \{\alpha_k \in A_k : k \in \mathbf{N}\}$ is a b-sequence of X at $(L,\ x)$. ■

E x a m p l e 3.5. Let $(X, \mathcal{A}, \gamma, p)$ be a first countable family where $(\gamma, p) = \{\gamma_n = \{U_\alpha :\ \alpha \in A_n\}, p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ and Y be a topological space. Let $Z = X \times Y$ and $\mathcal{A}^* = \{L \times \{y\} : L \in \mathcal{A},\ y \in Y\}$. Then $(Z, \mathcal{A}^*, \gamma^*, p)$ is a first countable family for the wA-sieve $(\gamma^*, p) = \{\gamma_n^* = \{U_\alpha \times Y :\ \alpha \in A_n\}, p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ of Z. ■

4. Applications

Theorem 4.1. *Let $\theta : Y \to 2^X$ be an l.s.c. mapping, Y be a paracompact space, $\mathcal{A} = \{\theta(y) : y \in Y\}$, the wA-sieve (γ, p) of X be such that $(X, \mathcal{A}, \gamma, p)$ is an A-family and $Y_0 = \{y \in Y : \theta(y)$ is not complete relatively to the wA-sieve $(\gamma, p)\}$ be a σ-discrete subset of Y (i.e. a countable union of discrete (in Y) subsets of Y). Then there exist a u.s.c. compact-valued mapping $\psi : Y \to 2^X$*

and an l.s.c. compact-valued mapping $\varphi : Y \to 2^X$ *such that* $\varphi(y) \subseteq \psi(y) \subseteq \theta(y)$ *for every* $y \in Y$*. Moreover, if* $L \subseteq Y$ *and* $dim L \leq n$ *then there exists a u.s.c. mapping* $\psi : L \to 2^X$ *such that* $\psi(y) \subseteq \theta(y)$ *and* $\mid \psi(y) \mid \leq n+1$ *for every* $y \in L$*.*

P r o o f. There exist a mertizable family $(Z, \mathcal{A}', \rho)$ and an open continuous mapping $g : Z \to X$ such that $\mathcal{A}' = \{g^{-1}(L) : L \in \mathcal{A}\}$. For every $y \in Y \backslash Y_0$ the metric space $(g^{-1}(\theta(y)), \rho)$ is complete. By virtue of Theorem 2 from [1] there exist an l.s.c mapping $\varphi_1 : Y \to 2^Z$and a u.s.c. mapping $\psi_1 : Y \to 2^Z$ such that $\varphi_1(y) \subseteq \psi_1(y) \subseteq g^{-1}(\theta(y))$ for every $y \in Y$ and the sets $\varphi_1(y)$ and $\psi_1(y)$ are compact for every $y \in Y$. Obviously the mappings $\varphi(y) = g(\varphi_1(y))$ and $\psi(y) = g(\psi_1(y))$ are the required ones. ∎

Corollary 4.2. *Let* $(X, \mathcal{A}, \gamma, p)$ *be a complete A-family of subsets of a space* X*,* $\theta : Y \to 2^X$ *be an l.s.c. mapping,* Y *be paracompact and* $\theta(y) \in \mathcal{A}$ *for every* $y \in Y$*. Then there exist a u.s.c. mapping* $\psi : Y \to 2^X$ *and an l.s.c. mapping* $\varphi : Y \to 2^X$ *such that* $\varphi(y) \subseteq \psi(y) \subseteq \theta(y)$ *for every* $y \in Y$*. Moreover, if* $dim Y = 0$*, then the mapping* φ *is single-valued.* ∎

Corollary 4.3. *Let* $(X, \mathcal{A}, \gamma, p)$ *be a first countable family of subsets of a space* X*,* $\theta : Y \to 2^X$ *be an l.s.c. mapping,* Y *be a* $\sigma-$*discrete, paracompact space and* $\theta(y) \in \mathcal{A}$ *for every* $y \in Y$ *. Then there exists a single valued continuous mapping* $f : Y \to X$ *such that* $f(y) \in \theta(y)$ *for every* $y \in Y$*.* ∎

5. Inverse theorems

Theorem 5.1. *Let* X *and* Y *be topological spaces,* $f : Y \to X$ *be an open continuous mapping onto* X*,* $\mathcal{A}$ *be a family of subsets of* X *and* $\mathcal{A}' = \{f^{-1}(L) : L \in \mathcal{A}\}$ *be such that* $(Y,\ \mathcal{A}',\ \rho)$ *is a metrizable family for some continuous pseudometric* ρ *on* Y*. Then if* L *is a* T_1*-space for every* $L \in \mathcal{A}$*, there exists an wA-sieve* $(\gamma,\ p)$ *for which* $(X, \mathcal{A}, \gamma, p)$ *is a first countable family .*

