Mathematica
Balkanica

New Series Vol. 22, 2008, Fasc. 1-2

Series in Generalized Bessel-Maitland Functions: Some Convergence Theorems in the Complex Plane

Jordanka Paneva-Konovska [1]

Presented by P. Kenderov

The classical Cauchy-Hadamard, Abel and Tauber theorems give some important properties about the convergence of the power series in complex plane. In this paper we prove same type theorems for series in the generalized Bessel-Maitland (Wright) functions. Special cases of them are related to the Bessel, Bessel-Maitland, Lommel and Struve functions.

AMS Subj. Classification: 30B10, 30B30; 33C10, 33C20

Key Words: Cauchy-Hadamard, Abel and Tauber theorems, generalized Bessel-Maitland function

1. Introduction

The classical Cauchy-Hadamard, Abel and Tauber theorems give some important properties about the convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ in the complex plane.

In general, by the classical Abel theorem, from the convergence of the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ at a point z_0 it follows the existence of the limit $\lim_{z \to z_0} f(z) = f(z_0)$ when z belongs to a suitable angle domain with a vertex at a point z_0. The example with the geometrical series $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$ at the point $z_0 = 1$ (Tchakalov [13], p.92) shows that the inverse proposition is not true in general. That is, the existence of this limit does not imply the

[1] Partially supported by National Science Research Fund - Bulgarian Ministry of Education and Science, under Grant MM 1305/2003

convergence of the series $\sum\limits_{n=0}^{\infty} a_n z_0^n$ without additional conditions on the growth of the coefficients.

The corresponding result is given by the following classical theorem.

Theorem (Tauber). *If the coefficients of the power series* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *satisfy the condition* $\lim\limits_{n\to\infty} n a_n = 0$ *and if* $\lim\limits_{z\to 1} f(z) = S$ *(*$z \to 1$ *radially), then the series* $\sum a_n$ *is convergent and* $\sum\limits_{n=0}^{\infty} a_n = S$.

It turns out that the Abel theorem fails even for series of the kind $\sum\limits_{k=1}^{\infty} a_{n_k} z^{n_k}$, where $(n_1, n_2, \ldots, n_k, \ldots)$ is a suitable permutation of the nonnegative integers (Tchakalov [13], p.92). Therefore, it is interesting to know if for series in a given sequence of holomorphic functions a statement like the Abel theorem is available. A positive answer to this question is given for series in Laguerre and Hermite polynomials in Rusev [11], §11.3; Rusev [12], Ch.4, §4; and Boyadjiev [1], and resp. for series in Bessel and Bessel-Maitland functions - in Paneva-Konovska [6], [7], [8].

Let $J^{\mu}_{\nu,\lambda}(z)$ be the generalized Bessel-Maitland (or Wright) function, introduced by Pathak [10] (for details see Kiryakova [3], p.353; Marichev [4], eq.(8.2)):

$$J^{\mu}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(\nu+k\mu+\lambda+1)}, \tag{1}$$

$$z \in \mathbb{C} \setminus (-\infty, 0], \quad \mu > 0, \quad \nu, \lambda \in \mathbb{C}.$$

In this paper we consider series of the form:

$$\sum_{n=0}^{\infty} a_n J^{\mu}_{n-2\lambda,\lambda}(z), \quad z \in \mathbb{C}, \quad \mu > 0, \quad \lambda \in \mathbb{C} \tag{2}$$

with complex coefficients a_n (n=0, 1, 2, ...) and prove for them some theorems, corresponding to the classical Cauchy-Hadamard, Abel and Tauber theorems.

In particular, these results lead to corresponding convergence theorems for series in some special functions of mathematical physics, like Bessel, Bessel-Maitland, Lommel and Struve functions.

