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The Normality of Meromorphic Functions along an Arbitrary Curve

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In the paper [5] O. Lehto and K.I. Virtanen showed that a normal meromorphic function has good locally boundary properties, that is, the existence of the asymptotic boundary value implies the existence of the angular boundary value. In the same paper, in the terms of growth of the spherical derivative, it was proved the assertion which gives necessary and sufficient conditions for a meromorphic function to be normal. In the paper [9] it is defined the normality of a function along the curve. It is showed that these functions have good locally boundary properties.

In this paper we prove the assertions which in the terms of growth of the spherical derivative and a P-sequences, give necessary and sufficient conditions for a meromorphic function to be normal along the curve. We give an example of a meromorphic function that is normal along an arbitrary curve, while it is not normal in the sense of Lehto and Virtanen.

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1. Introduction and preliminaries

Let $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbf{C}$, with the boundary Γ, and let $\overline{\mathbf{C}} = \mathbf{C} \bigcup \{\infty\}$ be the *Riemann sphere*. For any $w \in \mathbf{D}$, denote by $g_w(z)$ the bilinear mapping of the disk $\mathbf{D}$ defined as

$$g_w(z) = \frac{z + w}{1 + z\bar{w}}, \quad z \in \mathbf{D}.$$

Then the *pseudohyperbolic metric* d_{ph} on $\mathbf{D}$ is given by

$$d_{ph}(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - z_1 \bar{z}_2} \right|, \quad z_1, z_2 \in \mathbf{D}.$$

For a fixed $w \in \mathbf{D}$, the set $\mathbf{D}_{ph}(w,r)$ defined as

$$\mathbf{D}_{ph}(w,r) = \{z \in \mathbf{D} : d_{ph}(z,w) < r\}, \quad 0 < r < 1,$$

is called the *pseudohyperbolic disk*.

The function d_h defined on $\mathbf{D}$ as

$$d_h(z_1,z_2) = \ln \frac{1 + d_{ph}(z_1,z_2)}{1 - d_{ph}(z_1,z_2)}, \quad z_1, z_2 \in \mathbf{D},$$

is the *hyperbolic metric of Poincaré model of Lobachevski's geometry.*

For a fixed $w \in \mathbf{D}$, the set $\mathbf{D}_h(w,r)$ defined as

$$\mathbf{D}_h(w,r) = \{z \in \mathbf{D} : d_h(z,w) < r\}, \quad 0 < r < \infty,$$

is called the *hyperbolic disk* with the center w and hyperbolic radius r.

It is easy to verify that there holds

$$\mathbf{D}_h(w,r) = \mathbf{D}_{ph}(w,r') \quad \text{with} \quad r = \ln \frac{1+r'}{1-r'}.$$

From the above we see that the pseudohyperbolic disk may be always replaced by the corresponding hyperbolic disk, and conversely. In the sequel, we will use one of those two disks that gives the possibility of simpler proof of the formulated assertions.

Given a set $A \subseteq \mathbf{D}$ and a function $f : \mathbf{D} \to \overline{\mathbf{C}}$, denote by $C(f,A)$ the *boundary set* of the function f with respect to the set A. Namely, $C(f,A)$ is a set of all points $\omega \in \overline{\mathbf{C}}$ for which there exists a sequence (z_n) in A such that $z_n \to \xi$ for some $\xi \in \Gamma$ and $f(z_n) \to \omega$ as $n \to \infty$.

The *spherical derivative* of a meromorphic function $f : \mathbf{D} \to \overline{\mathbf{C}}$ is a function $f^{\#}$ defined as

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}, \quad z \in \mathbf{D}.$$

Here, as always in the sequel, we will consider that each mentioned curve is a *Jordan's curve* such that $\gamma \subset \mathbf{D} \cup \Gamma$ with $\gamma \cap \Gamma = \{e^{i\theta}\}$.