P r o o f. Put $\mu_1 = \{O_\rho(y,\ \frac{1}{2^k}) : y \in Y,\ k \in \mathbf{N}\} = \{U_\alpha : \alpha \in A_1\}$. Then μ_1 is an open cover of Y. Put $A_n = \{(\alpha_1, \alpha_2, ..., \alpha_n) \in A_1^n : U_{\alpha_n} \subseteq U_{\alpha_{n-1}} \subseteq ... \subseteq U_{\alpha_1}\}$, $\gamma_n = \{U_{(\alpha_1,\alpha_2,...,\alpha_n)} = U_{\alpha_n} : (\alpha_1, \alpha_2, ..., \alpha_n) \in A_n\}$ and $p_n((\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1})) = (\alpha_1, \alpha_2, ..., \alpha_n)$ for every $(\alpha_1, \alpha_2, ..., \alpha_{n+1}) \in A_{n+1}$ and for every $n \in \mathbf{N}$. Clearly, $(\mu,\ p) = \{\mu_n = \{U_\alpha : \alpha \in A_n\},\ p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ is an wA-sieve of Y. Next, put $\gamma_n = \{f(U_\alpha) : \alpha \in A_n\}$. Thus γ_n is an open cover of X for every $n \in \mathbf{N}$ (f is an open mapping). Clearly, $(\gamma,\ p) = \{\gamma_n = \{f(U_\alpha) : \alpha \in A_n\},\ p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ is an wA-sieve of X. Let $L_0 \in \mathcal{A}$, $x_0 \in L_0$ and $x_0 \in f(U_\alpha)$ for some $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m) \in A_m$

and $m \in \mathbf{N}$. Thus $U_\alpha = U_{\alpha_m} = O_\rho(y_1,\ \frac{1}{2^k})$ for some $y_1 \in Y$ and $k \in \mathbf{N}$. There exists $y_0 \in U_\alpha$ such that $f(y_0) = x_0$. Hence $y_0 \in f^{-1}(L_0) \in \mathcal{A}'$.

Take $l \geq k,\ l > m,\ l \in \mathbf{N}$ such that $U_\beta = O_\rho(y_0,\ \frac{1}{2^l}) \subseteq O_\rho(y_1,\ \frac{1}{2^k})$ $(\beta \in A_1)$. For every $i \in \mathbf{N}$ there exists $\alpha_{m+i+1} \in A_1$ such that $U_{\alpha_{m+i+1}} = O_\rho(y_0,\ \frac{1}{2^{l+i}})$. Therefore $a = \{a_n = (\alpha_1, \alpha_2, ..., \alpha_n) : n \in \mathbf{N}\}$ where $\alpha_{m+1} = \beta$ is an s-sequence. Obviously $\alpha = a_m$. To prove the Theorem it is sufficient to show that a is a b-sequence of X at $(L_0,\ x_0)$. Take $x_1 \in \bigcap\{L_0 \bigcap f(U_{\alpha_n}) : n \in \mathbf{N}\}$. There exists $y_n \in U_{\alpha_n}$ such that $f(y_n) = x_1$ for every $n \in \mathbf{N}$. Thus $y_n \in f^{-1}(L_0)$ and $\rho(y_n,\ y_0) < \frac{1}{2^{n+j}}$ for $j = l-m-1$ and every $n > m$. Let V be an open subset of Y such that $y_0 \in V$. There exist $\varepsilon > 0$ and an open subset W of Y such that $y_0 \in W$ and $O_\rho(y_0, \varepsilon) \bigcap f^{-1}(L) \subseteq V$ provided $f^{-1}(L) \bigcap W \neq \emptyset,\ L \in \mathcal{A}$. Thus for every open subset V of Y such that $y_0 \in V$ there exists $n_k \in \mathbf{N}$ such that $y_{n_k} \in f^{-1}(x_1) \cap V$. Therefore $y_0 \in Cl_{f^{-1}(L_0)} f^{-1}(x_1) = f^{-1}(x_1)$. Thus $x_0 = x_1$ and the Condition BS1 for a b-sequence holds.