2. A Cauchy-Hadamard Type Theorem

The following asymptotic formula with respect to the index holds for the generalized Bessel-Maitland functions (1):

$$(3)\qquad J^{\mu}_{n-2\lambda,\lambda}(z)=\frac{(z/2)^n}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)}(1+\theta^{\mu}_{n-2\lambda,\lambda}(z)),\ z\in\mathbb{C},\ \mu>0,\lambda\in\mathbb{C},$$

$$\theta^{\mu}_{n-2\lambda,\lambda}(z)\to 0 \ \text{ as } \ n\to\infty \quad (n\in\mathbb{N}).$$

The detailed proof of (3) is given in Paneva-Konovska [9].

Theorem 1 (Cauchy-Hadamard type) . *The domain of convergence of the series (2) is the circle domain $|z|<R$ with a radius of convergence*

$$(4)\qquad R=1/\Lambda,\qquad \Lambda=2^{-1}\limsup_{n\to\infty}(|a_n||\Gamma(\lambda+1)\Gamma(n-\lambda+1)|^{-1})^{1/n}.$$

The cases $\Lambda=0$ and $\Lambda=\infty$ are included in the common case, if $1/\Lambda$ is meant as ∞, respectively as 0.

P r o o f. Let us denote

$$u_n(z)=a_nJ^{\mu}_{n-2\lambda,\lambda}(z),\quad b_n=2^{-1}(|a_n||\Gamma(\lambda+1)\Gamma(n-\lambda+1)|^{-1})^{1/n}.$$

Using the asymptotic formula (3), we get

$$u_n(z)=a_n\frac{(z/2)^n}{\Gamma(\lambda+1)\Gamma(n-\lambda+1)}(1+\theta^{\mu}_{n-2\lambda,\lambda}(z)).$$

The proof goes separately in the three cases:

1. $\Lambda=0$. Then $\lim_{n\to\infty}b_n=\limsup_{n\to\infty}b_n=0$. Let us fix $z\neq 0$. Obviously, there exists a number N_1 such that for every $n>N_1$: $|1+\theta^{\mu}_{n-2\lambda,\lambda}(z)|<2$ and $2b_n<1/|z|$ which is equivalent to $|u_n(z)|=b_n^n|z|^n|1+\theta^{\mu}_{n-2\lambda,\lambda}(z)|<2^{1-n}$. The absolute convergence of (2) follows immediately from this inequality.

2. $0<\Lambda<\infty$. First, let z be inside the domain $|z|<R$ $(z\in\mathbb{C})$, i.e $|z|/R<1$. Then $\limsup_{n\to\infty}|z|b_n<1$. Therefore, it exists a number $q<1$ such that $\limsup_{n\to\infty}|z|b_n\leq q$, whence $|z|^nb_n^n\leq q^n$. By using the asymptotic formula (3) for the common member $u_n(z)$ of the series (2), we obtain $|u_n(z)|=b_n^n|z|^n|1+\theta^{\mu}_{n-2\lambda,\lambda}(z)|\leq q^n|1+\theta^{\mu}_{n-2\lambda,\lambda}(z)|$. Since $\lim_{n\to\infty}\theta^{\mu}_{n-2\lambda,\lambda}(z)=0$ there exists N_2: such that for every $n>N_2$ it holds $|1+\theta^{\mu}_{n-2\lambda,\lambda}(z)|<2$ and hence

$|u_n(z)| \leq 2q^n$. Because the series $\sum\limits_{n=0}^{\infty} 2q^n$ is convergent, the series (2) is also convergent, even absolutely.

Now, let z lie outside this domain. Then $|z|/R > 1$ and $\limsup\limits_{n\to\infty} |z| b_n > 1$. Therefore there exists infinite number of values n_k of n: $|z|^{n_k} b_{n_k}^{n_k} > 1$. Since $\lim\limits_{n\to\infty} \theta^{\mu}_{n-2\lambda,\lambda}(z) = 0$, there exists N_3 so that for $n_k > N_3$; $|1+\theta^{\mu}_{n_k-2\lambda,\lambda}(z)| \geq 1/2$, i.e. $|u_{n_k}(z)| \geq 1/2$ for infinite number of values of n. The necessary condition for convergence is not satisfied. Therefore the series (2) is divergent.