In [5] O. Lehto and K.I. Virtanen investigated normal meromorphic functions. A function f meromorphic on the disk $\mathbf{D}$ is said to be *normal* in $\mathbf{D}$ if the family $\mathcal{F}_f := \{f \circ g_w : w \in \mathbf{D}\}$ is *normal* in $\mathbf{D}$ *in the sense of Montel.*

Definition 1. A function $f : \mathbf{D} \to \overline{\mathbf{C}}$ is said to be *normal along the curve* γ if the family $\mathcal{F}_f := \{f \circ g_w : w \in \gamma\}$ is *normal* in the disk $\mathbf{D}$ *in the sense* *of Montel*, i.e., if every sequence in $\mathcal{F}_f$ contains a subsequence which converges uniformly on compact subsets of $\mathbf{D}$.

Definition 2. Given $0 < r < 1$, the set $\bigcup_{w\in\gamma} \mathbf{D}_{ph}(w,r)$ is called the *γ-curvilinear angle* with the vertex at a point $e^{i\theta}$. By the invariance of the hyperbolic metric under bilinear mapping $g_w(z)$, it follows that

$$\bigcup_{w\in\gamma} \mathbf{D}_{ph}(w,r) = \bigcup_{w\in\gamma} g_w(\mathbf{D}_{ph}(0,r)) =: \Delta_\gamma(e^{i\theta}, r).$$

If γ is a radius of the disk $\mathbf{D}$ that ends at a point $e^{i\theta}$, then the γ-curvilinear angle is a set bounded by parts of the circle $K_r = \{z : |z| = r\}$ and hypercycles that pass through points $e^{i\theta}$ and $-e^{i\theta}$. Such a set contains the angles in $\mathbf{D}$ with vertex $e^{i\theta}$ or $-e^{i\theta}$.

If γ is the *orocycle* $O(e^{i\theta}) = \left\{z : \left|z - \frac{1}{2}e^{i\theta}\right| = \frac{1}{2}\right\}$, then the γ-curvilinear angle is a set bounded by two orocycles that passe through the point $e^{i\theta}$.

Definition 3. Given the γ-curvilinear angle defined above, we say that a function $f : \mathbf{D} \to \overline{\mathbf{C}}$ has a *γ-curvilinear angular boundary value* if it holds

$$C\left(f, \bigcup_{w\in\gamma} g_w(\mathbf{D}_{ph}(0,r)\right) = \{c\},$$

or equivalently if

$$\lim_{z(\in\cup_{w\in\gamma}\mathbf{D}_{ph}(w,r))\to e^{i\theta}} f(z) = c$$

for each $r \in (0,1)$ and some $c \in \overline{\mathbf{C}}$. Note that this value does not depend on r.

2. The normality of a meromorphic function along a curve

By a well known result of O. Lehto and K.I. Virtanen from [5], a meromorphic function $f : \mathbf{D} \to \overline{\mathbf{C}}$ is a normal function if and only if

$$\sup_{z\in\mathbf{D}} (1 - |z|^2) f^{\#}(z) < \infty.$$

We prove here the following result.

Theorem 1. *Let $f : \mathbf{D} \to \overline{\mathbf{C}}$ be a meromorphic function on the disk $\mathbf{D}$. Then the following statements are equivalent.*

(i) f *is a normal function along the curve* γ*;*

(ii) $\sup_{z\in\bigcup_{w\in\gamma}\mathbf{D}_{ph}(w,r)} (1-|z|^2)f^{\#}(z) \leq C(r) < \infty$, *for all* $0 < r < 1$, *where* $C(r)$ *is a positive constant depending on* r*.*

Proof. (i)⇒(ii). Let $f : \mathbf{D} \to \overline{\mathbf{C}}$ be a normal meromorphic function along the curve γ. Then the family $\mathcal{F}_f := \{f \circ g_w : w \in \gamma\}$ is a normal family in the disk $\mathbf{D}$ in the sense of Montel (see [7]). By Marty's normality criterion ([4]), it follows that for a compact set $\overline{\mathbf{D}_{ph}(0,r)} = \{z \in \mathbf{D} : |z| \leq r\}$ with $0 < r < 1$, there is a constant $M(r)$ depending on r such that

$$(1)\qquad (f\circ g_w)^{\#}(z) \leq M(r) \quad \text{for each } z \in \overline{\mathbf{D}_{ph}(0,r)}, \text{ and for each } w \in \gamma.$$