Take an open subset U of X such that $x_0 \in U$. Then $f^{-1}(U)$ is an open subset of Y and $y_0 \in f^{-1}(U) \bigcap f^{-1}(L_0)$. There exist $\varepsilon > 0$ and W an open subset of Y such that $y_0 \in W$ and $O_\rho(y_0, \varepsilon) \bigcap f^{-1}(L) \subseteq f^{-1}(U)$ provided $f^{-1}(L) \bigcap W \neq \emptyset,\ L \in \mathcal{A}$. Put $W' = f(W)$. Then W' is an open subset of X and $x_0 \in W'$. Take an $r \in \mathbf{N}$ such that $\frac{1}{2^r} < \varepsilon$. There exists $s \in \mathbf{N}$ such that $\frac{1}{2^{s+l}} \leq \frac{1}{2^r}$ and $U_{\alpha_{m+1+s}} = O_\rho(y_0,\ \frac{1}{2^{s+l}})$. If $L \in \mathcal{A}$ is such that $L \bigcap f(W) \neq \emptyset$ then $O_\rho(y_0, \frac{1}{2^{s+l}}) \bigcap f^{-1}(L) \subseteq f^{-1}(U)$. Thus $f(U_{\alpha_{m+s+1}}) \bigcap L \subseteq U$. ■

R e m a r k 5.2. Theorem 5.1 combined with Theorem 2.19 and Proposition 1.5 gives another proof of the statement in Example 3.4.

Theorem 5.3. *Let X and Y be topological spaces, $f : Y \to X$ be an open continuous mapping onto X, $\mathcal{A}$ be a family of subsets of X and $\mathcal{A}' = \{f^{-1}(L) : L \in \mathcal{A}\}$ be such that there exists a continuous pseudometric ρ on Y for which $(Y,\ \mathcal{A}',\ \rho)$ is a metrizable family. Then if $(f^{-1}(x), \rho)$ is a complete metric space for every $x \in X$ and every $L \in \mathcal{A}$ is a T_1-subspace of X there exists an wA-sieve $(\gamma,\ p)$ for which $(X, \mathcal{A}, \gamma, p)$ is an A-family.*

P r o o f. Put $\mu_1 = \{O_\rho(y,\ \frac{1}{2^k}) : y \in Y,\ k \in \mathbf{N}\} = \{U_\alpha : \alpha \in A_1\}$. Then μ_1 is an open cover of Y. Put $A_n = \{(\alpha_1, \alpha_2, ..., \alpha_n) \in A_1^n : U_{\alpha_n} \subseteq U_{\alpha_{n-1}} \subseteq ... \subseteq U_{\alpha_1},\ diam U_{\alpha_i} \leq \frac{1}{2^i},\ i = 1, ...n\}$, $\gamma_n = \{U_{(\alpha_1, \alpha_2, ..., \alpha_n)} = U_{\alpha_n} : (\alpha_1, \alpha_2, ..., \alpha_n) \in A_n\}$ and $p_n((\alpha_1, \alpha_2, ..., \alpha_n, \alpha_{n+1})) = (\alpha_1, \alpha_2, ..., \alpha_n)$ for every $(\alpha_1, \alpha_2, ..., \alpha_{n+1}) \in A_{n+1}$ and for every $n \in \mathbf{N}$. Clearly, $(\mu,\ p) = \{\mu_n = \{U_{\alpha_n} : \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in A_n\},\ p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ is an wA-sieve of Y.

Put $\gamma_n = \{f(U_{\alpha_n}) : \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in A_n\}$. Thus γ_n is an open cover of X for every $n \in \mathbf{N}$. Clearly, $(\gamma,\ p) = \{\gamma_n,\ p_n : A_{n+1} \to A_n : n \in \mathbf{N}\}$ is an wA-sieve of X.