3. $\Lambda = \infty$. Let $z \in \mathbb{C}\backslash\{0\}$. Then $b_{n_k} > 1/|z|$ for infinite number of values n_k of n. But, from here $|u_{n_k}(z)| = |z|^{n_k}\ b_{n_k}^{n_k}\ |1+\theta^{\mu}_{n_k-2\lambda,\lambda}(z)| \geq 1/2$ and the necessary condition for the convergence of the series (2) is not satisfied, whence we conclude that the series (2) is divergent for every $z \neq 0$. ■

3. An Abel Type Theorem

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angle domain with size $2\varphi < \pi$ and with vertex at the point $z = z_0$, which is symmetric in the straight line defined by the points 0 and z_0. The following theorem is valid.

Theorem 2 (Abel type) . *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, Λ be a real number defined by (4), $0 < \Lambda < \infty$. Let K be the circle domain $|z| < 1/\Lambda$. If $f(z)$ is the sum of the series (2) on the domain K and this series is convergent at the point z_0 of the boundary of K, then $\lim\limits_{z\to z_0} f(z) = \sum\limits_{n=0}^{\infty} a_n J^{\mu}_{n-2\lambda,\lambda}(z_0)$, when $|z| < R$ and $z \in g_\varphi$, i.e.*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n J^{\mu}_{n-2\lambda,\lambda}(z_0), \quad z \in g_\varphi. \tag{5}$$

P r o o f. Let us consider the difference

$$\Delta(z) = \sum_{n=0}^{\infty} a_n J^{\mu}_{n-2\lambda,\lambda}(z_0) - f(z) = \sum_{n=0}^{\infty} a_n (J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)) \tag{6}$$

and represent it in the form

$$\Delta(z) = \sum_{n=0}^{k} a_n (J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)) + \sum_{n=k+1}^{\infty} a_n (J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)).$$

Let $p > 0$. By using the notations

$$\beta_m = \sum_{n=k+1}^{m} a_n J^{\mu}_{n-2\lambda,\lambda}(z_0), \quad m > k, \quad \beta_k = 0,$$

$$\gamma_n(z) = 1 - J^{\mu}_{n-2\lambda,\lambda}(z)/J^{\mu}_{n-2\lambda,\lambda}(z_0),$$

and the Abel transformation (see in Markushevich [5]), we obtain consequently:

$$\sum_{n=k+1}^{k+p} a_n(J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1})\gamma_n(z)$$

$$= \beta_{k+p}\gamma_{k+p}(z) - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1}(z) - \gamma_n(z)),$$

i.e.

$$\sum_{n=k+1}^{k+p} a_n(J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z))$$

$$= (1 - J^{\mu}_{k+p-2\lambda,\lambda}(z)/J^{\mu}_{k+p-2\lambda,\lambda}(z_0)) \sum_{n=k+1}^{k+p} a_n J^{\mu}_{n-2\lambda,\lambda}(z_0)$$

$$- \sum_{n=k+1}^{k+p-1} (\sum_{s=k+1}^{n} a_s J^{\mu}_{s-2\lambda,\lambda}(z_0))(\frac{J^{\mu}_{n-2\lambda,\lambda}(z))}{J^{\mu}_{n-2\lambda,\lambda}(z_0))} - \frac{J^{\mu}_{n+1-2\lambda,\lambda}(z))}{J^{\mu}_{n+1-2\lambda,\lambda}(z_0))}).$$

From the asymptotic formula (3) it follows that there exists a natural number M such that $J^{\mu}_{n-2\lambda,\lambda}(z_0) \neq 0$ when $n > M$. Let $k > M$. Then, for every natural $n > k$:

$$(7) \qquad J^{\mu}_{n-2\lambda,\lambda}(z)/J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n+1-2\lambda,\lambda}(z)/J^{\mu}_{n+1-2\lambda,\lambda}(z_0) = (z/z_0)^n \times$$

$$\frac{(1+\theta^{\mu}_{n-2\lambda,\lambda}(z))(1+\theta^{\mu}_{n+1-2\lambda,\lambda}(z_0))-(z/z_0)(1+\theta^{\mu}_{n+1-2\lambda,\lambda}(z))(1+\theta^{\mu}_{n-2\lambda,\lambda}(z_0))}{(1+\theta^{\mu}_{n-2\lambda,\lambda}(z_0))(1+\theta^{\mu}_{n+1-2\lambda,\lambda}(z_0))}.$$