Since

$$(f\circ g_w)^{\#}(z) = \frac{|f'(g_w(z))|}{1+|f(g_w(z))|^2}\cdot|g_w'(z)| = \frac{|f'(g_w(z))|}{1+|f(g_w(z))|^2}\cdot\frac{1-|w|^2}{|1+\bar{w}z|^2},$$

by (1) we obtain

$$\frac{|f'(g_w(z))|}{1+|f(g_w(z))|^2}\cdot\frac{1-|w|^2}{|1+\bar{w}z|^2}\cdot\frac{1-|z|^2}{1-|z|^2} \leq M(r) \text{ for each } z \in \overline{\mathbf{D}_{ph}(0,r)}, \text{ and } w\in\gamma.$$

If we suppose that $z \in \overline{\mathbf{D}_{ph}(0,r)}$, $w \in \gamma$, and $u = g_w(z)$, then

$$\frac{1-|w|^2}{|1+\bar{w}z|^2}\cdot(1-|z|^2) = 1-|u|^2,$$

and hence

$$\frac{|f'(u)|}{1+|f(u)|^2}\cdot(1-|u|^2)\cdot\frac{1}{1-|z|^2} \leq M(r)$$

for each $u \in \overline{\mathbf{D}_{ph}(w,r)}$ and $z \in \overline{\mathbf{D}_{ph}(0,r)}$.
Multiplying the last inequality by $1-|z|^2$ we obtain

$$\begin{aligned}\frac{|f'(u)|}{1+|f(u)|^2}\cdot(1-|u|^2) &\leq M(r)(1-|z|^2)\\ &\leq M(r) < +\infty.\end{aligned}$$

Therefore, for a fixed $0 < r < 1$, and for each $u \in \overline{\mathbf{D}_{ph}(w,r)} \subset \bigcup_{w\in\gamma}\mathbf{D}_{ph}(w,r)$ it holds

$$(1-|u|^2)\frac{|f'(u)|}{1+|f(u)|^2} \leq M(r) < +\infty,$$

and therefore

$$\sup_{u\in\cup_{w\in\gamma}\mathbf{D}_{ph}(w,r)}(1-|u|^2)\frac{|f'(u)|}{1+|f(u)|^2}\le M(r)<+\infty,$$

i.e.,

$$\sup_{u\in\cup_{w\in\gamma}\mathbf{D}_{ph}(w,r)}(1-|u|^2)f^{\#}(u)\le M(r)<+\infty,$$

as desired.

(ii)⇒(i). Suppose that for a meromorphic function $f:\mathbf{D}\to\overline{\mathbf{C}}$ it holds

$$\sup_{u\in\cup_{w\in\gamma}\mathbf{D}_{ph}(w,r)}(1-|u|^2)f^{\#}(u)\le C(r)<+\infty \quad \text{for each } 0<r<1.$$

Then

$$(1-|u|^2)f^{\#}(u)\le C(r)<+\infty \quad \text{for each} \quad u\in\overline{\mathbf{D}_{ph}(w,r)} \text{ and } w\in\gamma.$$

Given $w\in\gamma$ and $u\in\overline{\mathbf{D}_{ph}(w,r)}$ there exists $z\in\overline{\mathbf{D}_{ph}(0,r)}$ such that $g_w(z)=u$. Thus, for each $z\in\mathbf{D}_{ph}(0,r)$ and each $w\in\gamma$ we obtain

$$(1-|g_w(z)|^2)f^{\#}(g_w(z))\le C(r)<+\infty,$$

that is,

$$(1-|g_w(z)|^2)\frac{|f'(g_w(z))|}{1+|f(g_w(z))|^2}\le C(r)<+\infty.$$

Since $1-|g_w(z)|^2=\frac{1-|w|^2}{|1+\bar{w}z|^2}(1-|z|^2)$, it follows from the above inequality that

$$\frac{1-|w|^2}{|1+\bar{w}z|^2}(1-|z|^2)\cdot\frac{|f'(g_w(z))|}{1+|f(g_w(z))|^2}\le C(r)<+\infty.$$