To prove that $(X, \mathcal{A}, \gamma, p)$ is an A-family take $L_0 \in \mathcal{A},\ x_0 \in L_0$ and an s-sequence $a = \{a_n = (\alpha_1, \alpha_2, ..., \alpha_n) : n \in \mathbf{N}\}$ such that $x_0 \in \bigcap\{f(U_{\alpha_n}) : n \in \mathbf{N}\}$. Let $x' \in \bigcap\{L_0 \bigcap f(U_{\alpha_n}) : n \in \mathbf{N}\}$. Since $f^{-1}(x_0)$ and $f^{-1}(x')$ are complete with respect to ρ, $f^{-1}(x_0) \bigcap(\bigcap\{\overline{U_{\alpha_n}}^{\rho} : n \in \mathbf{N}\}) \neq \emptyset$ and $f^{-1}(x') \bigcap(\bigcap\{\overline{U_{\alpha_n}}^{\rho} : n \in \mathbf{N}\}) \neq \emptyset$. Therefore there exist $y_0 \in f^{-1}(x_0) \subseteq f^{-1}(L_0)$ and $y' \in f^{-1}(x') \subseteq f^{-1}(L_0)$ such that $\rho(y_0, y') \leq \frac{1}{2^n}$ for every $n \in \mathbf{N}$. Hence $\rho(y_0, y') = 0$. Now MF1 for $(Y,\ \mathcal{A}',\ \rho)$ implies that $y_0 = y'$. Therefore $x_0 = x'$ which shows that BS1 holds for the s-sequence a.

Take an open subset U of X such that $x_0 \in U$. It can be shown, as in the proof of the previous theorem, that $O_\rho(y_0, \frac{1}{2^l}) \bigcap f^{-1}(L) \subseteq f^{-1}(U)$ for some $l \in \mathbf{N}$, some open $W \subseteq Y$ neighborhood of y_0 and every $L \in \mathcal{A}$ with $f^{-1}(L) \cap W \neq \emptyset$. For every $n \in \mathbf{N}$, $U_{\alpha_n} = O_\rho(y_n^*, \frac{1}{2^{n+i_n}})$ for some $y_n^* \in Y$ and $i_n \in \mathbf{N}$. There exists $r \in \mathbf{N}$ such that $\frac{1}{2^{r+i_r}} < \frac{1}{2^{l+1}}$. Then $U_{\alpha_r} \subseteq O_\rho(y_0, \frac{1}{2^l})$. Thus r and $f(W)$ are the required from BS2 ones and a is an s-sequence. ∎

Theorem 5.4. *Let X and Y be topological spaces, $f : Y \to X$ be an open continuous mapping onto X, $\mathcal{A}$ be a family of subsets of X and $\mathcal{A}' = \{f^{-1}(L) : L \in \mathcal{A}\}$ be such that there exists a continuous pseudometric ρ on Y for which $(Y,\ \mathcal{A}',\ \rho)$ is a metrizable family. Then if $(f^{-1}(L), \rho)$ is a complete metric space for every $L \in \mathcal{A}$, $(f^{-1}(x), \rho)$ is a complete metric space for every $x \in X$ and every $L \in \mathcal{A}$ is a T_1-subspace of X there exists an wA-sieve $(\gamma,\ p)$ for which $(X, \mathcal{A}, \gamma, p)$ is a complete A-family.*

Proof. To check if Condition 2.6 holds for the wA-sieve (γ, p) constructed in the proof of the previous theorem take $L \in \mathcal{A}$ and an s-sequence $a = \{\alpha_n : n \in \mathbf{N}\}$ such that $L \bigcap f(U_{\alpha_n}) \neq \emptyset$ for every $n \in \mathbf{N}$. There exists $z \in f^{-1}(L) \bigcap(\bigcap\{\overline{U_{\alpha_n}}^{\rho} : n \in \mathbf{N}\}) = f^{-1}(L) \bigcap(\bigcap\{U_{\alpha_n} : n \in \mathbf{N}\})$. ∎

Corollary 5.5. *Every (complete) metrizable family is a (complete) A-family.* ∎

Corollary 5.6. *Let $(X_j, \mathcal{A}_j, \gamma^j, p^j),\ j \in \mathbf{N}$ be A-families, $X = \prod\{X_j : j \in \mathbf{N}\}$ and $\mathcal{A} = \{\prod\{L_j : j \in \mathbf{N}\} : L_j \in \mathcal{A}_j$ for every $j \in \mathbf{N}\}$. Then there exists an wA-sieve $(\gamma,\ p)$ of X such that*

1) $(X,\ \mathcal{A},\ \gamma,\ p)$ is an A-family;

2) The family $(X,\ \mathcal{A},\ \gamma,\ p)$ is complete if and only if the families $(X_j, \mathcal{A}_j, \gamma^j, p^j),\ j \in \mathbf{N}$ are complete. ∎

6. On a theorem of V. I. Ponomarev

In [10] V. I. Ponomarev has proved the following theorem:

Theorem 6.1. *For a T_1-space X the following are equivalent:*
1) X is a first countable space.
2) X is an open continuous image of a metric space.