For the right hand side of (7) we apply the Schwartz lemma. Then we get that there exists a constant C, so that:

$$|J^{\mu}_{n-2\lambda,\lambda}(z)/J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n+1-2\lambda,\lambda}(z)/J^{\mu}_{n+1-2\lambda,\lambda}(z_0)| \leq C|z - z_0||z/z_0|^n.$$

Analogously, there exists a constant B:

$$|1 - J^{\mu}_{k+p-2\lambda,\lambda}(z)/J^{\mu}_{k+p-2\lambda,\lambda}(z_0)| \leq B|z - z_0| \leq 2B|z_0|.$$

Let ε be an arbitrary positive number and choose $N(\varepsilon)$ so large that for $k > N(\varepsilon)$ the inequality

$$|\sum_{s=k+1}^{n} a_s J^{\mu}_{s-2\lambda,\lambda}(z_0)| < \min(\varepsilon\cos\varphi/(12B|z_0|), \varepsilon\cos\varphi/(6C|z_0|))$$

holds for every natural $n > k$. Therefore, for $k > \max(M, N(\varepsilon))$:

$$|\sum_{s=k+1}^{\infty} a_s J^{\mu}_{s-2\lambda,\lambda}(z_0)| \le \min(\varepsilon\cos\varphi/(12B|z_0|), \varepsilon\cos\varphi/(6C|z_0|),$$

and

$$|\sum_{n=k+1}^{\infty} a_n (J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z))|$$

$$\le (\varepsilon\cos\varphi/6)(1 + \sum_{n=k+1}^{\infty} |z_0|^{-1}|z - z_0||z/z_0|^n)$$

$$\le (\varepsilon\cos\varphi/6)(1 + |z - z_0|/(|z_0| - |z|)).$$

But near the vertex of the angle domain g_φ in the part d_φ closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle, we have $|z - z_0|/(|z_0| - |z|) < 2/\cos\varphi$, i.e. $|z - z_0|\cos\varphi < 2(|z_0| - |z|)$. That is why the inequality

$$(8) \qquad |\sum_{n=k+1}^{\infty} a_n (J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z))| < (\varepsilon\cos\varphi)/6 + \varepsilon/3 \le \varepsilon/2$$

holds for $z \in d_\varphi$ and $k > \max(M, N(\varepsilon))$. Fix some $k > \max(M, N(\varepsilon))$ and after that choose $\delta(\varepsilon)$ such that if $|z - z_0| < \delta(\varepsilon)$ then the inequality

$$(9) \qquad |\sum_{n=0}^{k} a_n (J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z))| < \varepsilon/2$$

holds inside d_φ. We get

$$|\Delta(z)| = |\sum_{n=0}^{\infty} a_n (J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z))|$$

for the module of the difference (6). From (8) and (9) it follows that the equality (5) is satisfied. ■

4. A Tauber Type Theorem

Let us consider the series $\sum\limits_{n=0}^{\infty} a_n,\ a_n \in \mathbb{C}$ and

$$z_0 \in \mathbb{C}, \quad |z_0| = R, \quad 0 < R < \infty, \quad J^{\mu}_{n-2\lambda,\lambda}(z_0) \neq 0 \ \text{ for } n = 0, 1, 2, \ldots .$$

In the rest of this section, we denote

$$J^{*}_{n,\lambda,\mu}(z; z_0) = \frac{J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)}.$$

Let the series $\sum\limits_{n=0}^{\infty} a_n J^{*}_{n,\lambda,\mu}(z; z_0)$ be convergent for $|z| < R$ and

$$F(z) = \sum_{n=0}^{\infty} a_n J^{*}_{n,\lambda,\mu}(z; z_0), \quad |z| < R.$$

Theorem 3 (Tauber type) . *If $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers with*