Since $|g_w'(z)|=\frac{1-|w|^2}{|1+\bar{w}z|^2}$, the above inequality implies

$$\frac{|f'(g_w(z))|}{1+|f(g_w(z))|^2}\cdot|g_w'(z)|(1-|z|^2)\le C(r)<+\infty,$$

whence we have

$$(f\circ g_w)^{\#}(z)(1-|z|^2)\le C(r)<+\infty$$

or

$$(f\circ g_w)^{\#}(z)\le\frac{C(r)}{1-|z|^2}\le\frac{C(r)}{1-r^2}=M(r)<+\infty.$$

Therefore, $(f \circ g_w)^{\#}(z) \le M(r) < +\infty$, $w \in \gamma$, $z \in \overline{\mathbf{D}_{ph}(0,r)}$. Thus by Marty's normality criterion, we conclude that $\mathcal{F}_f = \{f \circ g_w : w \in \gamma\}$ is a normal family in the sense of Montel ([7]), and so by Definition 1, f is a normal function along the curve γ. ■

Theorem 1 and the properties of the pseudohyperbolic metric imply the following assertion.

Theorem 2. *If $f : \mathbf{D} \to \overline{\mathbf{C}}$ is a meromorphic function that is normal along the curve γ, then f is normal along any curve $\gamma_1 \subset \bigcup_{w\in\gamma} \mathbf{D}_{ph}(w,r)$ with $0 < r < 1$.*

V.I. Gavrilov introduced in [2] the notion of P-sequences which is used for investigation of the normality of meromorphic functions.

Definition 4. Let $f : \mathbf{D} \to \overline{\mathbf{C}}$ be a function. A sequence $(z_n) \subset \mathbf{D}$ is called a *P-sequence* of the function f if the following conditions are satisfied:

(i) $\lim_{n\to\infty} |z_n| = 1$, and

(ii) for every subsequence (z_{n_k}) of (z_n), and for any $\varepsilon > 0$, in the union $\bigcup_{k\in\mathbf{N}} D_h(z_{n_k}, \varepsilon)$ the function f takes infinitely many times every value in $\overline{\mathbf{C}}$, with at most two exceptions.

Theorem A. *([1]). Let $f : \mathbf{D} \to \overline{\mathbf{C}}$ be a meromorphic function on $\mathbf{D}$, and let $(z_n) \subset \mathbf{D}$ be a sequence such that $\lim_{n\to\infty}(1-|z_n|^2)f^{\#}(z_n) = +\infty$. Then (z_n) is a P–sequence of the function f.*

Theorem B. *([1]). A sequence $(z_n) \subset \mathbf{D}$ such that $\lim_{n\to\infty} |z_n| = 1$ is a P-sequence of a meromorphic function f on $\mathbf{D}$ if and only if there is a sequence of positive numbers (ε_n) such that $\lim_{n\to\infty} \varepsilon_n = 0$ and*

$$\lim_{n\to\infty} \left(\sup_{z\in D_h(z_n,\varepsilon_n)} (1-|z|^2) f^{\#}(z) \right) = +\infty.$$

By Lemma 1 and Theorem 2 from [1], follows the following assertion.

Theorem C. *Let $f : \mathbf{D} \to \overline{\mathbf{C}}$ be a meromorphic function on $\mathbf{D}$, and let $(z_n) \subset \mathbf{D}$ be a sequence such that $\lim_{n\to\infty} |z_n| = 1$ and $\lim_{n\to\infty} f(z_n) = a$ for some $a \in \overline{\mathbf{C}}$. Further, let $(z'_n) \subset \mathbf{D}$ be a sequence such that $\lim_{n\to\infty} |z'_n| = 1$, $\lim_{n\to\infty} d_h(z_n, z'_n) = 0$, and $f(z'_n)$ does not converge to a as $n \to \infty$. Then (z_n) and (z'_n) are both P-sequences of the function f.*

We are now ready to prove the following result.