P r o o f. Let X be a first countable space. Then $A = \{X\}$ is a first countable family. From Theorem 2.19 if follows that there exist a space Z, a continuous pseudometric ρ on Z and an open continuous mapping $\pi : Z \to X$ onto X such that $A' = \{\pi^{-1}(L) : L \in A\}$ is a metrizable family. Thus $(Z, \{Z\}, \rho)$ is a metrizable family and (Z, ρ) is a metric space. Implication $1 \to 2$ is proved.

Implication $2 \to 1$ follows from Theorem 5.1 and Remark 2.13. ■

Corollary 6.2. *Let $\theta : Y \to 2^X$ be an l.s.c. mapping of a $\sigma-$discrete paracompact space into a first countable T_1-space. Then there exists a continuous single valued mapping $f : Y \to X$ such that $f(x) \in \theta(x)$ for every $y \in Y$.*

P r o o f. Follows from Corollary 4.3 and Example 3.1. ■

7. On a theorem of Worrell and Wicke

A space X is a space with a base of countable order if $\{X\}$ is an A-family of subsets of X (see [11] and [12], or [2] Corollary 6.7).

Theorem 7.1. *Let $\varphi : Y \to X$ be an open continuous mapping of a space Y onto a space X, $\mathcal{A}$ be a family of T_1-subsets of X, $\mathcal{A}' = \{\varphi^{-1}(L) : L \in \mathcal{A}\}$ and the family of all fibers $\{\varphi^{-1}(x) : x \in X\}$ be a complete A-family of Y with respect to an wA-sieve (γ, p). If $\mathcal{A}'$ is an A-family with respect to (γ, p), then $\mathcal{A}$ is an A-family for some wA-sieve, too.*

P r o o f. Follows from Theorem 2.19, Corollary 2.23 and Theorem 5.3. ■

For the family $\mathcal{A} = \{X\}$ we obtain:

Corollary 7.2. *(Worrell-Wicke). A T_1-space X is a space with a base of countable order if and only if X is a complete (i.e. all fibers are complete) open continuous image of some metric space.* ■

8. On one theorem of Isbell and one of Junnila

In [6] J. R. Isbell has proved that every space is an open quotient of a paracompact space in which, for every family of open sets, there is a disjoint family of open subsets having the same union. In [7] H. J. K. Junnila has proved that every (first countable) space is an open continuous image of some (metrizable) $\sigma-$discrete paracompact space.

Theorem 8.1. *Let $(X, \mathcal{A}, \gamma, p)$ be a first countable family of subsets of a T_1-space X. Then there exist a $\sigma-$discrete paracompact space Z and an open continuous mapping $\varphi : Z \to X$ onto X such that $\mathcal{A}' = \{\varphi^{-1}(L) : L \in \mathcal{A}\}$ is a metrizable family of Z for some continuous metric ρ on Z.*

Proof. There exists a space Y, and a continuous pseudometric ρ_1 on Y and an open continuous mapping φ_1 from Y onto X such that $(Y, \mathcal{A}'', \rho_1)$, where $\mathcal{A}'' = \{\varphi_1^{-1}(L) : L \in \mathcal{A}\}$ is a metrizable family.

Let Y^N be a space with the metric $d((y_1, y_2, ...), (y_1', y_2'...)) = \sum\{2^{-n} : n \in \mathbf{N}$ such that $y_n \neq y_n'\}$.

Put $n((y_1, y_2, ...)) = \min\{m : y_m = y_{m+k}$ for every $k \in N\}$, $Z = \{(y_1, y_2, ...) \in Y^N : n((y_1, y_2, ...)) < \infty\}$ and $\varphi_2((y_1, y_2, ...)) = y_{n_0}$ where $n_0 = n((y_1, y_2, ...))$. Let $\varphi(z) = \varphi_1(\varphi_2(z))$ for all $z \in Z$ and $\rho(z, z') = d(z, z') + \rho_1(\varphi_2(z), \varphi_2(z'))$. On Z consider the topology generated by the basis $\{\varphi_2^{-1}(U) \cap O : U$ is open in Y and O is open in $(Z, \rho)\}$.

If $\mathcal{A}' = \{\varphi^{-1}(L) : L \in \mathcal{A}\} = \{\varphi_2^{-1}(L) : L \in \mathcal{A}''\}$, then $(Z, \mathcal{A}', \rho)$ is a metrizable family of Z. By construction, ρ is a continuous metric. ■

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Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia, Bulgaria
E-Mail: katiamih@math.bas.bg

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