(10) $$\lim\{n a_n\} = 0,$$

and there exists

$$\lim_{z \to z_0} F(z) = S \quad (|z| < R,\ z \to z_0 \ \textit{radially}),$$

then the series $\sum\limits_{n=0}^{\infty} a_n$ is convergent and

$$\sum_{n=0}^{\infty} a_n = S.$$

P r o o f. For a point z of the segment $[0, z_0]$ we have

$$\sum_{n=0}^{k} a_n - F(z) = \sum_{n=0}^{k} a_n - \sum_{n=0}^{\infty} a_n J^{*}_{n,\lambda,\mu}(z; z_0)$$

$$= \sum_{n=0}^{k} a_n \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} - \sum_{n=0}^{\infty} a_n \frac{J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)}$$

$$= \sum_{n=0}^{k} a_n \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} - \sum_{n=k+1}^{\infty} a_n J^{*}_{n,\lambda,\mu}(z; z_0),$$

and therefore,

$$
(11)\qquad |\sum_{n=0}^{k} a_n - F(z)| \le \sum_{n=0}^{k} |a_n| \left| \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} \right|
$$

$$
+ \sum_{n=k+1}^{\infty} |a_n| \left| J^{*}_{n,\lambda,\mu}(z;z_0) \right| .
$$

By using the asymptotic formula (3) for the generalized Bessel-Maitland functions, we obtain:

$$
a_n \frac{J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \frac{1+\theta^{\mu}_{n-2\lambda,\lambda}(z)}{1+\theta^{\mu}_{n-2\lambda,\lambda}(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1+\widetilde{\theta}_{n,\lambda,\mu}(z;z_0)\right).
$$

Let ε be an arbitrary positive number. We choose a number N_1 so large that the inequalities $|1+\widetilde{\theta}_{k,\lambda,\mu}(z;z_0)| < 2, |ka_k| < \frac{\varepsilon}{6}$ hold as $k \ge N_1$. If $k > N_1$ and z is on the segment $[0, z_0]$, then for the second summand in (11) the following estimate is valid:

$$
(12)\qquad \sum_{n=k+1}^{\infty} |a_n| \left| J^{*}_{n,\lambda,\mu}(z;z_0) \right| = \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^n |1+\widetilde{\theta}_{n,\lambda,\mu}(z;z_0)|
$$

$$
\le 2 \left| \frac{z}{z_0} \right|^{k+1} \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^{n-k-1} \le 2 \sum_{n=0}^{\infty} |a_{n+k+1}| \left| \frac{z}{z_0} \right|^n
$$

$$
= 2 \sum_{n=0}^{\infty} \frac{|(n+k+1)a_{n+k+1}|}{n+k+1} \left| \frac{z}{z_0} \right|^n < 2 \sum_{n=0}^{\infty} \frac{\varepsilon/6}{n+k+1} \left| \frac{z}{z_0} \right|^n
$$

$$
< \frac{2}{k} \frac{\varepsilon}{6} \frac{1}{1-|z/z_0|} = \frac{\varepsilon}{3} \frac{1}{k} \frac{|z_0|}{|z_0|-|z|} .
$$

Now let us consider the first summand in (11). We have:

$$
\sum_{n=0}^{k} |a_n| \left| \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} \right|
$$

$$
= \sum_{n=0}^{m} |a_n| \left| \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} \right| + \sum_{n=m+1}^{k} |a_n| \left| \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} \right| .
$$

According to Schwarz's lemma, there exists a constant C such that

$$
\left| \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} \right| < C|z-z_0|.
$$

Moreover, there exists a number N_2 such that the following inequality

$$\sum_{n=0}^{m} |a_n| \left| \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} \right| \le C\,|z - z_0|\,k\,\frac{\sum\limits_{n=0}^{m} |a_n|}{k} \tag{13}$$

$$< C\,|z - z_0|\,k\,\frac{\varepsilon}{3RC} = |z - z_0|\,k\,\frac{\varepsilon}{3R}.$$

holds as $k > N_2$. It remains to estimate the sum

$$\sum_{n=m+1}^{k} |a_n| \left| \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} \right|.$$