Theorem 3. *A meromorphic function $f : \mathbf{D} \to \overline{\mathbf{C}}$ is normal along the curve γ if and only if for each $r \in (0, +\infty)$ f does not possess a P-sequence in domain $\bigcup_{w\in\gamma} \mathbf{D}_h(w, r)$.*

Proof. $\Rightarrow$. Let $f : \mathbf{D} \to \overline{\mathbf{C}}$ be a normal meromorphic function along the curve γ. We will prove that the function f does not possess a P-sequence in $\bigcup_{w\in\gamma} \mathbf{D}_h(w, r)$ for none $0 < r < 1$. Namely, we will show that for every sequence (ε_k) such that $\varepsilon_k > 0$ for all $k = 1, 2, \ldots$, and $\lim_{k\to\infty} \varepsilon_k = 0$, and for every sequence (z_k) with $z_k \in \bigcup_{w\in\gamma} \mathbf{D}_h(w, r)$ for all $k = 1, 2, \ldots$, such that $|z_k| \to 1$ as $k \to \infty$, there is a positive constant C such that

$$\overline{\lim}_{k\to\infty} \left(\sup_{z\in\mathbf{D}_h(z_k,\varepsilon_k)} (1 - |z|^2) f^{\#}(z) \right) \le C.$$

Let (ε_k) be a sequence of positive numbers such that $\lim_{k\to\infty} \varepsilon_k = 0$, and let (z_k) be an arbitrary sequence in $\bigcup_{w\in\gamma} \mathbf{D}_h(w, r)$ for which $\lim_{k\to\infty} |z_k| = 1$. Further, let $\varepsilon = \sup\{\varepsilon_k : k \in \mathbf{N}\}$. Since $z_k \in \bigcup_{w\in\gamma} \mathbf{D}_h(w, r)$, it follows that $z_k \in \mathbf{D}_h(w_k, r)$ with $w_k \in \gamma$. If we assume that $r_1 = r + \varepsilon$, then $\mathbf{D}_h(z_k, \varepsilon_k) \subset \mathbf{D}_h(w_k, r_1)$ for all $k = 1, 2, \ldots$. Indeed, if $z \in \mathbf{D}_h(z_k, \varepsilon_k)$, then

$$\begin{aligned} d_h(z, w_k) &\le d_h(z, z_k) + d_h(z_k, w_k) \\ &< \varepsilon_k + r \le \varepsilon + r = r_1, \end{aligned}$$

whence we see that $z \in \mathbf{D}_h(w_k, r_1)$. Thus $\mathbf{D}_h(z_k, \varepsilon_k) \subset \bigcup_{w\in\gamma} \mathbf{D}_h(w, r_1)$ for all $k = 1, 2, \ldots$. Since the function f is normal along the curve γ, in view of the fact that $\mathbf{D}_h(w, r_1) = \mathbf{D}_{ph}(w, r_1')$ with $r_1 = \ln \frac{1+r_1'}{1-r_1'}$, $r_1' \in (0, 1)$, it follows by Theorem 1 that there is a positive constant C such that

$$\sup_{z\in\cup_{w\in\gamma}\mathbf{D}_h(w,r_1)} (1 - |z|^2) f^{\#}(z) \le C,$$

and hence

$$\sup_{z\in D_h(z_k,\varepsilon_k)} (1 - |z|^2) f^{\#}(z) \le C \quad \text{for all} \quad k = 1, 2, \ldots.$$

This shows that

$$\overline{\lim}_{k\to\infty} \left(\sup_{z\in\mathbf{D}_h(z_k,\varepsilon_k)} (1 - |z|^2) f^{\#}(z) \right) \le C. \tag{2}$$

By (2) and Theorem B, we conclude that (z_k) is not a P-sequence of the function f.

$\Leftarrow$. Suppose that a meromorphic function $f : \mathbf{D} \to \overline{\mathbf{C}}$ does not possess a P-sequence in each domain $\bigcup_{w\in\gamma} \mathbf{D}_h(w,r)$ with $0 < r < \infty$. We will prove by a contradiction that f is a normal function along the curve γ. Indeed, let $r_1 \in (0,1)$ such that

$$\sup_{z\in\cup_{w\in\gamma}\mathbf{D}_h(w,r_1)} (1-|z|^2)f^{\#}(z) = +\infty,$$

where the sup is taken over the disks $\mathbf{D}_h(w,r_1) = \mathbf{D}_{ph}(w,r_1')$ with $r_1 = \ln \frac{1+r_1'}{1-r_1'}$, $r_1' \in (0,1)$. This implies that there is a sequence (z_n) with $z_n \in \bigcup_{w\in\gamma} \mathbf{D}_h(w,r_1)$ for all $n = 1,2,\ldots$, such that $\lim_{n\to\infty}(1-|z_n|^2)f^{\#}(z_n) = +\infty$. Then by Theorem A, it follows that (z_n) is a P-sequence of the function f. This contradiction completes the proof. ∎