To this end, using asymptotic formula (3) for (1), we find consequently:

$$\frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} = \frac{(z_0)^n(1 + \theta^{\mu}_{n-2\lambda,\lambda}(z_0)) - z^n(1 + \theta^{\mu}_{n-2\lambda,\lambda}(z))}{z_0^n(1 + \theta^{\mu}_{n-2\lambda,\lambda}(z_0))}$$

$$= 1 - \left(\frac{z}{z_0}\right)^n \frac{1 + \theta^{\mu}_{n-2\lambda,\lambda}(z)}{1 + \theta^{\mu}_{n-2\lambda,\lambda}(z_0)} = 1 - \left(\frac{z}{z_0}\right)^n \left[1 + \frac{\theta^{\mu}_{n-2\lambda,\lambda}(z) - \theta^{\mu}_{n-2\lambda,\lambda}(z_0)}{1 + \theta^{\mu}_{n-2\lambda,\lambda}(z_0)}\right]$$

$$= 1 - \left(\frac{z}{z_0}\right)^n - \left(\frac{z}{z_0}\right)^n \frac{\theta^{\mu}_{n-2\lambda,\lambda}(z) - \theta^{\mu}_{n-2\lambda,\lambda}(z_0)}{1 + \theta^{\mu}_{n-2\lambda,\lambda}(z_0)}.$$

Therefore,

$$\left| \frac{J^{\mu}_{n-2\lambda,\lambda}(z_0) - J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)} \right| \tag{14}$$

$$\le \left| 1 - \left(\frac{z}{z_0}\right)^n \right| + \left|\frac{z}{z_0}\right|^n \left| \frac{\theta^{\mu}_{n-2\lambda,\lambda}(z) - \theta^{\mu}_{n-2\lambda,\lambda}(z_0)}{1 + \theta^{\mu}_{n-2\lambda,\lambda}(z_0)} \right|.$$

We obtain the following inequalities

$$\left| 1 - \left(\frac{z}{z_0}\right)^n \right| = \left| 1 - \frac{z}{z_0} \right| \left| 1 + \frac{z}{z_0} + \left(\frac{z}{z_0}\right)^2 + \ldots + \left(\frac{z}{z_0}\right)^{n-1} \right| \le n \left| 1 - \frac{z}{z_0} \right|$$

for the first summand of (14). According to Schwarz's lemma, there exists a constant ρ such that

$$\left| \frac{\theta^{\mu}_{n-2\lambda,\lambda}(z) - \theta^{\mu}_{n-2\lambda,\lambda}(z_0)}{1 + \theta^{\mu}_{n-2\lambda,\lambda}(z_0)} \right| \le 1 \qquad \text{as} \qquad |z - z_0| < \rho.$$

Then, for such $|z|$, we obtain for the second summand of (14):

$$\left|\frac{z}{z_0}\right|^n \left|\frac{\theta^{\mu}_{n-2\lambda,\lambda}(z)-\theta^{\mu}_{n-2\lambda,\lambda}(z_0)}{1+\theta^{\mu}_{n-2\lambda,\lambda}(z_0)}\right| \le \left|\frac{z}{z_0}\right|^n \; |z-z_0|.$$

From (10) it follows that

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} n|a_n|}{k} = 0, \quad \lim_{k\to\infty} \frac{\sum\limits_{n=1}^{k} |a_n|}{k} = 0.$$

Then a number N_3 exists such that

$$\frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{and} \quad \frac{\sum\limits_{n=m+1}^{k} |a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{as} \quad k > N_3.$$

Therefore,

$$\text{(15)} \qquad \sum_{n=m+1}^{k} |a_n| \left|\frac{J^{\mu}_{n-2\lambda,\lambda}(z_0)-J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)}\right| \le \sum_{n=m+1}^{k} n|a_n| \left|1-\frac{z}{z_0}\right|$$

$$+\sum_{n=m+1}^{k} |a_n| \left|\frac{z}{z_0}\right|^n |z-z_0| \le k\,\frac{|z-z_0|}{R}\,\frac{\sum\limits_{n=m+1}^{k} n|a_n|}{k} + k\,|z-z_0|\,\frac{\sum\limits_{n=m+1}^{k} |a_n|}{k}$$

$$< k\,|z-z_0|\,\frac{1+R}{R}\,\frac{\varepsilon}{3(1+R)} = k\,|z-z_0|\,\frac{\varepsilon}{3R}.$$