3. The normality of a meromorphic function in D

It is proved in [1] that a meromorphic function f on the disk $\mathbf{D}$ is normal in $\mathbf{D}$ if and only if the function f does not possess a P-sequence in $\mathbf{D}$. Obviously, each normal meromorphic function on the disk $\mathbf{D}$ is normal along every curve $\gamma \subset \mathbf{D}$.

In the similar manner as in [2], by using P-sequences, we will construct the function f which is normal along an arbitrary given curve $\gamma \subset \mathbf{D}$ with $\gamma \bigcap \Gamma = \{e^{i\theta}\}$, such that f is not *normal in the sense of Lehto and Virtanen.*

Example. For the Jordan's curve γ such that $\gamma \subset \mathbf{D} \cup \Gamma$ and $\gamma \cap \Gamma = \{e^{i\theta}\}$, and for $0 < r < 1$, denote by L_r^d and L_r^l with the parts of the boundary of γ-curvilinear angle $\Delta_\gamma(e^{i\theta}, r)$ that lie on the right and on the left of the curve γ, respectively.

If the curve γ is the radius $[0, e^{i\theta}]$ of the disk $\mathbf{D}$, then L_r^d and L_r^l lie on the circle $K_r = \{z : |z| = r\}$ and on the hypercycles generated by the circle K_r and by bilinear mappings φ_r of the disk $\mathbf{D}$ defined as $\varphi_r(z) = \frac{z-re^{i\theta}}{1-re^{-i\theta}z}$.

If the curve γ is the orocycle $O(e^{i\theta})$, then L_r^d and L_r^l lie on the circle K_r and on the orocycles generated by the circle K_r and by bilinear mappings φ_w of the disk $\mathbf{D}$ defined as $\varphi_w(z) = \frac{z-w}{1-\bar{w}z}$ with $w \in O(e^{i\theta})$.

Now choose a sequence (z_n) such that $z_n \in L_{1-1/n}^d$ for all even n, and $z_n \in L_{1-1/n}^l$ for all odd n, such that $\lim_{n\to\infty} d_{ph}(z_n, z_{n+1}) = \infty$. Let ε_n be a sequence of positive numbers such that

1) $\varepsilon_{n+1} < \varepsilon_n$ for all $n = 1,2,\ldots$;

2) $\lim_{n\to\infty} \varepsilon_n = 0$;

3) The disks $B_n = \{z : |z - z_n| < \varepsilon_n\}$ with $n = 1, 2, \ldots$, are pairwise disjoint;

4) $\lim_{n\to\infty}(\sup_{z\in B_n} d_{ph}(z, z_n)) = 0$;

5) $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.

Further, we put $a_n = \varepsilon_n^3$ for all $n = 1, 2, \ldots$, and consider the function $f(z) = \sum_{n=1}^{\infty} a_n(z - z_n)^{-1}$. Then by the property 3), for any fixed $k \in \mathbf{N}$ and each $z \in B_k$, we have

$$|\sum_{n\neq k} a_n(z - z_n)^{-1}| \leq \sum_{n=1}^{\infty} \varepsilon_n^2 < +\infty.$$

Hence, f is a meromorphic function on $\mathbf{D}$ whose poles are points z_n, $n = 1, 2, \ldots$.

Since $|f(z_n + \varepsilon_n)| < +\infty$ and $\lim_{n\to\infty} d_{ph}(z_n, z_n + \varepsilon_n) = 0$, it follows by Theorem C that (z_n) is a P-sequence of the function f. Hence, the terms of P-sequence (z_n) of the function f lie on the boundaries of γ-curvilinear angles $\Delta_\gamma(e^{i\theta}, 1 - 1/n)$ with $n = 1, 2, \ldots$.