Finally, let us note that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| \le \sum_{n=0}^{m} |a_n| \left|\frac{J^{\mu}_{n-2\lambda,\lambda}(z_0)-J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)}\right|$$

$$+\sum_{n=m+1}^{k} |a_n| \left|\frac{J^{\mu}_{n-2\lambda,\lambda}(z_0)-J^{\mu}_{n-2\lambda,\lambda}(z)}{J^{\mu}_{n-2\lambda,\lambda}(z_0)}\right| + \sum_{n=k+1}^{\infty} |a_n| \left|J^{*}_{n,\lambda,\mu}(z;z_0)\right| .$$

Let $N=\max(N_1,N_2,N_3), k>N$ and $|z-z_0|<\rho$. Then by using (12),(13),(15), we can conclude that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| < |z-z_0|\,k\,\frac{\varepsilon}{3R} + k\,|z-z_0|\,\frac{\varepsilon}{3R} + \frac{\varepsilon}{3}\,\frac{1}{k}\,\frac{|z_0|}{|z_0|-|z|}$$

$$= \frac{\varepsilon}{3}\left[\frac{2k}{R}\,|z-z_0| + \frac{1}{k}\,\frac{|z_0|}{|z_0|-|z|}\right].$$

If we substitute z by $z_0(1-\frac{1}{k})$, then

$$\left|\sum_{n=0}^{k} a_n - F\left(z_0(1-\frac{1}{k})\right)\right| < \frac{\varepsilon}{3}3 = \varepsilon.$$

This proves that $\lim\limits_{k\to\infty}\sum\limits_{n=0}^{k} a_n$ exists and equals $\lim\limits_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right)$, i.e.

$$\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} F\left(z_0(1-\frac{1}{k})\right) = S.$$

Thus the theorem is proved. ∎

5. Special Cases

A. Let $\lambda = 0$, then special function (1) turns into the generalization of the Bessel function $J_\nu(z)$, introduced by E.M. (Maitland) Wright [14], and called Wright function or misnamed in the literature also as Bessel-Maitland function:

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu+\mu k+1)}\,, \quad \mu > -1, \tag{16}$$

for details, see Marichev [4], p.109; Kiryakova [3], p.336.

Namely,

$$J_{\nu,0}^\mu(z) = (z/2)^\nu J_\nu^\mu(z^2/4), \tag{17}$$

therefore our results, for $\lambda = 0$, yield the corresponding theorems in Paneva-Konovska [8]. Additionally, if $\mu = 1$, then $J_{\nu,0}^1(z) = J_\nu(z)$ and we get the theorems for convergence of series in Bessel functions, see Paneva-Konovska [6], [7].

B. Let $\mu = 1$ in (1), then (see Marichev [4], p.109; Kiryakova [3], p.336):

$$J_{\nu,\lambda}^1(z) = \frac{2^{2-2\lambda-\nu}}{\Gamma(\lambda)\Gamma(\lambda+\nu)}\, s_{2\lambda+\nu-1,\nu}(z), \tag{18}$$

where $s_{\alpha,\nu}(z)$, $\alpha, \nu \in \mathbb{C}$ denotes the Lommel function (Erdélyi *et al.* [2], Vol. 2, p.50, (69)):

$$s_{\alpha,\nu}(z) = \frac{z^{\alpha+1}}{(\alpha-\nu+1)(\alpha+\nu+1)}\,{}_1F_2\left(1; \frac{\alpha-\nu+3}{2}; \frac{\alpha+\nu+3}{2}; -\frac{z^2}{4}\right). \tag{19}$$