However, since for each $z', z'' \in \mathbf{D} \setminus \bigcup_{k=1}^{\infty} B_k$ holds

$$|f(z') - f(z'')| \leq |z' - z''| \sum_{n=1}^{\infty} \varepsilon_n,$$

and since each γ-curvilinear angle $\Delta_\gamma(e^{i\theta}, r)$, $0 < r < 1$, contains a finite number of terms of the sequence (z_n), it follows that the function f is bounded on the set $\Delta_\gamma(e^{i\theta}, r) \cap U(e^{i\theta})$ where $0 < r < 1$ and $U(e^{i\theta})$ is a neighborhood of the point $e^{i\theta}$. This fact and the assertion that every uniformly bounded sequence of holomorphic functions is locally compact (see [3, Theorem 1, p. 19]) yield that f is a normal function along the curve γ.

Note that the above function f shows that γ-curvilinear angles $\Delta_\gamma(e^{i\theta}, r)$, $0 < r < 1$, are maximal sets on which a meromorphic function f is a normal function along the curve γ.

R e m a r k . The above construction of the function f shows that Theorems 1 and 3 generalize results from [2] and [6] to a larger class of meromorphic functions than those investigated in these papers.

The significance of the normality of a function along the curve is contained in the fact that the normality is a necessary condition for the existence of γ-curvilinear angular boundary values of a function.

Theorem D. *([9]). Let $f : \mathbf{D} \to \overline{\mathbf{C}}$ be a meromorphic function on the disk $\mathbf{D}$. Then the following statements are equivalent.*

(i) f is a normal function along the curve γ, and there holds

$$\lim_{z(\in\gamma)\to e^{i\theta}} f(z) = \alpha;$$

(ii)

$$\lim_{z(\in\cup_{w\in\gamma}\mathbf{D}_{ph}(w,r))\to e^{i\theta}} f(z) = \alpha \quad \text{for all} \quad 0 < r < 1,$$

that is, α is a γ-curvilinear angular boundary value of the function f at a point $e^{i\theta}$.

Theorem 4. *Let f be a meromorphic function on $\mathbf{D}$ that has the asymptotic boundary value along the curve γ. Then f has a γ-curvilinear angular boundary value if and only if*

$$\sup_{z\in\cup_{w\in\gamma}\mathbf{D}_{ph}(w,r)} (1-|z|^2)f^{\#}(z) \leq C(r) < \infty \quad \text{for all} \quad 0 < r < 1.$$

Proof. The proof follows immediately from Theorems 1 and D, and note that Theorem 4 generalizes Theorem 5 in [5]. ■

Finally, recall that boundary properties of meromorphic functions that are normal on hyperbolic subgroups of the group of conformal automorphisms of the unit disk are investigated in [2], while boundary properties for normality of meromorphic functions on semigroups are investigated in [8]. Theorems D and 4 generalize results from these papers to a larger class of functions.

References

[1] V.I. Gavrilov, On the distribution of values of functions meromorphic in the unit disk which are not normal, *Math. Sbor.*, **67** no. 3, 1965, 408-427 (Russian).

[2] V.I. Gavrilov and E.F. Burkova, On meromorphic functions generating normal families on subgroups of conformal automorphisms of the unit disk, *Dokl. Akad. Nauk SSSR*, **245** no. 6, 1979, 1293-1296 (Russian).

[3] G.M. Goluzin, *Geometric theory of functions of complex variables*, Nauka, Moscow, 1952 (Russian).

[4] W.K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[5] O. Lehto and K.I. Virtanen, Boundary behavior and normal meromorphic functions, *Acta Math.*, **97**, 1957, 47-65.

[6] G. D. Levshina, On meromorphic functions normal and hypernormal with respect to a subgroup of a group of conformal automorphisms of the unit disk, *Dokl. Akad. Nauk SSSR*, **252** no. 3, 1980, 438-440 (Russian).

[7] P. Montel, *Leçons sur les familles normales de functions analytiques et leurs applications*, Paris, 1927.

[8] Ž. Pavićević and J. Šušić, An application of cyclic properties of dinamical systems to the investigations of boundary limits of arbitrary functions, *Dokl. Akad. Nauk*, **387** no. 1, 2002, 16-18 (Russian).

[9] Ž. Pavićević and J. Šušić, Boundary behavior of arbitrary and meromorphic functions on the unit disk (in preparation).

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