For $\nu = n+1-2\lambda$, relation (18) becomes

$$J^1_{n+1-2\lambda,\lambda}(z) = \frac{2^{1-n}}{\Gamma(\lambda)\Gamma(n+1-\lambda)}\, s_{n,n+1-2\lambda}(z),$$

and Theorems 1, 2, 3 provide, as special cases, results on the convergence of series in Lommel functions ($\widetilde{a}_n := c_n a_n$):

$$\sum_{n=0}^{\infty} \widetilde{a}_n J^1_{n+1-2\lambda,\lambda}(z) = \sum_{n=0}^{\infty} a_n s_{n,n+1-2\lambda}(z). \tag{20}$$

Additionally, let us take $\lambda = 1/2$, then we obtain the Struve functions (see Erdélyi *et al.* [2], Vol. 2, p.51, (84)):

$$\mathbf{H}_\nu(z) = \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)}\, s_{\nu,\nu}(z) \tag{21}$$

and our results turn into Cauchy-Hadamard, Abel and Tauber type theorems for series in Struve functions ($\widetilde{a}_n := c_n a^*_n = d_n a_n$):

$$\sum_{n=0}^{\infty} \widetilde{a}_n J^1_{n,1/2}(z) = \sum_{n=0}^{\infty} a^*_n s_{n,n}(z) = \sum_{n=0}^{\infty} a_n \mathbf{H}_n(z). \tag{22}$$

References

[1] L. Boyadjiev, Abel's theorems for Laguerre and Hermite series. Comptes Rendu de l'Academy Bulgare des Sciences, **39**, No 4 (1986), 13-15.

[2] A. Erdélyi et al. (Ed-s), *Higher Transcendental Functions*, Vols. **1 - 3**, McGraw-Hill, New York-Toronto-London, 1953.

[3] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman & J. Wiley, Harlow and N. York, 1994.

[4] O. I. Marichev, *Method of Evaluation of Integrals of Special Functions* (In Russian), Nauka i Technika, Minsk, 1978; Engl. translation: O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental*

Functions: Theory and Algorithmic Tables, Ellis Horwood Ltd., N. York etc.; Halsted Press, 1983.

[5] A. M a r k u s h e v i c h, *Theory of Analytic Functions,* Vol. **1** (In Russian), Nauka, Moscow, 1967.

[6] J. P a n e v a - K o n o v s k a, Cauchy-Hadamard and Abel type theorems for Bessel functions series, In: *Proc. 19-th Summer School "Applications of Mathematics in Engineering", Varna, 24.08-2.09, 1993*, Sofia (1994), 165-170.

[7] J. P a n e v a - K o n o v s k a, A Tauber type theorem for series in Bessel functions, Fractional Calculus & Applied Analysis, **2**, No 5 (1999), 683-688.

[8] J. P a n e v a - K o n o v s k a, On the convergence of series in Bessel-Maitland functions, To appear in: *Annuaire of Sofia University "St. Kl. Ohridski", Mathematics*, (2007).

[9] J. P a n e v a - K o n o v s k a, Index-asymptotic formulae for Wright's generalized Bessel functions, *Mathematical Sciences Research Journal*, **11**, No 5 (2007), 424-431.

[10] R. S. P a t h a k, Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations, Proc. National Acad. of Sciences of India, **A-36**, No 1 (1966), 81-86.

[11] P. R u s e v, *Analytic Functions and Classical Orthogonal Polynomials*, Publ. House Bulg. Acad. Sci., Sofia, 1984.

[12] P. R u s e v, *Classical Orthogonal Polynomials and Their Associated Functions in Complex Domain*, Publ. House Bulg. Acad. Sci., Sofia, 2005.

[13] L. T c h a k a l o v, *Introduction in Theory of Analytic Functions* (In Bulgarian), Nauka i Izkustvo, Sofia, 1972.

[13] E. M. W r i g h t, On the coefficients of power series having exponential singularities, *J. London Math. Soc.*, **8** (1933), 71-79.

Faculty of Applied Mathematics and Informatics
Technical University - Sofia
1156 Sofia, BULGARIA
Received: 18.04.2007
E-mail: yorry77@mail.bg